KONINKLIJKE AKADEMIE
VAN WETENSCHAPPEN
-:- TE AMSTERDAM -:-

PROCEEDINGS OF THE SECTION OF SCIENCES

VOLUME XXIII
— (2ND PART) —
— (Nos. 6—9) —

PUBLISHED BY
"KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN", AMSTERDAM
JUNE 1922

(Translated from: „Verslagen van de Gewone Vergaderingen der Wis- en Natuurkundige Afdeeling" Dl. XXVIII en XXIX).

CONTENTS.

KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN
TE AMSTERDAM.

PROCEEDINGS

VOLUME XXIII

N°. 6.

President: Prof. H. A. LORENTZ.

Secretary: Prof. P. ZEEMAN.

(Translated from: "Verslag van de gewone vergaderingen der Wis- en Natuurkundige Afdeeling," Vols. XXVIII and XXIX).

CONTENTS.

Physics. — "*On the determination of quanta-conditions by means of adiabatic invariants.*" By G. KRUTKOW. (Communicated by Prof. P. EHRENFEST.)

(Communicated at the meeting of September 25, 1920).

In a series of papers EHRENFEST has shown that only such functions of the general co-ordinates of a mechanical system may be quanticized as are *adiabatic invariants* [1]). These functions can always be found [2]). Moreover, as we shall see, theory may answer the question as to the number of *essential* adiabatic invariants, which in accordance with the quanta-hypothesis have to assume discontinuous values. If we suppose that the "density of probability" of the motion of the system, when not adiabatically acted upon, does not depend explicitly on the time, and if then by means of some hypothesis or some theorem which is derived from the properties of the system, we replace the *time-mean* of a phase-function by a *numerical mean*, it follows immediately that *the number of essential invariants is equal to the number of determining quantities of the system which is left after the numerical mean has been determined* (comp. equations (12) sqq. below). By the determination of the adiabatic invariants and the separation of the *essential* ones the uncertainty as to the choice of the forms of motion which are admissible on the quanta-hypothesis, becomes materially lessened. Still we must not expect that the adiabatic invariants which we have found are necessarily those which have to be quanticized: any arbitrary function of those quantities is again an adiabatic invariant and has thus equal claims to being selected. However, this liberty of choice can be somewhat restricted; there is a further condition to which we may subject the quanta-functions. This condition is of the nature of a hypothesis, but we may give it a simple statistical interpretation. In every case, where the theory of quanta has been applied with success [3]), the condition is fulfilled. It was introduced by PLANCK as a *fundamental theorem* for a complete determination of the quantities

[1]) P. EHRENFEST. These Proc. XIX N⁰. 3, p. 576. Ann. der Phys. 51 (1916) p. 327.

[2]) G. KRUTKOW. Proc. Amst XXI p. 1112. 1919.

[3]) My knowledge of the literature of the subject does not, however, extend beyond the beginning of 1917.

which have to be quanticized[1]). A new proof will be given by establishing a connection between the adiabatic invariants and the phase-space (below 18").

This connection, which will be found to arise in a natural way, with a concept derived from statistical mechanics, strengthens the bond between it and the theory of quanta, a bond which, as it seems to me, has gone into the background in the latest development of the theory or at least has not been sufficiently emphasized, although in my opinion it is of great importance. In view of this connection I think that the only justification of the expression "action-quantum" is the fact that it recalls to our mind the dimensions of the phase-extension.

Another conception of great importance to the theory of quanta which will find a place in our classification is PLANCK's [2]) *coherence of degrees of freedom.* To me it seems of fundamental importance. Its meaning will be found to appear very clearly by a juxtaposition of the properties of a conditionally periodic system and a BOLTZMANN "ergode".

This coherence of degrees of freedom must be very clearly distinguished from what is called "*degeneration*" [3]). For instance from our point of view an ergodic system is to the highest degree coherent, but could in no case be called degenerated. For a degenerated system the number of *essential* adiabatic invariants is *greater* than that of the degrees of freedom, for a coherent system it is *smaller*.

The question arises: must the *supernumerary* adiabatic invariants of a degenerated system be quanticised or, as suspected by SCHWARZSCHILD [4]), is the number of quanta-conditions smaller for such a system for the normal case without degeneration?

For the solution of these questions the three steps which have been taken viz. (1) establishment of the adiabatic invariants (2) selection of the *essential* ones and (3) "normalisation" of the latter are insufficient. In order to get nearer to the solution we must, I think, take into account, that *the quanta-functions must have a meaning which is independent of the system of co-ordinates.* We may undoubtedly postulate this: if the quanta-laws are really physical laws, they must necessarily satisfy this condition. The question is, how to formulate this new invariance of the quanta-functions? I shall not try to discuss it here in general; but only remark that

[1]) M. PLANCK. Ann. d. Phys. 50 (1916) p. 392.

[2]) M. PLANCK, l.c.

[3]) K. SCHWARZSCHILD. Sitzungsber. Berlin 1916. P. EPSTEIN Ann. d. Phys. 51 (1916). p. 168.

[4]) K. SCHWARZSCHILD, l.c.

we may return from the canonical equations which are so convenient in the theory of quanta to the equations on cartesian co-ordinates. Here the invariance in question means: invariance with respect to the groups of rotations and translations; vector-analysis thus provides the means of testing hypothetical quanta-quantities for the new postulata [1]).

The above mentioned means enable us in special cases to separate the quanta-quantities without ambiguity, for instance for the mechanical systems considered by PLANCK in the paper quoted. In some cases, however, an ambiguity remains, which we may get rid of in the following manner: by putting all but one of the quanta-quantities equal to nought, a "*singular motion*" must be obtained. In this manner we are able to make a connection between the methods sketched out above and PLANCK's theory on the physical structure of the phase-space, PLANCK's singular motions forming the last step in the series. We may recapitulate as follows:

The quanta-quantities are (α) functions of the integrals of the equations of motion (β) adiabatic invariants which (γ) must be "*normalized*" and (δ) have a meaning which is independent of the system of co-ordinates and finally (ϵ) yield singular motions in PLANCK's sense of the expression.

§ 1. *The fundamental equation.*

Let a mechanical system of n degrees of freedom be given by its canonical equations of motion

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \; ; \; \dot{q}_i = \frac{\partial H}{\partial p_i} \; (i = 1,2,\ldots.n) \; . \; . \; . \; . \; . \quad (1)$$

We shall consider a number of systems and introduce a function $\varrho(p_i,q_i,t)$ which may be called the *density of probability*: ϱ must satisfy the fundamental equation of statistical mechanics [2]):

$$\frac{\partial \varrho}{\partial t} + \sum_{i=1}^{n} \left(\frac{\partial \varrho \dot{p}_i}{\partial p_i} + \frac{\partial \varrho \dot{q}_i}{\partial q_i} \right) = 0 \; . \; . \; . \; . \; . \; . \quad (2)$$

or, using (1):

$$\frac{\partial \varrho}{\partial t} + \sum_{i=1}^{n} \left(\frac{\partial \varrho}{\partial p_i} \dot{p}_i + \frac{\partial \varrho}{\partial q_i} \dot{q}_i \right) = \frac{d\varrho}{dt} = 0 \; . \; . \; . \; . \quad (2')$$

ϱ is therefore a function of the integrals of the equation (1).

---

[1]) In the theory of the ZEEMAN-effect as given by SOMMERFELD and DEBIJE (Phys. Zschr. 17. (1916) a difficulty is met with here. This may, I think, be evaded in different ways, but I am not able to give a *uniquely* determined solution.

[2]) J. W. GIBBS. Scientific papers. II' p. 16; Statistical Mechanics. Chapter I.

If we suppose that the condition is stationary:

$$\frac{\partial \varrho}{\partial t} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

it follows that: ϱ is a function of those integrals which do not contain t explicitly, i.e. of $(2n-1)$ integrals, if only, as we shall suppose all the time, H does not depend on t explicitly.

We are at liberty to understand by ϱ the density of probability a posteriori or a priori. When applied to the theory of quanta our result expresses the fact, that *the quanta-quantities are functions of the* $(2n-1)$ *integrals of equations* (1) *which are independent of* t.

Replacing the $2n$-dimensional phase-space (p_i, q_i) by the corrresponding integral space (c_i, t_i)[1]) the "path" of the system is a straight line parallel to the t_1-axis. We can describe these lines either by making t increase, i.e. by following a definite system in its motion, or by keeping t constant and varying τ, i.e. considering together all the systems with given $c_1 \ldots c_n\, t_2 \ldots . t_n$ and all possible values of t_1.

§ 2. *H contains a variable parameter.*

If H contains a parameter which may either have a constant value as in the case just considered or vary slowly[2]), the quantities c_i and t_i are no longer constant, but variable; they have to satisfy the following "equations of motion"[3]):

$$\dot{c}_i = -\frac{\partial K}{\partial t_i} \quad ; \quad \dot{t}_i = \frac{\partial K}{\partial c_i} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where:

$$K = c_1 + \left(\frac{\partial V}{\partial a} \dot{a}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

[1]) As in a previous communication we write the integrals of equations (1) in the form:

$$H = c_1, \quad H_2 = c_2; \ldots ., \quad H_n = c_n.$$

and

$$\frac{\partial V}{\partial c_1} = t + \tau = t_1, \qquad \frac{\partial V}{\partial c_2} = t_2, \ldots ., \qquad \frac{\partial V}{\partial c_n} = t_n,$$

where $c_1, \ldots, c_n, \tau, t_2, \ldots, t_n$ represent the $2n$ integration-constants and V JACOBI's characteristic function. Comp. Proc. XXI, p. 1112, 1919.

[2]) The slowness is expressed in the fact, that H contains only a, not the correspondiug momentum.

[3]) Proc. l.c.

or putting $a =$ const. approximately and representing the derivatives with respect to a by dashes:

$$\left.\begin{aligned} c'_i = -\frac{\partial}{\partial t_i}\left(\frac{\partial V}{\partial a}\right) \quad ; \quad t'_x = \frac{\partial}{\partial c_x}\left(\frac{\partial V}{\partial a}\right) \\ t'_1 = \frac{1}{a} + \frac{\partial}{\partial c_1}\left(\frac{\partial V}{\partial a}\right) \end{aligned}\right\}\begin{pmatrix} i = 1, 2, \ldots, n \\ x = 2, 3, \ldots, n \end{pmatrix} \quad . \quad . \quad (3')$$

Since the equations have the canonical form, we have as the fundamental equation:

$$\frac{\partial \varrho}{\partial a} + \sum_{i=1}^{n}\left(\frac{\partial \varrho}{\partial c_i} c'_i + \frac{\partial \varrho}{\partial t_i} t'_i\right) = 0 \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Here we may not as before take $\frac{\partial \varrho}{\partial a} = 0$. A further difficulty presents itself: starting from a special line parallel to the t_1-axis in the (c_i, t_i) space — a special "stream-line" — if we now vary a, as equation (3) or (3′) show, the stream-line becomes broken up. If we then keep a constant again and take together the points, that lie on a straight line, ϱ will vary along this stream-line, since it contains points of different origin. Thus on the new line ϱ is not stationary, but explicitly dependent on t_1.

We now form [1]) the time-mean of ϱ, which we shall call $\bar{\varrho}$ and the difference $\varrho - \bar{\varrho}$. Since $\int dt_1 \, (\varrho - \bar{\varrho}) = 0$, the quantity $\varrho - \bar{\varrho}$ in its dependence on t_1 shows elevations and depressions round about $\bar{\varrho}$, the sum of the surfaces of the former being equal to that of the latter. Each point carries its $\varrho - \bar{\varrho}$ value along with it and hence the curve shifts regularly with the time t_1. A stationary curve represents the tendency towards condensation (in an elevation) or rarefaction (in a depression) for the points of the stream-line, on the supposition of the change of a being sufficiently small. If we make our moving curve slide along the stationary one, in the course of time elevations will cover depressions and vice versa. A further small change of a may therefore produce a diminution of the difference $\varrho - \bar{\varrho}$. By this reasoning it becomes clear that starting from a stationary density a *sufficiently slow* change of a will to a corresponding degree of approximation produce a stationary density [2]).

---

[1]) For the method now following comp. J. W. GIBBS. Statistical Mechanics. Chapter XIII.

[2]) Comp. J. M. BURGERS. Proc. Amst. XX (1916) 149, Ann. d. Phys. 1917 (2) and my paper in the Proc. Amst. l. c.

We will therefore suppose that a changes slowly in the sense of the theory of adiabatic invariants. Let Da be the total change of a, i.e.

$$Da = \dot{a}\int dt$$

and let Dc_i and Dt_x represent the corresponding changes of c_i and t_x; considering further that

$$\dot{a} = const.,$$

and hence

$$\dot{a} = \frac{Da}{Dt} \quad \text{with} \quad (Dt = \int dt):$$

we find

$$\frac{Dc_i}{Da} = -\overline{\frac{\partial}{\partial t^i}\left(\frac{\partial V}{\partial a}\right)} \quad ; \quad \frac{Dt_x}{Da} = \overline{\frac{\partial}{\partial c_x}\left(\frac{\partial V}{\partial a}\right)}, \quad . \quad . \quad . \quad . \quad (6)$$

where the horizontal line indicates the time-mean. If in equation (5) we take c'_i and t'_i to mean these time-means, we obtain

$$\frac{\partial \varrho}{\partial a} + \sum_{i=1}^{n}\left(\frac{\partial \varrho}{\partial c_i}\frac{Dc_i}{Da} + \frac{\partial \varrho}{\partial t_i}\frac{Dt_i}{Da}\right) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Since ϱ is independent of t, the corresponding term under the summation-sign in (7) mnst be omitted.

§ 3. *Phase-space and adiabatic invariants.*

The stationary density ϱ need not depend on all the variables

$$c_1, \ldots, c_n \quad ; \quad t_2, \ldots, t_n$$

For example in a *conditionally-periodic system* without commensurable relations between the periodicity-moduli ϱ depends on the quantities c_i only. This follows from the theorem which allows us to replace the time-average by an averaging over a Ω-cell [1]). For an *ergodic* (or quasi ergodic) system in consequence of the *ergode-hypothesis* ϱ depends on the energy c_1 only. We shall here suppose, that ϱ depends on k quantities $(k \leq n)$, which we shall indicate by

$$c_1, c_2, \ldots, c_k$$

These integrals may be called *essential* integrals. Our supposition with regard to ϱ comes to the same as assuming that for our system the time-mean may be replaced by a definite numerical mean. To compute this we proceed as follows.

Suppose the system of equations

$$H_1 = c_1, H_2 = c_2, \ldots, H_k = c_k \quad . \quad . \quad . \quad . \quad . \quad (8)$$

to be soluble with respect to $p_1, p_2, \ldots, p_k$, thus

[1]) J. M. Burgers. l.c. and my paper l.c.

$$p_\lambda = k_\lambda (q_1, \ldots, q_n, p_{k+1} \cdot p_n; c_1, \ldots c_k) \quad (\lambda = 1, 2, \ldots, k) \quad . \quad (8')$$

Introducing the differentials $dc_1 \ldots . dc_k$ instead of the differentials $dp_1 \ldots . dp_k$ into the phase-integral

$$I = \iint \ldots \iint dp_1 \ldots . dp_n \, dq_1 \ldots . dq_n \quad . \quad . \quad . \quad . \quad (9)$$

we find

$$I = \int \ldots \int dc_1 \ldots dc_k \int \ldots \int dp_{k+1} \ldots dp_n \, dq_1 \ldots dq_n \frac{\partial(p_1, \ldots p_k)}{\partial(c_1, \ldots c_k)} \quad (10)$$

or

$$I = \int \ldots \int dc_1 \ldots dc_k \, \omega(c_1, \ldots c_k) \quad . \quad . \quad . \quad . \quad (10')$$

where

$$\omega(c_1, \ldots, c_k) = \int \ldots \int dp_{k+1} \ldots dq_n \frac{\partial(p_1, \ldots, p_k)}{\partial(c_1, \ldots, c_k)} \quad . \quad . \quad (11)$$

In (11) the integration has to be carried out, the limits being determined by (8). From the (p_i, q_i)-space or the (c_i, t_i)-space we may pass to the k-dimensional $(c_i, \ldots c_k)$-space. A streamline of the former space corresponds to a fixed point in the latter. The density ϱ is replaced by $\varrho\omega$ in the c-space. Its elements therefore have the *weight* ω. For the iso-parametric motion ($a =$ const.) the c-space is *static* i.e. each point is fixed. The integral (11) gives us the numerical mean looked for, namely, if f is a phase-function, we have:

$$F = \int \ldots \int f \frac{\partial(p_1 \ldots p_k)}{\partial(c_1 \ldots c_k)} dp_{k+1} \ldots dq_n / \omega . \quad . \quad . \quad . \quad (11')$$

Returning to equation (7) we now have:

$$\frac{\partial \varrho}{\partial a} + \sum_{i=1}^{k} \frac{\partial \varrho}{\partial c_i} \frac{Dc_i}{Da} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

since ϱ is a function of $c_1, \ldots, c_k$ only. Similarly the quantities $Dc_i/_{Da}$ only depend on $c_1, \ldots, c_k$, as is easily seen from (6), if on the right hand side we replace the time-mean by the numerical mean (11'). Therefore ϱ retains its property $\varrho = \varrho(c_1, \ldots, c_k)$ when a changes. Equation (12) expresses, that ϱ is a function of those k integrals of the differential equations (6) which only contain the quantities $c_1, \ldots, c_k$. These integrals are obtained by integrating the set of k differential equations which on the left side contain the quantities $Dc_i/_{Da}$ $(i = 1, 2, \ldots, k)$. They are the *essential adiabatic invariants*, and we have thus proved that ϱ *is a function of the essential adiabatic invariants*.

Let us further consider the c-space. If a varies slowly, the fixed points in it begin to move. Since in this motion the points do not disappear nor new points are formed, the density $\rho\omega$ must satisfy the equation of continuity, i. e. our fundamental equation. As $\rho = \text{const.}$ is certainly a possible solution, ω itself must satisfy the equation

$$\frac{\partial \omega}{\partial a} + \sum_{i=1}^{k} \frac{\partial \omega \overline{c'_i}}{\partial c_i} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

where

$$\overline{c'_i} = \frac{D c_i}{D a}$$

Or with the notation

$$\frac{D\omega}{Da} = \frac{\partial \omega}{\partial a} + \sum_{i=1}^{k} \frac{\partial \omega}{\partial c_i} \frac{D c_i}{D a},$$

in the equivalent forms

$$\frac{D\omega}{Da} + \omega \sum_{i=1}^{k} \frac{\partial \overline{c'_i}}{\partial c_i} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (13')$$

or

$$\sum_{i=1}^{k} \frac{\partial \overline{c'_i}}{\partial c_i} = -\frac{1}{\omega} \frac{D\omega}{Da} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13'')$$

For the quantity on the left side — the "divergence" — we shall deduce another expression.

The essential adiabatic invariants — k in number — satisfy the equations

$$\frac{D v_i}{D a_i} = \frac{\partial v_i}{\partial a} + \sum_{i=1}^{k} \frac{\partial v_i}{\partial c_i} \overline{c'_i} \quad (i = 1, 2, \ldots, k) . \quad . \quad . \quad . \quad (14)$$

We shall suppose that the quantities v_i can be expressed in the quantities c_λ $(\lambda = 1, 2 \ldots k)$ or

$$\frac{\partial (v_1, \ldots, v_k)}{\partial (c_1, \ldots, c_k)} = \Upsilon \neq 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

The properties of our system can be equally well described by the quantities v_i as by the quantities c_i. The $(v_1 \ldots v_k)$-space has the advantage over the c-space of being *static*, also with respect to the action of adiabatic influences. Let us now examine the mutual relation of the two spaces.

To this end we shall consider the D-derivative of the determinant Υ:

$$\frac{D\Upsilon}{Da} = \sum_{i\mu} V_{i\mu} \frac{D v_{i\mu}}{Da}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

where $v_{i\mu}=\frac{\partial v_i}{\partial c_\mu}$ and $V_{i\mu}$ represents the corresponding sub-determinant. We have identically

$$\frac{Dv_i}{Da}=\frac{\partial v_i}{\partial a}+\sum_\lambda \frac{\partial v_i}{\partial c_\lambda}\overline{c'_\lambda}=0,$$

hence

$$\frac{\partial}{\partial c_\mu}\frac{Dv_i}{Da}=\frac{\partial^2 v_i}{\partial a\,\partial c_\mu}+\sum_\lambda \frac{\partial^2 v_i}{\partial c_\lambda \partial c_\mu}\overline{c'_\lambda}+\sum_\lambda \frac{\partial v_i}{\partial c_\lambda}\frac{\partial \overline{c'_\lambda}}{\partial c_\mu}=0 \quad . \quad . \quad . \quad (a)$$

and on the other hand

$$\frac{D}{Da}\frac{\partial v_i}{\partial c_\mu}=\frac{\partial^2 v_i}{\partial a\,\partial c_\mu}+\sum_\lambda \frac{\partial^2 v_i}{\partial c_\mu \partial c_\lambda}\overline{c'_\lambda} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

From (a) and (b) it follows that

$$\frac{D}{Da}\frac{\partial v_i}{\partial c_\mu}=\frac{Dv_{i\mu}}{Da}=-\sum_\lambda \frac{\partial v_i}{\partial c_\lambda}\frac{\partial \overline{c'_\lambda}}{\partial c_\mu} \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

or

$$\frac{Dv_{i\mu}}{Da}=-\sum_{\lambda=1}^{k} v_{i\lambda}\frac{\partial \overline{c'_\lambda}}{\partial c_\mu} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17')$$

Substituting in (16) we obtain

$$\frac{D\Upsilon}{Da}=-\sum v_{i\lambda} V_{i\mu}\frac{\partial \overline{c'_\lambda}}{\partial c_\mu} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16')$$

Since $\Sigma v_{i\lambda} V_{i\mu}=0$ for $\lambda \neq \mu$ and equal to Υ for $\lambda=\mu$, (16′) becomes

$$\frac{D\Upsilon}{Da}=-\Upsilon\sum \frac{\partial \overline{c'_\lambda}}{\partial c_\lambda} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16'')$$

Hence

$$\sum_\lambda \frac{\partial \overline{c'_\lambda}}{\partial c_\lambda}=-\frac{1}{\Upsilon}\frac{D\Upsilon}{Da} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16''')$$

Comparing this result with (13″) it follows that

$$\frac{1}{\Upsilon}\frac{D\Upsilon}{Da}=\frac{1}{\omega}\frac{D\omega}{Da} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

or

$$\frac{D}{Da}\log\frac{\Upsilon}{\omega}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18')$$

In other words: *Υ/ω is an adiabatic invariant* or

$$\Upsilon=\omega f(v_1,\ldots,v_k),\qquad \omega=\Upsilon F(v_1,\ldots,v_k) \quad . \quad . \quad . \quad (18'')$$

Substituting this value of ω in the integral (10′) we find

$$I=\int\ldots\int dc_1\ldots dc_k\,\Upsilon F(v_1,\ldots,v_k)=\int\ldots\int dc_1,\ldots,dc_k\frac{\partial(v_1\ldots v_k)}{\partial(c_1\ldots c_k)}F. \quad (19)$$

or

$$I=\int \ldots \int dv_1 \ldots dv_k\, F(v_1, \ldots, v_k) \quad . \quad . \quad . \quad . \quad (19')$$

Now we can always arrange, that F becomes equal to 1. We have only to introduce, instead of one of the v_i, the adiabatic invariant

$$\mathfrak{r}_1 = \text{funct.}\,(v_1, \ldots, v_k)$$

and submit it to the condition

$$\frac{\partial \mathfrak{r}_1}{\partial v_1} = F(v_1, \ldots, v_k) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

We then find

$$\frac{\partial \mathfrak{r}_1}{\partial c_\lambda} = \sum_x \frac{\partial \mathfrak{r}_1}{\partial v_x} \frac{\partial v_x}{\partial c_\lambda} = \sum_x \frac{\partial \mathfrak{r}_1}{\partial v_x} v_{x\lambda}$$

$$\Upsilon^* = \frac{\partial(\mathfrak{r}_1, v_2, \ldots, v_k)}{\partial(c_1, c_2, \ldots, c_k)} = \sum_\lambda \frac{\partial \mathfrak{r}_1}{\partial c_\lambda} V_{1\lambda} = \sum \frac{\partial \mathfrak{r}_1}{\partial v_x} V_{i\lambda} v_{x\lambda} = \frac{\partial \mathfrak{r}_1}{\partial v_1} \Upsilon,$$

or substituting for Υ its value ω/F:

$$\Upsilon^* = \omega \frac{\partial \mathfrak{r}_1}{\partial v_1} \frac{F}{1} = \omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Calling the thus *normalized set of essential adiabatic invariants* $v_1, \ldots, v_k$, we find

$$I=\int \ldots \int dv_1 \ldots dv_k . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

The *v-space* which is static with respect to adiabatic action is "*weightless*": its density ϱ is simply equal to $\varrho(v_1, \ldots, v_k)$. The quantities $v_1, \ldots, v_k$ *may* be quanticized. Its property which is expressed by eq. (22) is nothing but the fundamental law which according to PLANCK's hypothesis the quanta-quantities have to obey [1]. By our theorem (18″) this hypothesis is connected with the adiabatic invariants and thus finds a new confirmation. The property of the v-space being "weightless" displays the character of this fundamental law as a natural generalisation of the old quanta-hypothesis.

§ 4. *On the coherence of degrees of freedom.*

From the point of view now attained this very important conception appears as a natural consequence of our suppositions. If the number k — that of the essential integrals and adiabatic invariants — is smaller than the number of degrees of freedom n, as appears from (22), some of the quantities v_i must necessarily be of the dimen-

[1]) M. PLANCK. l.c.

sion h^p $(p > 1)$, since the dimension of I is h^n. In order to illustrate this and the previous results we shall contrast the properties of a BOLTZMANN *ergode* and a *conditionally periodic* system without commensurable relations:

ergode	*conditional periodic system*
essential integrals	
$H = c_1$	$H_1 = c_1,\ H_2 = c_2, \ldots,\ H_n = c_n$.
numerical mean	
$\int \ldots \int dp_1 \ldots dp_n\, dq_1 \ldots dq_n \dfrac{\partial p_1}{\partial c_1}$. (23)	$\int \ldots \int dq_1 \ldots dq_n \dfrac{\partial (p_1, \ldots, p_n)}{\partial (c_1, \ldots, c_n)}$ (23')
density	
$\varrho = \varrho (c_1, a)$	$\varrho = \varrho (c_1, \ldots, c_n; a)$
essential adiabatic invariants	
$V = \underset{H \leq c_1}{\int \ldots \int} dp_1 \ldots dq_n$	$v_i = \int_0 p_i\, dq_i\ (i = 1, 2, \ldots, n)$.
density	
$\varrho = \varrho (V)$	$\varrho = \varrho (v_1, v_2, \ldots, v_n)$

The conditionally periodic system is what BOLTZMANN calls a *subergode*. On the other hand the ergode appears as a *coherent* system with a smallest value of k, viz. $k = 1$. These short indications may suffice for the present.

§ 5. *Degeneration.*

A conditionally periodic system is called degenerated, if there are commensurable relations between the periodicity moduli. It is evident, that our system covers a *lower* set of points with its orbital curve everywhere densely, than when there are no such relations. Accordingly the numerical mean will be of a lower dimension and more quantities will remain free after the averaging process. Thus besides the quantities c the quantities t will play a part: the number of essential adiabatic invariants becomes larger than the number of degrees of freedom. The question, whether these supernumerary quantities have to be quanticized, we shall not discuss here. A good instance for the discussion of the questions which may arise here is afforded by the quanta-quantities in EPSTEIN's theory [1]) of the STARK-effect for an infinitely weak external electric field; the "parabolic" quanta-quantities which are found in this case cannot

[1]) P. EPSTEIN. Ann. d. Phys. 50. p. 490.

be represented as function of SOMMERFELD's "spherical" quanta-quantities alone; other adiabatic invariants containing the quantities t_2 and t_3 are essential in this case.

I am fully conscious of the fact, that by the above considerations the difficulties which still beset the theory of quanta are in no way removed, but only shifted. Still it seems to me that even the possibility of such displacement deserves attention. Moreover I expect that in *special* cases the general theory tentatively sketched out here may be found useful.

Physical Laboratory of the University. Petrograd, April 1, 1920.

Physiology. — "*On Sensibilization to Radioactivity by the action of Hormones*". By Prof. H. ZWAARDEMAKER.

(Communicated at the meeting of Sept. 25, 1920).

Sensibilization of organisms or organs to the energy of light has long since been a familiar process in physiology. It is especially H. VON TAPPEINER [1]) who has called attention to some fluorescent substances, which in the presence of oxygen largely increase the deleterious influence of light. This noxious influence resembles the influence of ultraviolet rays. Mr. and Mrs. HENRI [2]) have detected likewise an action of the colloidal selenium for the ultra-violet rays.

Long afterwards similar effects have been detected for the Röntgen-rays. This is instanced by the use of enzytol (10°/₀ solution of boric acid cholin) to increase the destruction of malignant tumors in Röntgenization [3])

In 1917 [4]) I have established sensibilization for Becquerel-rays. Here also there were fluorescent substances which brought it about, to wit fluorescein and eosin. The former had the stronger action for the α-rays, the latter for the β-rays. Their action took place irrespective of the presence or the absence of light. In virtue of standard experiments with adsorbentia (e.g. talcum venetum) I correlated the sensibilization in these cases with a reinforcement of the adsorptions, which the radioactive ions undergo through the action of the sensibilizers. In these experiments we found a supersession of fluorescein adsorption by eosin and not the reverse, running parallel to a supersession of the sensibilization of fluorescein by eosin and not the reverse [5]). In general the adsorptions play a prominent part in the action of radio-active atoms of the circulating fluids, because the ions, moving freely in the fluid, exert an influence

[1]) H. VON TAPPEINER, die photodynamische Erscheinung (Sensibilisierung durch fluoreszierenden Stoffe). Ergebnisse der Physiol. Bd. 8. S. 698. 1909.

[2]) M. ET MME. VICTOR HENRI, Action photodynamique du sélénium colloidal, Soc. de Biologie. C. R. du 24 févr. 1912.

[3]) A full list of the literature on cholin action is to be found in DORN Strahlentherapie Bd. 8. S. 499.

[4]) Kon. Akad. v. Wetenschappen, Amsterdam 27 Sept 1917.

[5]) See for the technique A. M. STREEF, Onderz. Physiol. Lab. Utrecht, 5e Reeks XVIII p. 59.

only when attached through adsorption to the surfaces of the cells, and not when they are located at various distances from the cells. An improved adsorption, by which a larger number of a certain group of ions attaches itself to the cells in a circulating fluid of a given composition, improves the result elicited by these ions.

There is a simple means to detect sensibilizers for radioactivity. One has only to start from radio-active antagonism [1]).

We, therefore, preferred to experiment on the heart of a cold-blooded animal, because the cells of this organ, the seat of automatic movement, are washed directly by the circulating fluid.

The heart of an eel or a frog beats only when, given the further necessary conditions, a radio-active component is present in the circulating fluid in the proper dosage. It does not matter whether the element under consideration is an α-rayer or a β-rayer (our normal β-rayer is potassium). When applying the α- and the β-rayer simultaneously, the quanta may be counterbalanced so as to inhibit each other's effect completely. At such a moment there is a standstill. A slight balance on the one side or the other will restore automaticity.

The dosage of α- and β-rayers in the circulating fluid must be much smaller in summer than in winter. When taken alone, 5 mgr of potassium chloride or 0,1 mgr of uranylnitrate per litre circulating fluid is in summer sufficient to maintain the automaticity of sensitive hearts. In winter at least 20 mgr of potassiumchloride or 10 mgr of uranylnitrate per litre is needed. Accordingly the summer-, and the winter-equilibria differ very much. In summer a combination of 20 to 30 mgr of potassiumchloride and 0,1 mgr of uranylnitrate (per litre) may arrest the heart's action; in winter this result can be achieved only by 40 mgr of potassiumchloride and 10 mgr of uranylnitrate.

After having secured an equilibrium, no matter in what season, a number of substances will give a shifting, which again restores automaticity. Among the anorganic components it is especially the calcium-ion to which we must ascribe a great influence; among the organic substances I found a number of substances having in common the property of considerable surface-activity (as observed for the boundary layer air-water).

Shiftings may be observed on either side. When on the α-side, so that a uranium-beat ensues, fresh potassium has to be added to obtain a standstill again. When the shiftings are on the β-side, uranium must be added to produce the same effect.

---

[1]) Kon. Akad. v. Wetensch. 27 April 1917.

The shifting does not at all prove that the substance, by which it is generated, is a sensibilizer; it only renders this probable. To ascertain this we have to find out the maximum-, and minimum-doses, between which the automaticity of the organ can be maintained; the potassium limits are the most important. In summer these extremes vary with the individuality of the animal, and range from 5 to 20 and from 10 to 300 mgr. of potassium chloride per Litre, a low threshold corresponding with a low upper-limit, etc.; in winter the extremes are more constant; 20—30 and 600—800 mgr. potassium chloride per litre. Similar observations were made for the other radio-active elements, but I pass them over in silence, since these elements do not occur in the animal organism. The same holds good for the majority of the sensibilizers found by us.

An exception is afforded by cholin and adrenalin, both hormones occurring in every organism. Their sensibilizing power for BECQUEREL rays is very strong, even when the dosis of cholin is one mgr. per litre of circulating fluid and of adrenalin 0,001 mgr. [1])

In the presence of one of these hormones the potassium-dosis that keeps up the heart's action, may be reduced to half the normal dosis, nay, to less even. In summer, therefore, these dosages are extremely small, even $^1/_2$ mgr. of KCl per litre. Then the greatest purity of chemicals is of the utmost importance.

However, there is a difference: Cholin shifts a potassium-uranium equilibrium towards the potassium-side, adrenalin towards the uranium-side. Whether this difference will also manifest itself in normal life is still an open question. For aught we know, there is nowhere in the organism an α-rayer, unless it be the trace of rest-activity left behind by emanation, when it is inspired and expired in minimal quanta as an indifferent gas together with the atmospheric air.

Potassium, cholin, adrenalin are normal constituents of the organism. Accordingly, the study of their mutual relations is a true physiological study.

The bio-radio-activity of potassium has no temperature-coëfficient. Both the velocity of effect and the dosage remain the same with 4°, 10° or 20° [2]).

The small differences lie within the latitude of the experimental errors. In this respect physiological radio-activity is analogous to photo chemical actions, whose temperature-coefficient is likewise

[1]) W. LIBBRECHT used for the same purpose but in another connection 0,05 mgr. per litre. (Arch. int. de Physol. T. 15, p. 357).

[2]) Summer- and winterdosage do not differ on account of the difference of temperature. We mention the difference in hormones as a possible cause.

insignificant. Still, the two are not identical. On the contrary, physically corpuscular raying differs fundamentally from light-rays. Nor have we succeeded, in spite of strenuous efforts, not even by means of the most concentrated visible or ultraviolet light, in achieving a recovery of automaticity in an organ perfused with a potassium-free fluid. As regards the temperature-coefficient the analogy between radio-biological and photo-chemical action is still a matter of surprise, even though we are told by modern researchers that in many cases the action of light rests on the liberation of electrons.

These phenomena might be correlated by assuming that the charged particles which send the radio-active radiation with great velocity through the lipoid films on the surface of the cells, evoke inside the cells a catalytic effect, which we usually call a stimulus [1]).

[1]) On physiological radio-activity. Journal of Physiology. Vol. 53 p. 286.

Histology. — "*An Inquiry into the Distribution of Potassium-compounds in the Electric Organ of the Thorn-back* (*Raja Clavata*)." By M. W. WOERDEMAN. (Communicated by Prof. H. ZWAARDEMAKER).

(Communicated at the meeting of October 30, 1920).

On Prof. ZWAARDEMAKER's suggestion I have latterly examined different tissues and organs microchemically on the presence of potassium-compounds. I intend to discuss the method and the results of this inquiry more in detail; for the present I will publish only my experience in the investigation of the electric organ of the Thorn-back (Raja clavata).

ZWAARDEMAKER's researches established that the function of organs, perfused artificially with a salt-solution, largely depends on the potassium-content of that solution, and that the potassium, being a weakly radio-active element, plays such a prominent part in the origin of the organic actions, on account of its very radio-activity. It may be supposed, therefore, that the potassium compounds which are normally to be found in the organs of animals or plants, take an important part in the normal functions of these organs. We presume, therefore, that information concerning the presence or the absence of potassium-compounds in certain cells, tissues, or organs will be gladly received.

Now, through chemical examination the quantum of potassium, contained in various organs, has already been determined with great accuracy. From this examination we do not learn, however, where in the organ the potassium-compounds are located. MACALLUM namely has detected that in a number of tissues and organs the potassium-salts are not distributed at random and irregularly, but that they often occur there at definite places, bound to quite definite structures. MACALLUM's [1]) reagent on potassium-compounds is a modification of the mixture used for the first time by DE KONINCK [2]). MACALLUM's mixture of cobalt-salt and sodium-nitrite, added to a potassium-salt

---

[1]) A. B. MACALLUM. On the distribution of potassium in animal and vegetable cells. Journ. of Physiol Vol. XXXII, 1905 and Die Methoden und Ergebnisse der Mikrochemie in der biologischen Forschung. Ergebn. der Physiologie, Jrg. 7 1908, p. 552.

[2]) DE KONINCK. Zeitschr. f. analyt. Chemie. Bnd. 20. 1881, p. 390.

solution precipitates FISCHER's crystalline salt (a potassium-cobalt sodium-nitrite). MACALLUM puts sections of frozen material or teased out preparations of fresh tissue in his cobalt-nitrite-sodium-nitrite mixture. In the tissues and in the cells the potassium-precipitate can now be formed. After being thoroughly washed with distilled water, by which the reagent is washed away, the tissue is subjected to an after-treatment with ammonium sulphide, which renders FISCHER's salt black (formation of cobalt-sulphide). Wherever microscopical examination reveals black precipitates or black decoloration, we may decide upon the presence of potassium-compounds.

If e.g. voluntary muscle-tissue is treated after MACULLUM's method, it will be seen that the potassium-compounds occur almost exclusively in the doubly refracting discs of the muscular fibres. The discs of single refraction do not contain any potassium. Now, because the electric organs of the so-called electric fishes are developed (with a few exceptions) from voluntary muscular tissue, and since nobody has as yet studied the distribution of potassium-compounds in this metamorphosed muscular tissue, we considered an inquiry into the distribution of potassium compounds in the electric organs of some consequence. I regret to say that, although Dr. C. KERBERT, Director of Natura Artis Magistra offered his kind assistance, the strong electric fishes could not be put at my disposal. But also the Thorn-back (Raja clavata), occurring at the Dutch North Sea-coast, has an electric organ, which, though its action is weak, largely resembles in structure the organs with stronger action. I, therefore, applied to the Zoological Station at Helder, where, through the kindness of Dr. REDEKE and Dr. VAN GOOR I was enabled to perform the potassium-reaction in the living electric organ of a thornback. The tissue was put in the reagent and afterwards made into sections with an ice-microtome (spray from liquid carbon dioxide). Although it is better to make the sections first and to treat them immediately after with the reagent, I followed the other way, because there was no freezing microtome at the Station. This altered working-method, however, did not lessen the value of the results. I took care to put very small pieces of tissue in the reagent in order to allow the reagent to permeate the tissue as effectually and as rapidly as possible. As I do not see any difference between the outer and the central parts of the preparations, the diffusion appears to have succeeded well. Before reporting the results of this inquiry I may as well remind the reader that the electric organ of the Thorn-back is situated within the lower two-thirds of the tail. There it lies at the side of the dorsal and ventral tail-muscles. It consists of a large

number of so-called electric platelets disposed in rows running parallel to the axis of the body. Each platelet (see fig. 1) consists of 1. a so-called anterior cortical layer composed of a single layer of cells; 2. an inner layer composed of numerous twisted lamellae

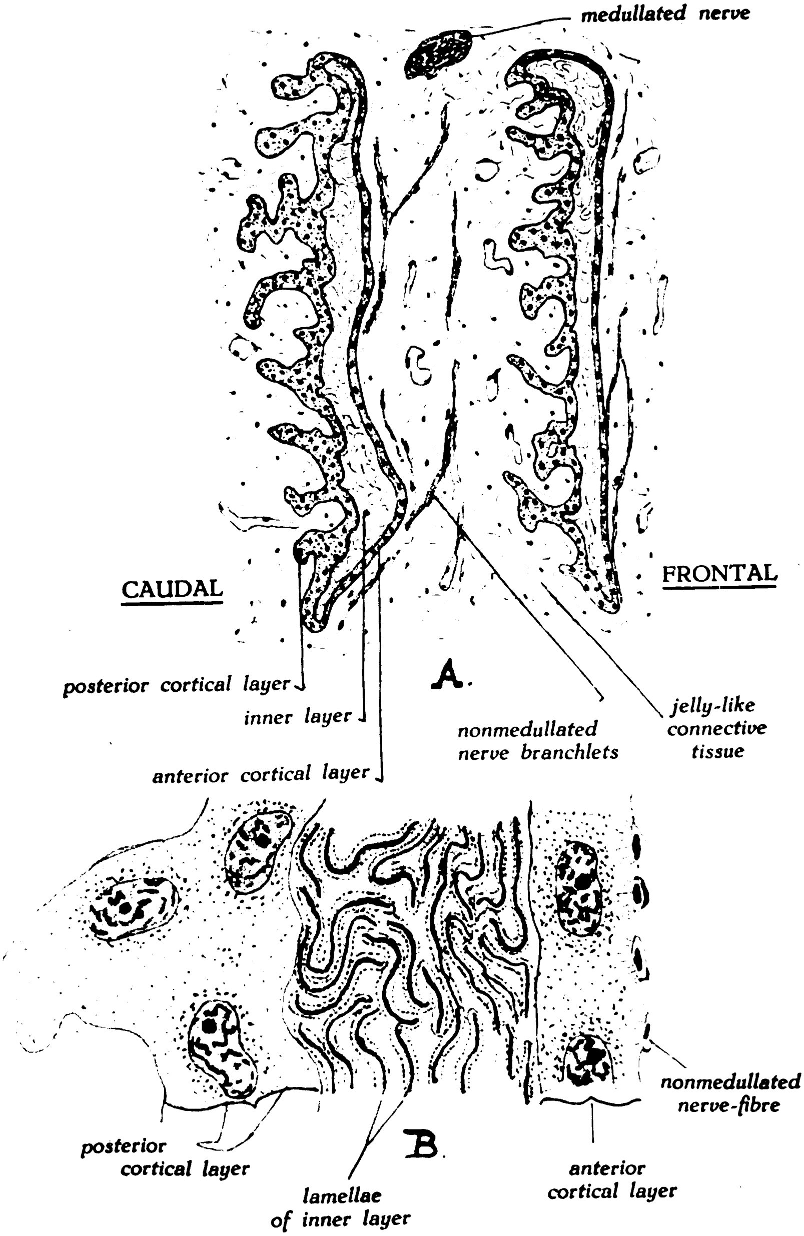


Fig. 1. A. Sagittal section through electric organ of thorn-back. (only two electric platelets have been represented).
B. Part of an electric platelet, more enlarged. (Diagrams, made after picture of frozen sections).

and 3. a posterior cortical layer. The latter consists, like the anterior layer, of a single layer of cells, but this layer is folded in so many ways, that in sections the impression is excited as if the posterior cortical layer is provided with fringe-like prolongations. The entire platelet is enveloped by a very thin homogeneous membrane (the electrolemma). A very fine network of non-medullated nerve-fibres lies against the anterior cortical layer. Where the fibres reach the cells of this layer, those cells present peculiar rods. The inner layer is the most complicated. To realize its structure we should bear in mind that each electric platelet must be considered as a modified muscular fibre.

The lamellae of the inner layer are derived from the anisotropous- and isotropous-discs of the striated muscle-fibre (see also ENGELMANN's researches in "Onderzoekingen Physiol. Lab. Utrecht 4e Reeks III, p. 307).

The doubly refracting layers become thicker, they lose their faculty of double refraction, while the isotropous discs remain visible as finer, dark stripes. The layers are sinuously disposed and thus originates the complicate structure of the inner layer of the electric platelet.

BABUCHIN could distinguish in young living rays the gradual transformation of the muscular fibres into electric platelets and was able to demonstrate that electric stimulation still elicited contractions in fibres which had not yet undergone a complete transformation. The electric plate once being formed, contractility is lost, but the generation of electricity, which is also a property of the muscular fibre, has far more become a principal function. So the electric organ may justly be looked upon as a highly interesting object with a view to potassium-researches.

Let it be finally observed that all electric platelets are located in a jelly-like connective tissue.

In successful preparations, treated after MACALLUM, I was now enabled to establish that the electric platelets contain a great many potassium compounds, whereas the jelly in which they are lying is almost destitute of potassium (see fig. 2). Whereas in the medullary sheath of the medullated nerves a distinct reaction is found, by which the neurokeratin-reticulum is disclosed, the non-medullated fibres appeared to be entirely potassium-free. It follows that in most preparations nothing is to be seen of the nerve-network which lies against the anterior cortical layer. This confirms MACALLUM's finding that no potassium-reaction occurs in the axis-cylinders of the nervefibres. The electrolemma is colourless and therefore apparently potassium-free. In the electric platelet itself the reaction reveals itself

most distinctly in the inner layer (Fig. 3). In the anterior and posterior cortical layers we observe fine-granular, black precipitates, above all round the nuclei. In the nuclei themselves there is no reaction. Indeed, MACALLUM could never observe a potassium-precipitate in a nucleus. This justifies the assumption that the nucleus is potassium-free, since so-called masked potassium-compounds (such as iron in haemoglobin) are unknown.

Cell-boundaries between the various nuclei I did not detect, consequently I would rather describe the anterior and posterior cortical layer as a syncytium. It is very striking that there is a considerable accumulation of black grains in that portion of the anterior syncytium which leans on the inner-layer of the platelet.

A regular granular layer exists on the boundary between the anterior syncytium and the inner layer. On the boundary between the inner layer and posterior syncytium my preparations do not reveal that accumulation of potassium-compounds. Although in some parts of the anterior syncytium also rods could be distinguished, they did not show any black coloration. The inner layer of the electric platelet is highly potassium-rich and it is remarkable that here also dark and light stripes occur, just as in voluntary muscular tissue. The dark stripes are diffusely black; I did not see any grains. — In pieces of voluntary muscular tissue taken from the tail of the ray, which were also treated with the potassium-reagent, the anisotropous discs were also diffusely black and the isotropous layers completely colourless. Now, because the inner layer of the platelet has arisen from the fibrillary part of a voluntary muscular fibre, we can hardly be mistaken in conceiving the alternation of dark and light stripes in the inner layer of the electric platelet as a remainder of the alternation of anisotropous, potassium-bearing and isotropous, potassiumfree discs of the muscular fibre.

In the jelly between the platelets I found only potassium deposits in the protoplasm of the starshaped connective-tissue cells. They, however, are poor in potassium and so the whole quantity of potassium-compounds in the jelly is very small; anyhow, strikingly small compared with the potassium-rich electric platelet. The same pictures recurring in various preparations as described above, I may be justified in considering the above-mentioned distribution of the potassium-compounds to be not a casual, but a typical phenomenon.

According to MACALLUM the forms under which potassium occurs in the tissues are the following:

1st. as a local precipitate;

2nd. as a series of local sharply outlined deposits;

M. W. WOERDEMAN: "An Inquiry into the Distribution of Potassium-compounds in the electric organ of the Thorn-back (Raja Clavata)".

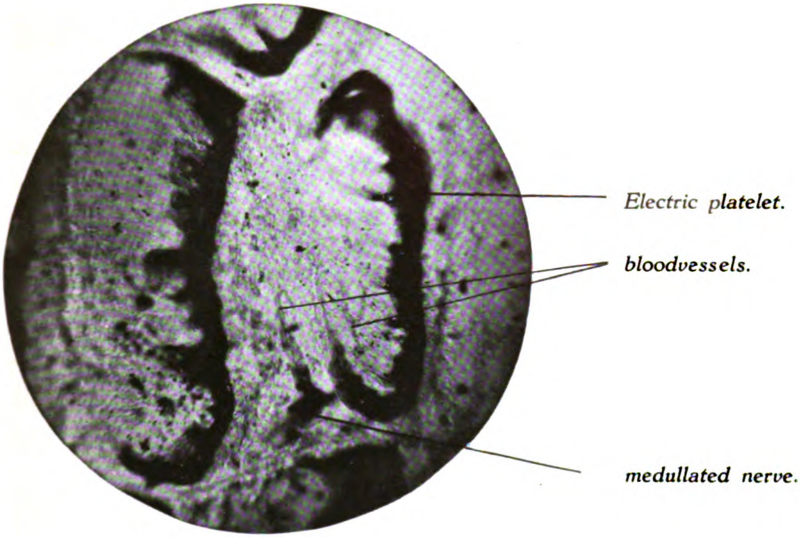


Fig. 2. Microphoto of two electric platelets of Raja punctata. (Section of frozen organ, treated with potassium-reagent) Exhibits a very black precipitate in the electric platelets and little or no precipitate in the jelly-like connective tissue between the platelets.
(Reichert. Objective 4. Ocul. 2).

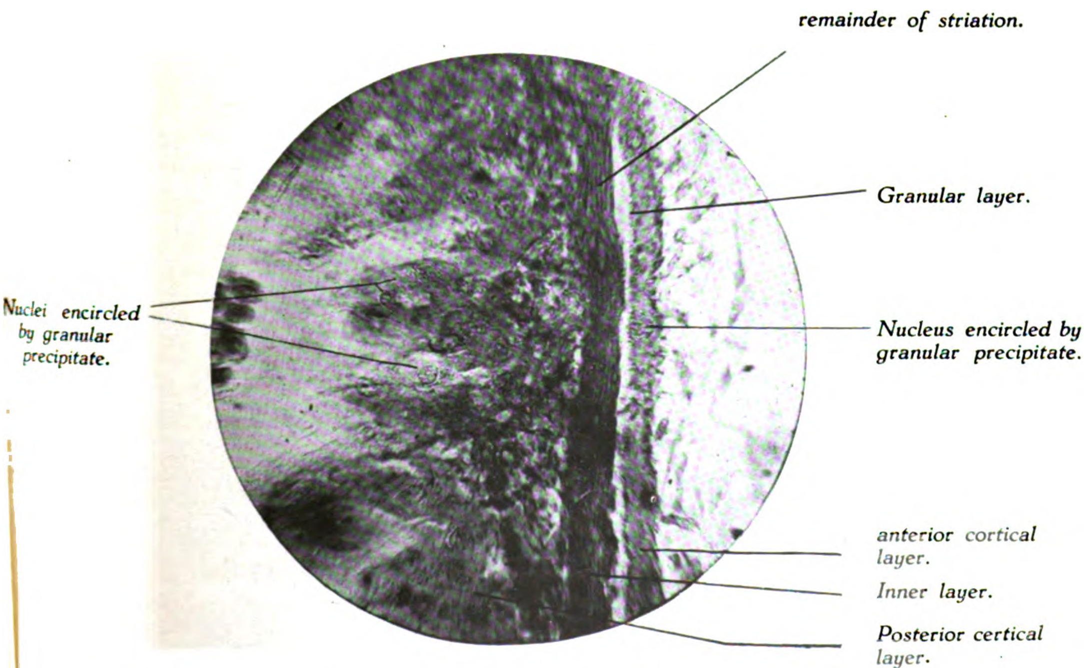


Fig. 3. Microphoto of an electric platelet of Raja punctata. (frozen section, treated with potassium-reagent).
(Oil-immersion 1/12 Leitz. Oculair 2).

3rd. as a so-called bio-chemical condensation.

According to this differentiation potassium occurs in the anterior, and the posterior cortical layer as a local precipitate, in the inner layer, however, in biochemical condensation.

The local precipitate is to be found especially round the nuclei of the two cortical layers and on the boundary of the anterior cortical layer and the inner layer, while in the latter the potassium has been condensed in the originally doubly refracting discs.

The most interesting facts brought to light by this experimentation are, in my judgment, 1st. the potassium-richness of the electric platelets and the slight quantity of potassium in the surrounding jelly; 2nd. the occurrence of á large precipitate of potassium on the boundary between the anterior cortical layer, and the inner layer, and 3rd. the fact, that in the inner layer the peculiar distribution of the potassium-compounds, found in voluntary muscular tissue, has been maintained. The physiological explanation of these facts will, as I hope, be given by those who are competent to do so.

Histological Laboratory of the Amsterdam University.

Physics. — "*Extinction by a Blackened Photographic Plate as Function of Wavelength, Quantity of Silver, and Size of the Grains*". By ALPH. DEUMENS. (Communicated by Prof. W. H. JULIUS).

(Communicated at the meeting of Sept. 25, 1920).

Introduction. For the right understanding of the photographic process knowledge of the final product: the blackened photographic plate, is indispensable. In this communication we shall give an account of a research of the blackened photographic plate.

We have set ourselves the task to investigate the blackening [1]) for different wave-lengths, to measure the quantity of silver present in the examined plates per unit of area, and we have further determined the size of the silver grains [2]).

The extinction of a photographic plate may be compared to that of a fixed colloidal solution, e.g. milk-glass or ruby-glass, where also small particles are steeped in another medium.

The quantities mentioned have been measured, because the extinction of a blackened photographic plate can be theoretically understood by considering it as the result of the action of a great number of irregularly distributed grains of silver, steeped in gelatine.

When the distribution and the nature of the silver in the plate is known, the action on radiation of given wave-lengths can be derived, and inversely something about the nature of the silver in the plate can be inferred from this action for given radiation on grains of given size.

In this communication we shall not enter further into the theoretical considerations, but reserve them for a following communication.

We now confine ourselves to the description of the experimental methods used and of the results obtained by them.

---

[1]) By blackening is understood Brigg's logarithm of $\frac{I_0}{I}$, in which I_0, resp. I, is the intensity of the light that has traversed an unblackened, resp. blackened part of the plate.

The name "blackening" is not appropriate, because the photographic plate is not black.

Hence it would be better to follow R. LUTHER (Zeitschr. f. phys. Chem. 1900) and speak of "the extinction" of the photographic plate.

[2]) Besides the extinction and silver content of a collargolsolution was, examined.

The investigation of the blackening for different colours has, besides, another practical use. Blackened photographic plates have often served as light-reducers. Of late they have been used as such in different researches in the Institute for Theoretical Physics. Compare Miss RIWLIN's investigation [1]). It was, therefore, necessary to find a method to gauge these reducers for different colours with regard to the quantity of light that they allow to pass.

The attention may also be drawn to the fact that the law of blackening that is found, can present differences when the plate is measured for different colours.

For the blackening is dependent on the method by which it is measured, in opposition to the number, the section, and the nature of the grains of silver present in the photographic plate per unit of area.

The micro-photometers work with light of different colours, thus e.g. that of HARTMANN [2]) and all other visual ones chiefly with yellow-green, that of PAUL KOCH [3]) chiefly with light of short wave-lengths, that of MOLL [4]) for the greater part with red and ultra-red.

Differences in results may, therefore, always be interpreted by the consideration that the blackened photographic plate is far from black.

§ 1. *Determination of the Blackening for Different Parts of the Spectrum.*

1. *Method of procedure.* The determination of the blackening takes place by means of the extinction meter of MOLL, modified according to the adjoined scheme. [5]) (fig. 1).

A Nitralamp Lp (25 candles, 4 Volts) throws a beam of light made parallel [6]) by lens Ls on the photographic plate Pt, which is always put at the same place in the apparatus. A dish Ct filled with a coloured liquid, and a colour-filter Fr enable us to throw a beam of the desired colour on the plate. After this the beam strikes the thermopile Th and causes a thermo-current proportional to the energy of the light that strikes it. The intensity is measured by the aid of a compensation method.

---

[1]) These Proc. Vol. XXIII N⁰. 5. p. 807.

[2]) Zeitschr. f. Instrum. Kunde 1899 p. 97.

[3]) Ann. der Physik Bd. 39, 1912, p. 705—751.

[4]) Dr. W. J. H. MOLL. Een nieuwe registreerende microfotometer. Versl. Kon. A. v. W. XXVIII (1919), p. 566.

[5]) Versl. Kon. Ak. v. W., 28, p. 1001—1006.

[6]) We choose a parallel beam to make the theory of the observed phenomenon as simple as possible. We avoid then also the Callier effect (Zeitschr. f. Wiss. Phot. Bd. 7, p. 257 etc.).

The decade resistance box *Bk*[1]) brings the current of an accumulator *Ar* to the strength at which the thermo-current is compen-

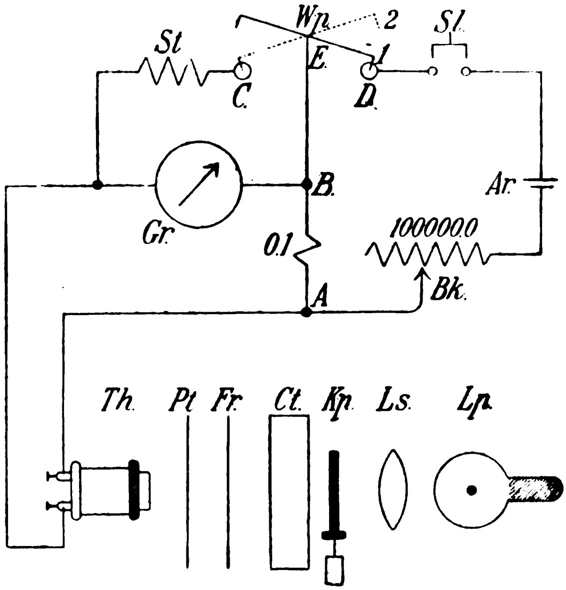


Fig. 1.

ated. Then the galvanometer *Gr* is without current. The circuits of accumulator and thermopile have the manganin wire *AB*, of about 0.1 ohm, in common. Shunt *St* protects the galvanometer Gr. against too strong currents. The zero-position is obtained by bringing valve *Kp* before the lamp, and by interrupting the current with the contact key *Sl*.

The blackening measurements are made in the following way: We place the unblackened part of the photographic plate *Pt* in the light path of the lamp *Lp* and determine the resistance R_0 in resistance box *Bk* necessary to compensate the thermocurrent. Then we shift *Pt*, till the blackened part of the plate is in the light path, and compensate again, let us suppose with R_1 Ohm. The blackening is then given by $log \frac{I_0}{I_1} = log \frac{R_1}{R_0}$.

In the measurements the following sources of error should be taken into account:

1. The spectral distribution of energy of the lamp (*Lp*) changes on protracted use. The same blackenings measured with an interval of some hours' burning of the lamp never gave differences greater than the accuracy allowed by the method.

[1]) Every resistance of from 0,1 to 100.000 ohms can be inserted on the decade resistance box.

2. The strength of current of the lamp may fluctuate; it must, therefore, be continually controlled. Care has been taken that the error in consequence of the fluctuations in the intensity of the lamp remains at the most of the order of 1°/₀.

As after a lamp has been lighted, the stationary and maximum intensity is not reached until some time after, the observations are always started half an hour after the lighting of the lamp, and the lamp remains burning all through the series of observations.

3. When we allow light to fall on the thermopile *Th* [1]), the galvanometer *Gr* reaches its position of equilibrium within 2 seconds. When the radiation is continued, we see the deflection, after having been constant for a moment, slowly diminish. This is a consequence of the getting warm of the surroundings of those places of contact in the thermopile that are not directly irradiated. [2])

In order to minimize the influence of this error, the successive manipulations in the measurements are performed with constant intervals.

These intervals are marked by an electric clock, which every ten seconds closes a mercury contact for a moment. This brings about that via a relay through the key *Wp*, which can turn round a horizontal axis at *E*, the circuit is closed and broken at *C* and *D* alternately for ten seconds. Only during the 10 seconds that the shunt *St* is cut out at *C*, and the accumulator current put in at *D*, is the valve *Kp* open, and is the light admitted on the thermopile.

It will be at once clear that in these ten seconds the resistance box *BK* cannot be adjusted so that the thermo-current is totally compensated. Therefore it is ascertained through a preliminary investigation what value of the resistance is about required for every blackening to compensate the thermo-current. The galvanometer shows, therefore, a small deviation. As deviation is taken the difference between the unshunted zero position (1), i. e. when valve *Kp* is shut, key *Wp* is in position 1 and key *Sl* is out of the circuit, and the unshunted state of equilibrium (3), i. e. when valve *Kp* is open, key *Wp* in position 1 and key *Sl* is in the circuit [3]).

---

[1]) The thermopile is provided with a cylindrical tube in order to prevent obliquely incident light.

[2]) This phenomenon is met with to a much smaller degree in the later improved construction of the thermopile.

[3]) As has already been said there is every time a period of ten seconds; this time is amply sufficient to allow the galvanometer with or without thermopile to resume its state of equilibrium, and to read the position reached on the graduated scale down to 0.1 m.m.

The correction of the resistance is calculated from the deviation in mm.

The influence of the error mentioned under 3 has been minimised by this procedure, for:

1. the thermopile is irradiated always *in the same way*, and *during the same short time*,

2. every observation is the mean of several measurements.

Besides every photographic plate is at least measured twice.

II. *The measurements.* The first observations showed at once that as was to be expected, the blackening is a function of the wave-length of the light with which it is measured. Frequently repeated observations confirmed the first results.

The following plates were used: Wellington plates, Speed 100, and Speed 400, dimension 13 × 18 cm., always of the same emulsion number [1]).

Every period (1) and (3) of 10 seconds is succeeded by a period (2), in which the key *Wp* is in position (2); hence the galvanometer is protected by the shunt from too great deviations.

In this period (2) of ten seconds the necessary manipulations, as the opening and closing of valve *Kp* and key *Sl*, the adjustment of the resistance box *BK* to the resistance approximated before, and the noting down of the observation, may be performed.

Every observation consists in the reading of 3 zero-positions (1) and 2 positions (3), and lasts therefore 90 seconds. The course of the observation may be characterized by giving the periods (1) (2) and (3) in the right succession, viz.:

(1)—(2)—(3)—(2)—(1)—(2)—(3)—(2)—(1). By averaging the 3 positions (1) and the 2 positions (3) and by subtraction, the sought small deviation is found in m.m.

[1]) The plates Speed 100 are developed with hydroquinone according to the recipe:

1st solution:	Hydroquinone	36	grms
	Potassium meta bi-sulphite	36	„
	Bromide of potassium	36	„
	Water	3000	„
2nd solution:	Potassium hydroxide	144	„
	Water	3000	„

Used were equal volume parts of the two solutions.

The plates Speed 400 were developed with glycin according to the recipe:

Distilled water	3000	grms
Sodium sulphite (powder)	150	„
Glycin	30	„
Potassium carbonate	150	„

As fixing bath is used in both cases:

1st solution:	Hyposulphite	1500	grms
	Water	3000	„
2nd solution:	Ammonium chloride	600	„
	Water	3000	„

Mix *A* and *B*.

1. Plates developed with hydroquinone [1]).

The plates N°. 1—4 have been developed for 7 1/2 minutes with 1 part of developer and 6 parts of water, plate 1 at 14.5°, plate 2—4 at 13°, the plates 5—10 have been developed at 13° for four minutes with 1 part of developer and 3 parts of water. After development all the plates were rinsed for 1/2 minute, fixed for 15 minutes, and rinsed for 2 hours.

In the plates 5—10 the two most uniformly blackened pieces of 15 cm^2 were chosen on each plate. These two pieces are denoted by A and B in the adjoined tables. Of each piece of 15 cm^2 the blackening is measured at three different spots. The area of the beam of rays on the photographic plate amounting to about 4 cm^2, the blackening is directly determined on 12 cm^2 of the 15 cm^2. The blackening given is a mean of these three.

In the plates 2—4 two pieces of 65 cm^2 are taken, because on account of the slight blackening a larger area must be used in the determination of silver to be described later. The blackening is measured in four places in every piece. The blackening given is the mean of these four.

In plate 1 the different blackenings — marked α, β, γ, and ∂ in the table — are so slight, that no determinations of silver can be made with them.

Accordingly the blackening is measured only at one place, but always twice.

2 Plates developed with glycin [2]).

The plates N°. 11—19 have been developed at 18° for 7 1/2 minutes with 1 part of glycin and 2 parts of water, and they have further been treated in the same way as the plates 5—10 with hydroquinone. As with glycin a cloud occurs near the blackened part, a very broad region of the plate is left unilluminated in the plates developed with glycin, and the resistance R_0 of the unblackened plate is measured at a place sufficiently far from the clouded part of the plate.

The blackening is measured for all plates for the following colours.

a. For the whole spectrum except for ultra-red.

The dish Ct in the light path of the lamp Lp is filled with a 3 % copper-chloride solution; filter Fr is superfluous here.

b. For ultra-red; centre of the intensity estimated at 1,25 μ. The

[1]) We take Wellington plates, Speed 100, developed with Hydroquinone and bromide of potassium to get plates with small grains.

[2]) We wish to make the blackening and the filter investigation for two different developers. As second developer we take glycin without potassium of bromide, because we then get a fairly constant and not too large grain.

dish in the light path of the lamp *Lp* is filled with an asphalt solution; filter *Fr* is superfluous here.

c. For rays of 0,640 μ to 0,535 μ; centre of the intensity estimated at 0,58 μ; we shall therefore indicate this wave-length region by "yellow".

d. For rays of from 0,590 μ to 0,435 μ, with the centre of intensity estimated at 0,54 μ. This region is indicated by "green".

At *c* and *d* in the light path is placed a 3 °/₀ copper-chloride solution, and besides a coloured gelatine filter as is sold by the firm Kipp, at *c* the filter 3*Cc*, at *d* the filter 5*Ac*.

e. For the whole spectrum. Neither dish *Ct* nor *Fr* is placed in the light path.

We find in fig. 2 both for hydroquinone and for gelatine the ratio $\frac{b}{d}$ and $\frac{c}{d}$ plotted against *d* as independent variable. Thus two curves

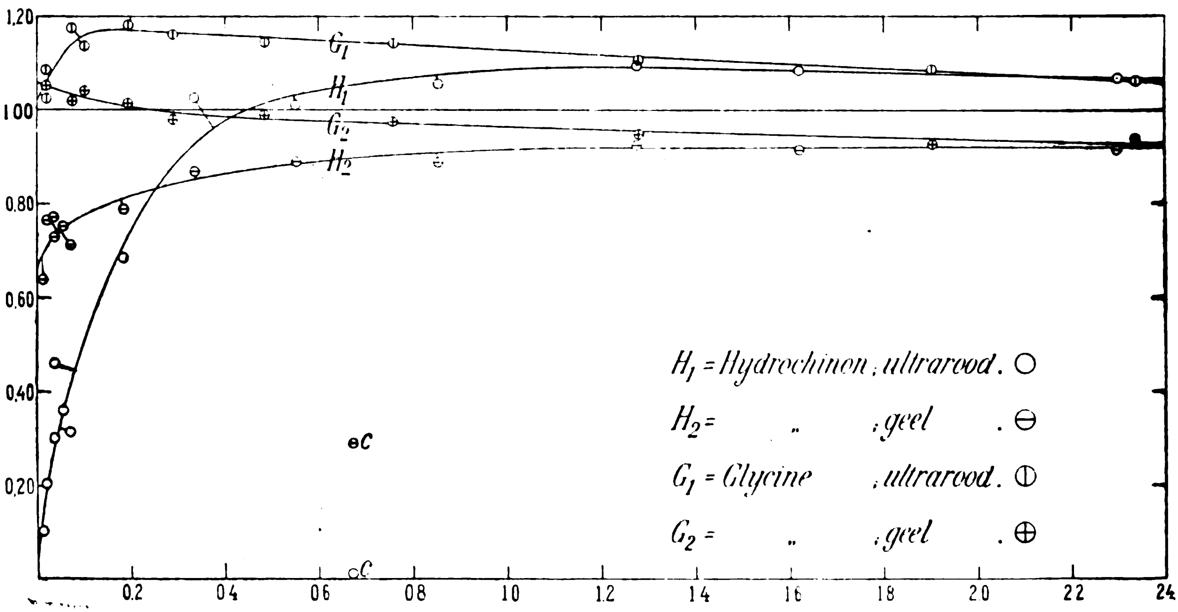


Fig. 2.

are obtained for glycin, denoted by the letters G_1 and G_2 and two for hydroquinone, indicated by H_1 and H_2 [1]).

---

[1]) Only the four blackenings of plate 1 have been measured, besides for *b*, *c* and *d*, also for the whole spectrum and for the whole spectrum except ultra-red.

In fig. 2 only $\frac{b}{d}$ and $\frac{c}{d}$ of plate 1 have been plotted against *d*. All the measurements, however, are found in the table 1 adjoined to this communication.

The points marked in fig. 2 have been obtained by taking the mean everywhere of the values A and B [1]) recorded in table I (see the end of this communication).

It appears convincingly from fig. 2 that the blackening found is a function of the wave-length with which the blackening has been measured.

As we hope to demonstrate further in a following communication and as appears from the measurements in § 3 — the strong deviation of H_1 for blackening from 0 to 0,6 will have to be attributed to the influence of the size of the grain of the silver.

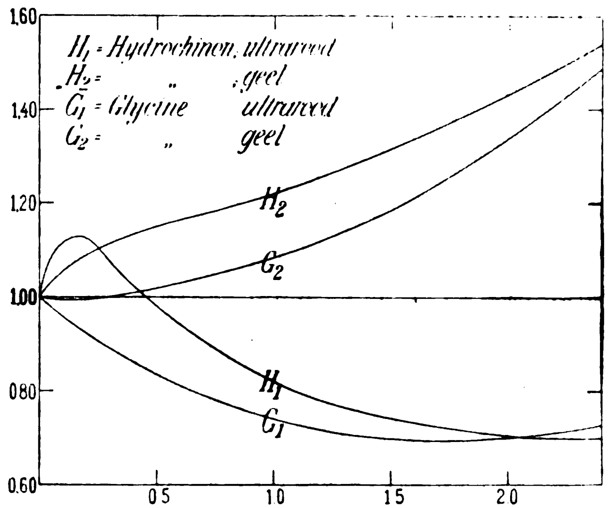


Fig. 3.

In order to make the survey of the efficiency of the photographic plate as light-reducer clearer, the transparency for the measured blackenings from fig. 2 has been derived in ratio to the transparency for green, by the aid of the relation:

$$10^{z_b - z_d} = \frac{I_b/I_0}{I_d/I_0} = \frac{D_b}{D_d}, \quad . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

in which D represents the transparency and the index b and d means ultra-red and green.

(1) results immediately from the definition of the blackening.

[1]) Finally the same measurements b, c and d have been carried out for collargol. As analogues of the blank plate serves a dish filled with pure water, as analogues of the blackened plate serves a same dish filled with collargol-solution. These observations are indicated in fig. 2 by C. 0.65×10^{-3} grams of collargol are taken per cm^3 of water.

In fig. 3 the ratio of the transparency for yellow, resp. ultra red to green has been plotted as ordinate against the blackening for green as abscissa.

Hence it appears that for the region yellow-green the plates developed with glycin with a blackening of from 0 to 1.0 to 8°/₀ may be safely used as reducers. For the rest it is, however, advisable always to gauge the plates with regard to their transparency for the wave-length region for which they must be used. In the investigation by Miss RIWLIN cited above this has been done by the method worked out by us [1]).

The mean error of the observations of blackening has the value of 0,7°/₀ for blackenings of 2.0 to 0.1, the value of 3.3°/₀ [2]) of the blackening for blackenings of 0.1 to 0.001. Also with blackenings above 2.0 the accuracy becomes less.

§ 2. *Determination of the Quantity of Silver Present per Unit of Area on a Blackened Photographic Plate.*

1. *Method of procedure.* It appears from the results of § 1 that the quantity of silver present per unit of area on the photographic plate cannot be proportional to the blackening.

The estimation of the silver was made with the extinction meter of MOLL [3]) in the form as it was used in Mr. DITMARSCH's still unpublished investigation of the flaking of colloids. For particulars of the research we refer to the publication of Mr. DITMARSCH's results. For our purpose it is sufficient to observe what follows.

The deviation of the galvanometer caused by change of the turbidity in one of the dishes, is registered photographically [4]). Accordingly

---

[1]) In our investigation it is impossible to reproduce all the blackenings that are to be investigated, on one photographic plate. This is preferable especially in a search for the law of blackening, properties of the blackening etc. As the way in which the problem has been put, necessitates the determination of the quantity of silver of the plates, and this requires a black plate of sufficiently large area, we meet with practical difficulties when we wish to represent all the blackenings on one plate. The same investigations have, however, first been carried out with the different blackenings in small squares on the same plate. The results obtained were the same. Hence the differences found cannot be attributed to the use of different plates.

[2]) These slight blackenings can be measured with the same apparatus by a somewhat more elaborate, but also more accurate way, which reduces the mean error to 0.9 %.

[3]) Versl. Kon. Akad. v. Wetensch. XXVIII 1920, p. 1001—1006.

[4]) The error resulting from the changes in the dimensions of the registering paper, which may appear after the development, appears to be smaller than 1 %.

the compensation switch described by MOLL, has been omitted[1]). The variation of the turbidity with the time may be read from the registered curve.

The use of the extinction meter of MOLL for the estimation of the silver-content of a solution[2]) is based on the fact that it becomes turbid when a definite quantity of a sodium-chloride solution of a given strength is added to it. The process that then takes place, consists of two parts. First of all the silver and sodium ions join to molecules of silver chloride, which takes place very quickly, and then the molecular disperse solution which can be compared with a finely divided sol, begins to flake slowly, and consequently becomes turbid. After a considerable time the turbidity becomes constant.

Hence a method worked out with a view to the study of the process of flaking, can also render good services for the investigation.

For liquids with equal silver content this turbidity is only dependent on the time. When we, however, start from liquids with different silver content, it appears from the shape of the registered curve that this turbidity depends in a great degree on the silver concentration. We have here, therefore, an accurate means to determine the content of a silver solution. This method has been applied in the following way:

The registered deviation of the galvanometer is plotted as function of the time and the area described in ten minutes[3]) — called area of flaking — is calculated. This is done for different solutions of known silver content. When we now plot the calculated areas against the resp. silver contents, we get a gauging curve[4]), from which the content of a silver solution that is to be examined, can be read after calculation of the area of flaking.

When the sketched method is applied unmodified to the photographic plate, we are confronted by the difficulty that the grains of silver lie embedded in gelatine. When the silver of the plate is dissolved in nitric acid, and when sodium chloride is added, the gelatine or its reaction products appear to prevent the flaking for the greater part. By varying all the circumstances and registering

[1]) Versl. Kon. Akad. v. Wetensch. XXVIII 1920, p. 1002—1003.

[2]) In our investigation this is always a silver nitrate solution.

[3]) In general the deviation of the galvanometer has already got near its maximum after 10 minutes.

[4]) Instead of the area of flaking also the deviation reached after then minutes might be used, but this method of procedure is less accurate. For the calculation of the area of flaking comes to the same thing as the use of different deviations separated from each other by equal time intervals.

the areas of flaking corresponding to them, we found as the most favourable conditions boiling of the pieces of the plate of 15 cm^2, resp. 65 cm^2 with nitric acid of 50 % for 60 min. resp. 90 min. The influence of the gelatine or of its reaction products then disappears at 8 %[1]) for the pieces of 15 cm^2, and at 17 %[2]) for those of 65 cm^2.

The content of the silver nitrate solutions thus obtained from the photographic plate, can however not be read from the above described curve of gauging. For this purpose we must first make *comparable* silver-nitrate solutions of known content. This is done by developing, fixing, and rinsing unilluminated photographic plates of the same kind as has been used, in the same way as the blackened photographic plates for the estimation of silver of which they must serve.

Then pieces of 15 cm^2 resp. 65 cm^2 of these unilluminated plates are heated with the same quantity of nitric acid and a known quantity of silvernitrate for 60 minutes resp. 90 minutes at 100°. The silver solutions thus obtained contain, therefore, the gelatine and its reaction products in the same form as the silver nitrate solutions obtained from the blackened plates, hence they can serve for the construction of the curves of gauging.

Fig. 4 gives two of the curves of gauging used; the area of flaking in cm^2 has been plotted as ordinate against the number of mg. $AgNO_3$ in 14 cm^3 of solution as abscissa; *I* serves in the determination of the quantity of silver in plates of 15 cm^2, *II* in those of 65 cm^2. As was to be expected, *I* lies higher than *II* on account of the smaller influence of the gelatine. That with zero silver a small value is still found for the area of flaking (cf. fig. 4 pieces *AB* 8 times enlarged), is a consequence of the turbidity through small particles of dust, which are raised by the stirring after the addition of the sodium chloride. The method has been used for silver solutions with a silver content of 0.5×10^{-3} to 156×10^{-3} grams per liter; the sensitiveness of the method to detect traces of silver extends, however, much further than 0.5×10^{-3} grams of Ag per liter.

---

[1]) The circumstances made it necessary for us to manufacture the dishes ourselves. In this we met with the difficulty that all the well-known acid-proof cementing substances were attacked by the acid after shorter or longer time. At last we found a suitable cement in bakelite. When the recipe of BERTRAND—GAUTHIER is applied, the influence of the gelatine can be quite eliminated, hence for 100 %. The acid-concentration required for this, however, attacks also the bakelite in less than an hour.

[2]) More prolonged heating — which would enable us to carry up the percentage higher — met with practical difficulties.

The mean error in the estimations of silver amounts to 3 %. In the measurements attention should be paid to the following points.

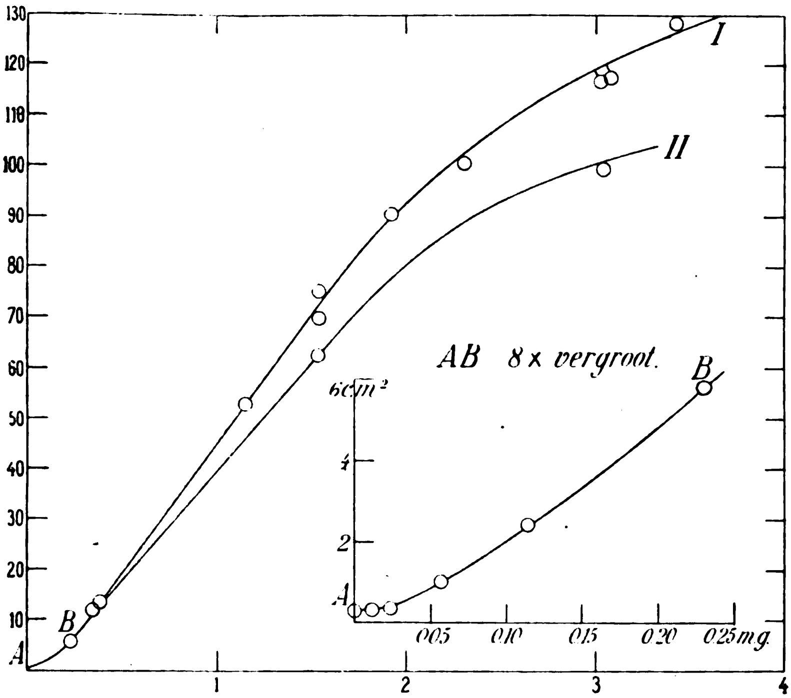


Fig. 4.

1. It appears experimentally that the area of flaking becomes larger by ultra-violet rays. To obviate this complication a dish of quinine-bisulphate has been placed in the light-path towards the liquid to be measured.

2. The area of flaking is greatly dependent on the concentration of the acid. Therefore the same acid concentration has always been used.

3. The area of flaking increases with the temperature. At 17° the area increases by 2,7 % per degree of temperature increase. Therefore the temperature of the liquid is always measured before and after the flaking, and all the observations have always been reduced to the same temperature.

4. In consequence of specks of dirt and premature flaking through the everywhere present sodium chloride, the liquid which is to be examined, will already exhibit a beginning turbidity. This error can become pretty great, especially with small concentrations.

By means of suitable manipulations the extent of this effect can be registered separately for every flaking. With only a few exceptions we find that this error is always smaller than 5 %. In many

cases it is no more than 2 or 3 °/₀ owing to the many precautions taken in the treatment of the plates. One of these is that all the plates after having been fixed and rinsed for a long time, are once more rinsed with distilled water for 10 or 12 hours, before they are dissolved in nitric acid.

The extent of this error is then so perfectly accidental, both for the gauging liquids and for the liquids under examination, that we have not taken it into account. [1])

§ 3. *Measurements of the Cross Section of the Silver Grains in the Photographic Plate.*

The way in which the silver of the photographic plate acts on the light, and the blackening observed in consequence of this, depends, as is immediately seen, to a great extent on the way in which the silver is distributed. The dimensions of the silver grains with respect to the wave-length of the light used in the measurement of the blackening, is of the greatest importance. If it is, therefore, required to obtain results which are liable to theoretic discussion, it is necessary to investigate, besides the blackening and the total quantity of silver, also the distribution of the silver. In this we have confined ourselves to the determination of the mean size of the silver grains — for so far as this is possible — in the plates of which the blackening and the total silver content was investigated before.

The measurement takes place with a Zeiss-microscope with oil-emersion of numerical aperture of 1.30, combined with an ocular N°. 12. An ocular micrometer is gauged with a Zeiss-object glass micrometer. Thus it appears that a scalar division of the ocular micrometer corresponds to 1.0 μ.

The value of the cross-section of the grain is the mean of 40 observations. [2])

With the smallest blackenings of the plates developed with *hydroquinone* N°. 1, 2, and 3 the grain cannot be measured, it being so small, that nothing is to be seen in the microscope but a faint greyish tint. In plate 4 the grain has been measured, but it is already very diffuse. Of the plates 5, 6, and 7 the size of the grain can very well be measured. For plate 8 — 0.49 μ was found for the cross-section — the result was more or less uncertain on account

---

[1]) The quantity of silver of the collargol used has also been determined.

[2]) Of these 40 observations 20 were always by Dr. H. C. Burger and 20 by me. The mean difference between the two series of observations amounted to 5 °/₀.

of the great blackening. For this reason we have repeated the measurements, after the grains of the plates have been mechanically pressed out into thinner layers. After this operation we find 0.48μ for the cross section of the grain. It seems therefore that the pressing out does not change the cross-section in the plates developed with hydroquinone.

The size of the grain of the strongly blackened plates 9 and 10 has been determined in the same way.

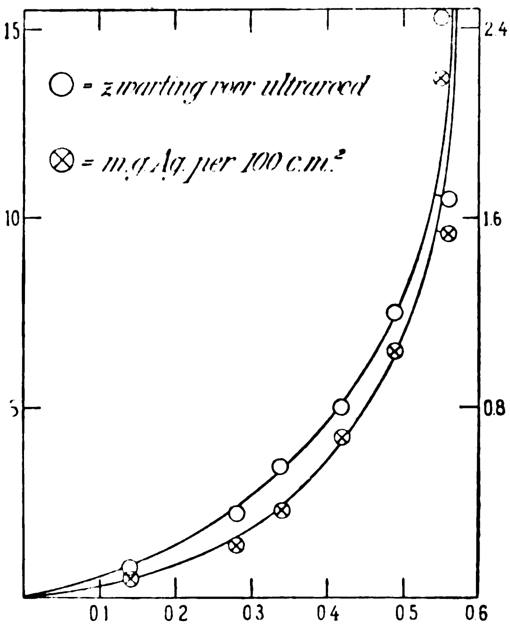


Fig. 5.

Fig. 5 gives the measured cross-sections of the grain of the plates 4--10 plotted against the quantity of silver found in § 2 in mg. per 100 cm², and against the blackening measured in § 1 for ultra-red. The lefthand vertical line is the axis of ordinates for the quantity of silver, the righthand vertical line that for the blackening.

When of the plates N°. 3 and 2 the quantity of mg. of silver per 100 cm.² and the blackening of ultra-red are derived from table 1 (see the end of this paper), and the values A and B averaged, two values of the size of the grain are found of every plate by substitution in the curves of fig. 5, which must both pass through the origin, and have therefore been continued to the origin.

The mean of this is 0.05 μ for plate 3 and 0.03 μ for plate 2. As these quantities have been obtained by interpolation, they are printed in italics in table I.

For the theory and the understanding of the action of the

blackened photographic plate on radiation of given wave-length especially investigations on plates with small and increasing size of the grains will be the most interesting. It is in order to get those plates with small, increasing grains that we have chosen the plates and the developer used. It is seen from fig. 5 that this purpose has been perfectly attained.

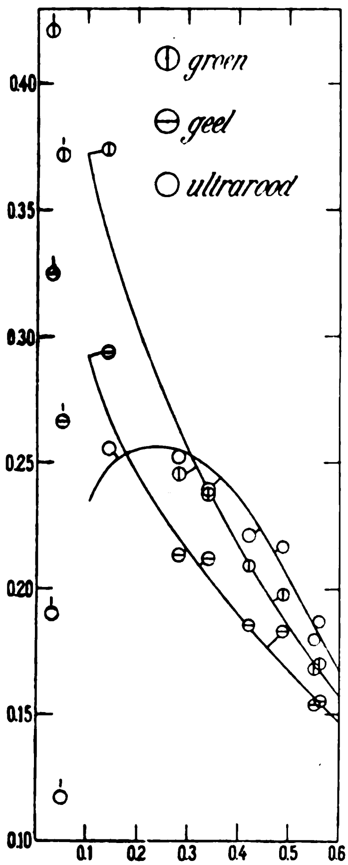


Fig. 6.

In fig. 6 the whole observing material is brought in connection. As independent variable the cross-section of the grain has been chosen, as ordinate the ratio of the resp. blackening and the quantity of silver in mg. per 100 cm.[2] of plate.

As the grain of plates 3 and 2 has been found by interpolation, — these points are indicated on fig. 6 by a vertical line over the circle — and as we accordingly do not know in how far this cross-section of the grain is accurate, the curves have only been continued up to the points obtained from the observations of plate 4.

In the plates 10—4 the mean of the values A and B has always been taken for fig. 6.

It is self-evident that with equal size and nature of the grains the blackening must be proportional to the quantity of silver per cm². Sheppard and Mees [1]), Hurter and Driffield [2]), Eder [3]) found this result; in our experiments this appears not to be the case. This contradiction must find its explanation in the fact that the said investigators had equal grains in the different blackenings, whereas this was not the case with us.

It appears clearly that really $\frac{z}{m}$ depends in a high degree on the size of the grain, as was derived by Nutting [4]).

[1]) Zeitschr. f. Wiss. Phot. Bd. 3 1905, p. 282—289.

[2]) Jahrbuch f. Photographie und Reproduktionstechnik 1899, p. 219.

[3]) Beitrage zur Photochemie und Spectralanalyse Eder und Valenta II, p. 57—58.

[4]) Phil. Mag. 1913, Vol 26 p. 425.

The measurements of the grain of the plates developed with *glycin* show that the grain is pretty well constant, and oscillates within the errors of measurement round a value of 0.96 μ with a blackening of 1.3 to 0.15 (measured for green); for slighter blackenings the value descends to 0.9 μ.

No measurements have been possible for greater blackenings, because the grain itself seems to be broken up into smaller pieces, when the layer of gelatine is pressed out.

In table II [1]) is found the ratio of blackening and silver content for the three colours used. It is seen that this ratio decreases with the blackening.

SUMMARY.

1. A method has been given to measure blackenings of photographic plates up to a blackening of 0.001 for different wave-lengths with an accuracy of 0.7 % for the blackenings of from 2.0 to 0.1, and of 3.3 % for the blackenings of from 0.1 to 0.001.

2. It is found that the blackening of the blackened photographic plate depends in a very great degree on the colour of the radiated light; hence when used as reducer the plate must first be gauged for the different colours.

3. MOLL's extinction meter has been applied to the silver analysis of the photographic plate. The method has been used from 0.5×10^{-3} to 156×10^{-3} grams of silver per Liter. The accuracy amounts to 3 % on an average.

4. The mean cross-sections of the grains of the examined plates have been determined. As the ratio of blackening and silver content present per unit of area in the photographic plate must be constant for grains of the same size and nature, the curves of this ratio and of not constant grain have been plotted.

Besides this ratio has been determined for plates with grains of constant cross-section of the grains.

With great pleasure we acknowledge here our indebtedness to Prof. L. S. ORNSTEIN, Dr. W. J. H. MOLL, and Dr. H. C. BURGER for the great encouragement and assistance they gave us in the execution of these researches.

[1]) The observations for collargol have been put together at the end of Table II.

TABLE I. (Hydroquinone).

Number of the plate.	Cross-section of the grain in μ.	Ratio of the blackening for ultra-red and m.g. Ag per 100 c.m.².	Ratio of the blackening for yellow and m.g. Ag. per 100 c.m².	Ratio of the blackening for green and m.g. Ag per 100 c.m.².
10 A	0.55 pressed out	$\frac{2.424}{13.2} = 0.184$	$\frac{2.060}{13.2} = 0.156$	$\frac{2.215}{13.2} = 0.168$
10 B		$\frac{2.482}{14.15} = 0.175$	$\frac{2.144}{14.15} = 0.152$	$\frac{2.381}{14.15} = 0.168$
9 A	0.56 pressed out	$\frac{1.787}{9.56} = 0.187$	$\frac{1.483}{9.56} = 0.155$	$\frac{1.636}{2.56} = 0.170$
9 B		1.726	1.483	1.602 [1]
8 A	0.49	$\frac{1.391}{6.60} = 0.211$	$\frac{1.178}{6.32} = 0.178$	$\frac{1.270}{6.60} = 0.192$
8 B		$\frac{1.409}{6.32} = 0.223$	$\frac{1.191}{6.32} = 0.188$	$\frac{1.282}{6.32} = 0.203$
7 A	0.42	$\frac{0.908}{4.31} = 0.211$	$\frac{0.777}{4.31} = 0.180$	$\frac{0.869}{4.31} = 0.202$
7 B		$\frac{0.943}{4.09} = 0.231$	$\frac{0.781}{4.09} = 0.191$	$\frac{0.880}{4.09} = 0.215$
6 A	0.34	$\frac{0.559}{2.51} = 0.223$	$\frac{0.491}{2.51} = 0.196$	$\frac{0.547}{2.51} = 0.218$
6 A		$\frac{0.554}{2.17} = 0.255$	$\frac{0.492}{2.17} = 0.227$	$\frac{0.656}{2.17} = 0.256$
5 A	0.28	$\frac{0.361}{1.70} = 0.212$	$\frac{0.307}{1.70} = 0.181$	$\frac{0.350}{1.70} = 0.206$
5 B		$\frac{0.332}{1.14} = 0.291$	$\frac{0.278}{1.14} = 0.244$	$\frac{0.323}{1.14} = 0.283$
4 A	0.14	$\frac{0.128}{0.532} = 0.241$	$\frac{0.148}{0.532} = 0.279$	$\frac{0.188}{0.532} = 0.354$
4 B		$\frac{0.126}{0.465} = 0.271$	$\frac{0.143}{0.465} = 0.308$	$\frac{0.183}{0.465} = 0.394$
3 A	*0.05*	$\frac{0.0208}{0.205} = 0.101$	$\frac{0.0485}{0.205} = 0.237$	$\frac{0.0703}{0.205} = 0.343$
3 B		$\frac{0.0248}{0.186} = 0.133$	$\frac{0.0547}{0.186} = 0.294$	$\frac{0.0745}{0.186} = 0.401$
2 A	*0.03*	$\frac{0.0159}{0.0824} = 0.193$	$\frac{0.0277}{0.0824} = 0.336$	$\frac{0.0364}{0.0824} = 0.442$
2 B		$\frac{0.0182}{0.0980} = 0.186$	$\frac{0.0307}{0.0980} = 0.313$	$\frac{0.0390}{0.0980} = 0.398$

[1]) The estimation of silver failed.

TABLE I *(Continuation)*.

Number of the plate.	Blackening for ultra-red.	Blackening for visible spectr.	Blackening for yellow.	Blackening for green.	Blackening for the whole spectrum.
1 α	0.0013	0.0110	0.0088	0.0129	0.0025
1 β	0.0042	0.0206	0.0158	0.0206	0.0066[5]
1 γ	0.0111	0.0361	0.0267	0.0367	0.0130
1 δ	0.0204	0.0542	0.0427	0.0567	0.0238

TABLE II. (Glycin).

Number of the plate.	Ratio of blackening for ultra-red and m.g. Ag per 100 c.m.².	Ratio of blackening for yellow and m.g. Ag per 100 c.m.².	Ratio of blackening for green and m.g. Ag per 100 c.m.².
19 A	$\frac{2.647}{14.6} = 0.181$	$\frac{2.204}{14.6} = 0.151$	$\frac{2.363}{14.6} = 0.162$
19 B	$\frac{2.316}{14.25} = 0.163$	$\frac{2.183}{14.25} = 0.153$	$\frac{2.308}{14.25} = 0.162$
18 A	$\frac{2.059}{11.2} = 0.184$	$\frac{1.756}{11.2} = 0.157$	$\frac{1.879}{11.2} = 0.168$
18 B	$\frac{2.080}{11.65} = 0.170$	$\frac{1.776}{11.65} = 0.152$	$\frac{1.929}{11.65} = 0.166$
17 A	$\frac{1.428}{10.05} = 0.142$	$\frac{1.225}{10.05} = 0.122$	$\frac{1.280}{10.05} = 0.127$
17 B	$\frac{1.408}{9.50} = 0.149$	$\frac{1.208}{9.50} = 0.127$	$\frac{1.279}{9.50} = 0.135$
16 A	$\frac{0.859}{6.86} = 0.125$	$\frac{0.731}{6.86} = 0.107$	$\frac{0.750}{6.86} = 0.109$
16 B	0.874	0.750	0.767 [1])
15 A	$\frac{0.229}{2.29} = 0.100$	$\frac{0.195}{2.29} = 0.085$	$\frac{0.193}{2.29} = 0.084$
15 B	$\frac{0.234}{2.05} = 0.114$	$\frac{0.203}{2.05} = 0.099$	$\frac{0.199}{2.05} = 0.097$
14	$\frac{0.1163}{0.936} = 0.124$	$\frac{0.106}{0.936} = 0.113$	$\frac{0.102}{0.936} = 0.109$
13	$\frac{0.090}{0.855} = 0.105$	$\frac{0.078}{0.855} = 0.091$	$\frac{0.076}{0.855} = 0.089$
12	$\frac{0.0218}{0.330} = 0.066$	$\frac{0.0224}{0.330} = 0.068$	$\frac{0.0212}{0.330} = 0.064$
11	$\frac{0.0203}{0.343} = 0.059$	$\frac{0.0196}{0.343} = 0.057$	$\frac{0.0187}{0.343} = 0.055$
Collargol.	$\frac{0.0105}{6.50} = 0.0015$	$\frac{0.211}{6.50} = 0.031$	$\frac{0.673}{6.50} = 0.110$

[1]) The estimation of silver failed.

Utrecht, Aug. 1920. *Institute for Theoretical Physics.*

Astronomy. — "*On the possibility of statistical equilibrium of the universe*". By Prof. W. DE SITTER.

(Communicated at the meeting of Nov. 27, 1920).

EINSTEIN has on several occasions expressed the opinion that the existence of a finite amount of matter in the universe must necessarily lead to the adoption of a finite three-dimensional space. In his inaugural address at Leiden [1]) he says:

"Wir können aber auf Grund der relativistischen Gravitations-"gleichungen behaupten, dass eine Abweichung vom euklidischen "Verhalten bei Räumen von kosmischer Grössenordnung dann vor-"handen sein muss, wenn eine noch so kleine positive mittlere "Dichte der Materie in der Welt existiert. In diesem Falle muss die "Welt notwendig räumlich geschlossen und von endlicher Grösse "sein, wobei ihre Grösse durch den Wert jener mittleren Dichte "bestimmt wird."

It appears to me that this statement cannot be accepted unreservedly. The gravitational field-equations are:

$$G_{\mu\nu} - \tfrac{1}{2}\, g_{\mu\nu}(G - 2\lambda) = -\varkappa T_{\mu\nu} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If we suppose all matter to be at rest and free from any strain or internal forces, then the tensor $T_{\mu\nu}$ has the value

$$T_{44} = g_{44}\,\rho, \text{ all other } T_{\mu\nu} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

ρ being the density in natural measure. We can put $\rho = \rho_0 + \rho_1$, where the average value of ρ_1 is zero; ρ_0 is then the average density. If we neglect ρ_1 the equations (1) are satisfied by the $g_{\mu\nu}$ implied by the line-element:

$$ds^2 = -\,dr^2 - R^2 \sin^2\frac{r}{R}\,[d\psi^2 + \sin^2\psi\, d\theta^2] + c^2\,dt^2, \quad . \quad . \quad (3A)$$

if we take

$$\varkappa\rho_0 = \frac{2}{R^2}, \quad \lambda = \frac{1}{R^2}, \text{ (EINSTEIN)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4A)$$

or by those of the line-element:

$$ds^2 = -\,dr^2 - R^2 \sin^2\frac{r}{R}\,[d\psi^2 + \sin^2\psi\, d\theta^2] + \cos^2\frac{r}{R}\,c^2dt^2, \quad . \quad (3B)$$

with

[1]) Aether und Relativitätstheorie. Berlin, Julius Springer, 1920, p. 13.

$$\rho_0 = 0, \qquad \lambda = \frac{3}{R^2} \quad \text{(DE SITTER)} \quad . \quad . \quad . \quad . \quad . \quad (4B)$$

For $R = \infty$ both (A) and (B) degenerate into:

$$ds^2 = - dr^2 - r^2 \left[d\psi^2 + \sin^2 \psi \, d\theta^2 \right] + c^2 dt^2, \quad . \quad . \quad . \quad (3C)$$

with

$$\rho_0 = 0, \qquad \lambda = 0 \quad \text{(NEWTON)} \quad . \quad . \quad . \quad . \quad . \quad (4C)$$

It thus appears that EINSTEIN's solution (A), in which three-dimensional space is finite and closed, is the only one which admits of a finite average density ρ_0. But this is only true, if the tensor $T_{\mu\nu}$ has the value (2), i. e. if the matter is at rest and in equilibrium. If the matter is either in motion, or subjected to stresses or pressures, the value (2) cannot be used; the equations (3) and (4) no longer represent the exact solution, and we can have finite values of ρ_0 also in the systems (B) and (C) [1]). EINSTEIN's assertion can thus only be maintained if we make the additional hypothesis that for the whole universe, or for regions of very large, or "cosmical", size, we can still use the value (2) of the tensor $T_{\mu\nu}$, i. e. if for such regions we assume the matter to be in statistical equilibrium.

This result can also be expressed thus: If the system (A) is the true one, then it is possible for the universe, or for large portions of it, to be in statistical equilibrium. If either (B) or (C) is the true system, then this is not possible. Now the possibility of statistical equilibrium of large portions of the universe is, to my mind at least, by no means self-evident, or even probable. The idea of evolution in a determined sense appears to me to be rather opposed to the actual existence, if not to the possibility, of equilibrium.

The systems (A) and (B), involving the introduction of the constant λ, originated from the wish to make the three-dimensional world finite [2]). At the present time the choice between the systems (A),

---

[1]) Similarly in the system (A) the value of ρ_0 differs from that given by $(4A)$. See also: DE SITTER, On EINSTEIN's theory of gravitation and its astronomical consequences, Monthly Notices of the R. A. S. Vol. LXXVII, pp. 6—7, 18 and 20—23.

[2]) If we assume the three dimensional line-element to be

$$d\sigma^2 = dr^2 + R^2 \sin^2 \frac{r}{R} \left[d\psi^2 + \sin^2 \psi \, d\theta^2 \right], \quad . \quad . \quad . \quad . \quad (5)$$

and $g_{i4} = 0$, then no other solutions than (A) and (B) exist. Of the two possible three-dimensional spaces of constant curvature having the line-element (5) we must choose the so-called elliptical space. The analogy with two-dimensional geometry suggests the spherical space, but this analogy is misleading. The elliptical space is really the one of which our ordinary euclidian geometry is the limiting case for $R = \infty$. In our common geometry a plane has a line (and not a point) at

(B) and (C) is purely a matter of taste. There is no physical criterion as yet available to decide between them. It is true that the systems (B) and (C) do not satisfy MACH's postulate that inertia must be traceable to a material source. But this postulate is a purely metaphysical one, and has no physical foundation whatever. It appears to me to be the last remnant of the desire for a purely *mechanical* interpretation of nature, which logically and historically is based on the belief in forces at a distance, and the impossibility of which has been so clearly demonstrated by EINSTEIN in his Leiden address.

The three systems differ however in their physical consequences at large distances, and an experimental discrimination between them may be possible in the future. The decision between (B) on the one, and (A) and (C) on the other hand may be brought about by the study of systematic radial motions of spiral nebulae [1]). The distinction between (A) and (C) is more difficult, since they both have $g_{44} = 1$, and differ only in the g_{ij} with i and j different from 4, the values of which at great distances it is not so easy to ascertain. The decision between these two systems must, I fear, for a long time be left to personal predilection.

---

infinity; two straight lines have only one (and not two) point of intersection, which may be situated at infinity; if we go to infinity along one branch of a hyperbola, we return along the other branch on the other (and not on the same) side of the asymptote. All these are properties of the elliptical as contrasted with the spherical space. The spherical is only a quite unnecessary reduplication of the elliptical one.

[1]) See DE SITTER, l. c. pp. 27—28. At that time (1917) the radial velocities of only three spirals were known, of which one was negative; the mean being + 600 *km/sec*. Now the radial velocities of 25 spirals are known (see Mount Wilson Publications, Nr. 161, p. 19) of which only three are negative, the mean being + 560 *km/sec* (or + 677 *km/sec* if the four brightest are omitted). The system (B) requires a (spurious) positive radial velocity for distant objects.

Physics. — "*The Mechanism of the Automatic Current Interrupter*". By Prof. J. K. A. WERTHEIM SALOMONSON.

(Communicated at the meeting of November 27, 1920).

The mechanism of the automatic current interrupter as represented by HELMHOLTZ's tuningfork interrupter, by NEEFF-WAGNER's hammer-break, and by the ordinary electric bell, has not yet been explained in an entirely satisfactory way. Lord RAYLEIGH was the first to give an explanation, without, however, entering into details. Later on its mechanism was studied by LIPPMANN, DVORAK, GUILLET, BOUASSE and others although no new points of view were opened. In this paper I intend to submit a few considerations on this subject, principally based on a research into the attraction by the electromagnet on the armature during the working of the apparatus. As an indicator for the attraction I used the number of lines of force passing through the armature at each moment. These were measured by an oscillographic method. This might have been done by the new ABRAHAM-rheograph, but as I did not possess this instrument I employed DEGUISNE's method, described in the Physikalische Zeitschrift 1910, p. 513. The results of this method were compared with those obtained by a new method, which I shall describe in an appendix to this paper.

The interrupter used in my experiments has a horizontal horse-shoe magnet. The cores turned from a solid bar of swedish iron completely bored and slit lengthways, have a length of 5 cm and a diameter of 1 cm. They are screwed at a distance of 3.2 cm from each other into a yoke of 1.4 cm^2 transverse section, and are each wound with 200 turns of well insulated copper wire of 1.2 ohm resistance each. The armature measured $1.2 \times 0.75 \times 4.4$ cm. It is screwed to a strong steel spring of 0.12×1.0 cm. with a free length of 1.3 cm. Into the other end of the armature a brass bar 0.4 cm. in diameter and 5 cm. in length was fixed, on which, if desired, a small copper weight could be screwed. It was generally used without weight and then made about 47 complete vibrations per second, the platinum contact being so adjusted as to make and break the current during one half of the periodic time. The arma-

ture was wound in its middle part with 40 turns of copper wire, the ends of which were connected by means of two large spiral windings with a pair of fixed terminals, in such a way as not to hamper its vibrations. If the interrupter is connected into a circuit with an inductionless ballastresistance of about 1 Ohm and with two accumulator cells, the vibrations have an amplitude such as to render the distance of the armature from the cores taken together, variable from 2 millimeters to 7.6 millimeters. Without current the sum of the airgaps has a length of 4.8 millimeter. The selfinduction of the electromagnet, which of course is not constant, has during the passage of the current a mean value of about 9.3 millihenry.

Whilst the interrupter was in action, oscillograms were taken of the current through the electromagnet and at the same time the magnetic density in the armature was oscillographically recorded. For the current a high frequency DUDDELL oscillograph of the Cambridge Instrument Cy was used, whilst the magnetic density was recorded with a SIEMENS and HALSKE oscillograph, or with a striug galvanometer. On the oscillographic records time marks of 0.01 second were inscribed. For the stringgalvanometer records 0.001 second marks were used.

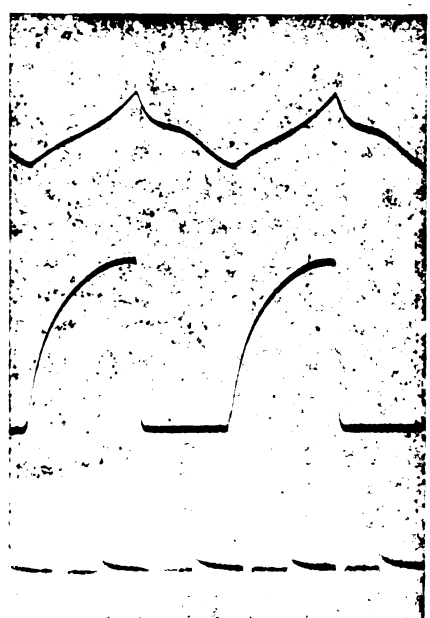


Fig. 1.

In this way curves, as given in fig. 1, were obtained (2 times enlargement of the original negative) with the oscillograph, or as in fig. 7 with the stringgalvanometer.

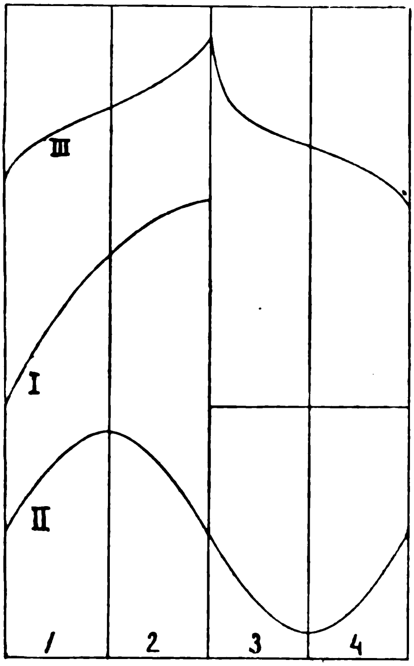


Fig. 2.

We can divide one complete period of the interrupter into 4 nearly equal parts. The two first quarter periods represent the time during which the circuit is closed, the two last ones the break period. During the 2nd and 3rd quarter period the armature moves towards the cores; during the 1st and 4th quarter period in an opposite direction. We know that the number of lines of force passing through the armature determines the force with which it is attracted by the electromagnet. We may even say that this attraction is very nearly proportional to the square of that number of lines of force.

Our curves show that the attraction during the second quarter period is very much greater than during the first. This fact was pointed out by Lord RAYLEIGH and has practically formed the basis of all later communications on this subject. But at the same time we see that during the 3rd quarter period, the current being broken, a strong attractive force still exists, which is notably stronger than the attraction which during the 4th quarter period works against the movement of the armature. Even when the interrupter works under very different conditions as to frequency, current-strength etc. this fact remains unchanged. We may say that the attraction during any part of the movement of the armature towards the pole pieces, greatly exceeds the attractive force in any point during its course away from the electromagnet. Consequently there is no need for any retarding device for making the current with respect to the movement of the armature — as suggested by Lord RAYLEIGH — in order to improve the working of the interrupter. Probably such a device would not only be inconvenient, but would hamper the working of the apparatus.

Can we explain the curve for the attraction? For the ascending part this is certainly possible. We can even calculate it approximately. We first suppose the selfinduction to be constant during the make

period. Applying the wellknown formula of HELMHOLTZ:

$$I = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

we compute the current strength in the magnet at every moment. The current strength being known we try to calculate the number of lines of force through the armature, assuming it to be proportional to the current strength and inversely proportional to the length of the airgap. We may do this as, practically, the total reluctance in the magnetic circuit is to be looked for in the airgaps. With small magnetizing forces the permeability of the iron is so great that this assumption is permittable. As an example we may take the moment just before the breaking of the current. Using HELMHOLTZ's formula and supplying the real value of the constants, we find $I = 1.13$ ampere, whilst from the oscillographic record we find $I = 1.17$ ampere. This makes the magnetising force: $0.4\,\pi \times 400 \times 1.17 = 590$. As the airgap has a length of 0.48 cm we get $590 \times 0.48 = 1225$ lines of force through 1 cm^2 air-section. These lines start from the pole pieces, which have a surface of 0.7 cm^2; hence we find for the magnetic density in the iron not more than 1750 lines per cm^2. This means that we may expect a permeability μ of the order of 3000. Taking $\mu = 3000$ we find that to force 1225 lines through 16.4 cm of iron of a section of 0.7 cm^2, not quite 5 ampere turns are needed. Consequently we have an error of not more than 1 %, if we consider the airgap only and disregard the ironpath.

In order to calculate the number of lines during the make-period, we assume that the armature vibrates in such a way as to vary the length of the airgaps periodically, according to the expression $a + b \sin 2\,\pi\, nt$. Then we get as an approximate expression for the number of lines of force:

$$B = \frac{0\,.\,4\pi\, N \frac{E}{R}\left(1 - e^{-\frac{R}{L}}\right)}{a + b \sin 2\,\pi\, nt} \quad \text{for} \quad 0 < t < {}^1/_2\, \pi$$

in which N is the number of turns of the magnetising coils, E the voltage of the galvanic battery, R the resistance of the circuit, L the mean selfinduction, n the frequency of the interruptions, a the mean length of the air-path, and b half the amplitude of the armature. If we put in this formula the value already given for each of the constants, we get as a result the curves in fig. 3, where I represents the current strength, II the length of the airgap and III the number of lines of force during the make period. If this last curve be

compared with the ascending part in the oscillographic record, we see that they correspond fairly well. The constructed curve shows

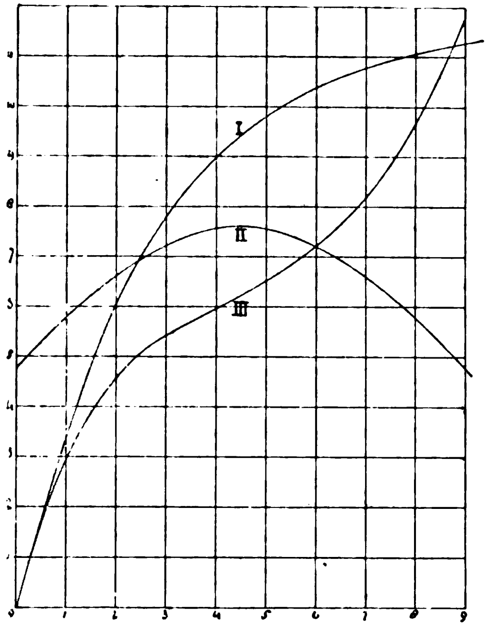


Fig. 3.

a somewhat more rapid ascent in its first part, and also some difference in the last part. But this can readily be explained. If we had calculated the current strength, taking into account that the selfinduction was greater at the beginning and at the end of the make period and smaller in the middle, the curves might have agreed better numerically: theoretically this point is of little or no interest.

The descending part of the curve, which embraces the two last quarter periods, represents the magnetic attraction during the break period. A quantitative explanation is as yet not possible, though qualitatively there seems to be no difficulty. We know that the less reluctance there is in the magnetic circuit, the longer will an electromagnet keep its magnetism after breaking the current. Immediately after breaking the current the air-path is rather large and consequently the reluctance is great and the magnetism disappears rapidly. As the armature approaches the core, the magnetic circuit improves and the magnetism disappears more slowly. The slope of the curve is indeed least at the end of the 3rd quarter period. From then to the end of the 4th quarter period the reluctance grows and the descent becomes more rapid again, becoming nearly as fast as in the commencement of the 3rd quarter period, though not quite, as at that moment the direct action of the magnetomotive force is taken away.

New method for making oscillographic records of the number of lines of force.

If we desire to make an oscillographic record of the number of lines of force in an iron path or an airgap, a few insulated copper-windings are laid round the iron or a small coil is placed in the air gap. When the number of the lines of forces B varies, an electromotive force $V = k\frac{dB}{dt}$ is generated. The terminals of the coil are connected with a condenser of a capacity C. This takes up a charge $q = VC$ and through the coil and the connecting wires with a total resistance r we have a current i.

Now we can state:

$$i = -\frac{dq}{dt} \quad \text{and} \quad k\frac{dB}{dt} + ri = V \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

After substitution we get:

$$k\frac{dB}{dt} = rC\frac{dV}{dt} + V$$

and putting $\frac{rC}{k} = A$:

$$\frac{dB}{dt} = A\left(\frac{dV}{dt} + \frac{1}{rC}V\right) . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

which gives after integration:

$$B = AV + \frac{A}{rC}\int V\,dt + Konst. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

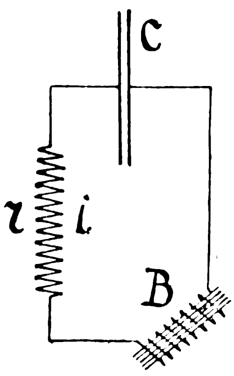


Fig. 4.

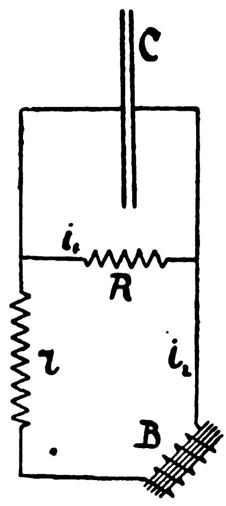


Fig. 5.

If we may disregard the expression $\frac{1}{rC} V$ with respect to $\frac{dV}{dt}$ the electromotive force V is proportional to the magnetic induction. Generally it will be impossible to measure V with an oscillographic electrostatic instrument. But we can use a galvanometric oscillograph at the terminals of C. We shall then get the connections shown in fig. 5 and the differential equations become:

$$-\frac{dq}{dt} = i_1 + i_2 \quad \text{and} \quad k\frac{dB}{dt} + ri_1 = Ri_2 \quad . \quad . \quad . \quad . \quad (4)$$

We eliminate i_1 and get

$$\frac{dB}{dt} = A\left\{\frac{dV}{dt} + \left(\frac{1}{RC} + \frac{1}{rC}\right)V\right\} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

in which $A = \frac{rC}{k}$. After integration this becomes:

$$B = AV + A\left(\frac{1}{RC} + \frac{1}{rC}\right)\int V\,dt + Konst. \quad . \quad . \quad . \quad . \quad (6)$$

We find a linear expression connecting B and V if the integral in (6) need not be considered. This is allowed if both RC and rC are very large and if also the frequency per second is high enough.

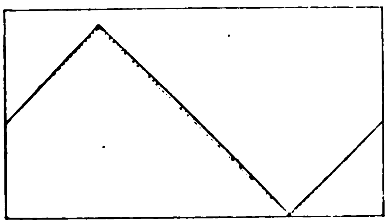


Fig. 6.

With a periodic change of B, which might be represented by a FOURIER series, the value for the integral during one period $= 0$. We have only to examine its value during one period. With a frequency of 50 per second and time constants CR and Cr of 0.2 second each we get for a potential curve as represented by the

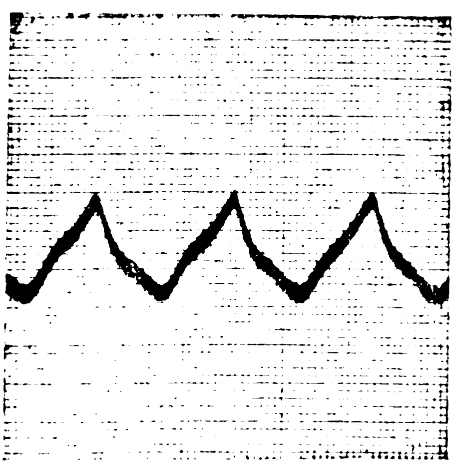


Fig 7.

broken-line curve in fig. 6 the correction indicated by the full-line curve. At the starting point and the end the correction is zero. At the highest point with an ordinate a we get a correction:

$$a\,(5 + 5) : 2 \times 200 = {}^{1}/_{40}\, a, \text{ or } 2.5\,\%$$

of the maximum ordinate.

In my experiments I used a condenser of 2 mikrofarad, R and r being 10^6 Ohm each. The oscillographic record was made with a stringgalvanometer. Fig. 7 gives an example of the curves obtained in this way.

Zoology. — "*The wing-design of mimetic butterflies*". By Prof. J. F. van Bemmelen.

(Communicated at the meeting of Nov. 27, 1920).

In a paper: On the phylogenetic significance of the wing-markings of Rhopalocera, read before the meeting of the second International Entomological Congress at Oxford in 1912, I made the casual remark that "while inspecting the series of butterflies in search for specimens showing the primitive colour-pattern, I was greatly impressed by the considerable percentage of mimetic forms among my harvest. So the idea occurred to me that perhaps Mimetism might, at least to a certain degree and for a limited number of cases, be explained by supposing the resemblance between two or more non-related forms to have started at an early period, when the ancestral types of different butterfly-families looked more like each other than nowadays, on account of the primitive colour-pattern common to them all".

Since those days I have tried to clear and widen my ideas about the real character of the primitive colour-pattern, especially by a detailed analysis of the wing-design in original forms such as the Hepialids, and by its comparison to the pattern of the body. These investigations have led me to a modified conception of primitiveness in pattern: the occurrence of sets of uniform spots, regularly arranged in rows between the wing-veins, and spread over the entire wing-surface, appearing to me as a still more original condition than the concentration of the markings in the shape of a stripe along the middle-line of the internervural cells. But this does not in the least weaken my conviction, that this latter arrangement has retained a considerable amount of primitiveness also, and that its origin lies far beyond the beginnings of genera, families, nay of the whole order of Lepidoptera.

Since then the Groningen Zoological Laboratory has acquired the magnificent collection of Lepidoptera left by the lamented Max Fürbringer in Heidelberg. Thereby I was enabled to study actual specimens of mimetic butterflies in nature and this made me wish to return to the question of Mimetism in general, but then considered exclusively from a purely morphological standpoint. I desire therefore to avoid carefully the biological side of the question, though I may be allowed to express my conviction that the often

striking superficial similarity between forms belonging to widely different groups, can hardly fail to provide certain advantages in the struggle for existence either to one or to both of them, or at least must have done so in former periods of their occurrence on earth.

I shall henceforth restrict myself to a careful comparative analysis of the colour-pattern. But before entering on this task, I wish to remark that the phenomenon of mimetic resemblance can never be ascribed to the influence of a general law, and consequently the different cases of Mimicry must be judged separately, quite independently of each other. That e. g. a Sesia resembles a wasp, cannot possibly stand in any genetic connection to the mimetic similarity between a Dismorphia and an Ithomiid or a Heliconid, or between a set of species of the latter families amongst each other. Nor can this occurrence of wasp-like Sphingids stand in any relation to the existence of other members of that same group, which seem to have assumed the habitus of humble-bees.

Mimetic resemblances consequently must be considered as of casual origin, and the considerable number of conditions, which had to be fulfilled before a real case of Mimicry could enter into existence, make us readily understand the relative rareness of the phenomenon, and its apparently capricious distribution over the animal kingdom (as REBEL has so judiciously pointed out).

Though, as mentioned before, I am inclined to acknowledge the high probability, that in many cases the close superficial and simulating resemblance existing between mimic and model is extremely useful either to the mimic only or to all the members of the mimetic set, I am also convinced that no impartial judgment can possibly be formed without carefully abstaining from all considerations about this hypothetical and problematic usefulness, and exclusively regarding the mimetic forms from a purely morphological standpoint, that is to say investigating them according to the very same principles and rules that have proved useful for the understanding of the colour-pattern of insects in general, and the laws that we could deduce from this study. To this conclusion we are logically led by the observation, that mimetic patterns do not differ in any special feature from colour-designs in general, but on the contrary agree with the non-mimetic patterns, at least when these are embraced in a general view. Solely when we compare the mimetic forms with their nearest allies: the non-mimetic members of the same genera, do we meet with certain cases where they seem to depart widely from the common generic type, though even this by no means can be called the general rule. By the adherents of the Mimicry-hypothesis this

apparent diversion from the original pattern is attributed to the influence of natural selection, leading gradually to a perfect though wholly superficial and spurious similarity with the model.

In order to be able to accept this hypothesis, it is obvious that we are obliged first to prove the assumed deviation from the primitive common type of the genus or family. We ought to abstain from accepting it a priori as a fixed truth, but should try to reconstruct the original common genus- or family-type of colour-design by a perfectly impartial comparative investigation of all the existing members of the group, mimetic as well as non-mimetic, judging them exclusively after the features of their markings, without the least regard to any biological profit these markings might possibly procure them.

The value of these considerations can best be appreciated by their application to a few concrete examples.

In the famous paper of BATES on the resemblance between members of the Pierid genus *Dismorphia* (*Leptalis*) and certain South-American Ithomiids and Heliconids, the author figures a perfectly white species of the said genus, side by side with the mimetic forms, and expressly states that this represents the original type of that family. It necessarily follows that he considers the mimetics as widely deviated from this type. PUNNETT, in the chapter on "Mimicry Batesian and Müllerian" of his valuable critical review "Mimicry in Butterflies", expressly puts forward that this is the current view among the supporters of the mimicry-theory, where he says: "We come back to our Pierine, which must be assumed to show the general characters and coloration of the family of whites to which they belong" .... and "If however they could exchange their n o r m a l dress for one resembling that of the Ithomiines". (The italics are mine).

Doubtless BATES did not for a moment presume that the case might as well be exactly the reverse: the mimetics representing the more original, least altered forms, while the whites, under the prevailing influence of albinism, have considerably departed from the primitive condition.

To make a choice between these two opposite views, we must in the first place undertake a careful and complete investigation of the various colour-patterns of all the members of the genus Dismorphia and different other genera of Pierids, and after that come to a clear understanding about the real nature of the differences between the mimetic and non-mimetic forms.

These differences can be summarized under three heads: those of pattern, of hue and of shape.

Beginning with the first, we may start with the assertion, that a really objective analysis of colour-patterns necessarily involves the exact consideration of the whole complex of markings in all its details. So we must as well pay attention to the underside as to the upper surface, and attribute the same importance to those features, in which the mimics differ from their models as to those in which they agree with them. Viewed from this standpoint (which up till now has very rarely been observed), we easily come to the conclusion, that all the elements, which enter in the composition of the pattern of mimetic forms, can be traced back to those of their non-mimetic congeners, and therefore may be counted among the characteristic features of the genus (or family) to which the mimics belong.

The same remark holds good for the particular hues the mimics display, and even for the apparently aberrant shapes they sometimes assume. When e. g. the mimetic Dismorphia's differ from the majority of the species belonging to the genus by the greater length and the more slender contour of their wings and body, the question if such a form of butterfly might really be regarded as aberrant, has carefully to be considered, instead of being accepted as solved. That it deviates from the "common" type, is obvious, but since when has mere commonness been regarded as a proof of primitivity? Do the Monotremes represent a widely aberrant and deeply modified type of Mammals, merely because they are (at present) restricted to two families? The broad square shape of the majority of Rhopalocera, with their rounded hind- and triangular forewings, including a short body, may far more probably be itself a modification of the narrow-winged form with slender body, such as we find in so many Sphingids and Heterocera, especially in an eminently primitive family as the Hepialids. Even among Rhopalocera themselves this latter habitus is no rare exception, for we find it prevailing in several families, e.g. the Ithomiids and Heliconids. So in matter of shape the resemblance between these "models" and their Dismorphian mimics can safely be attributed to their both having remained faithful to the more ancient form of Lepidopterous insects. Its antiquity may even reach far over the limits of this order, for the same contours prevail among many other, less specialised groups of insects, e.g. Odonata, Neuroptera or Trichoptera. Coming once more to the question of colours, it is easily conceivable that white need not at all be regarded as the most primitive hue in the Pierid family, several other colours: red, yellow, brown, black, occurring just as frequently, especially on the under-

side of the wings. Only its prevailing tendency to spread over large parts of the wing-surface and obliterate the original pattern by albinistic discoloration, gives to the white hue such a prominent place in the colour-scale of this family. But the same role is played by all the remaining shades in different cases. In this regard it deserves our attention that DIXEY, the eminent Pierid-specialist, in his paper on the phylogeny of their colour-pattern, does not start from a uniformly white groundform, but from a dark-hued regularly spotted type as Eucheira socialis.

Out of the numerous instances of Mimicry the astonishing case of *Papilio dardanus* "with his harem of different consorts, all tailless, all unlike (the male) himself, and often wonderfully similar to unpalatable forms found in the same localities" (PUNNETT), seems to to me especially fit to test the validity of my views. As PUNNETT states: "From (a) long series of facts it is concluded that the male of *P. dardanus* represents the original form of both sexes".

According to my standpoint the only "facts" on which such a conclusion should be based, are features relating to the colour-pattern of the male and that of the different females, compared to each other and to those of their fellow Papilionids. But the above-mentioned "facts" are of an entirely different and wholly inadequate character, for they are connected with the mimetic resemblance of the females to Danaid models, and their apparent divergence from the bulk of Papilionids.

An impartial scrutiny of the relation in pattern between the male form and the manyfold females should be undertaken entirely regardless of any such resemblances. When conscientiously remaining true to this principle, and exclusively applying the general rules for the consideration of the colour-pattern, we are forced to the conclusion, that the male form, instead of being the original, is by far the most-modified.

The opposite opinion seems chiefly to have root in the unconscious susceptibility of the human mind to first impressions. We are so accustomed to associate the type of a Papilionid butterfly with the swallow-tail-image, that we involuntarily consider those members of the family, which by their tails, their characteristic markings at the inner angle of the hind-wings, their yellow and black hues, come nearest to this apparent ground-form, as the original representatives of the family. But when we cast a general look over the whole of it, we encounter numbers of species in which the tails are absent, either in both sexes or in one of them, and in the latter case it need not exclusively be the female sex, which lacks tails: *P. memnon*

for instance showing the opposite case. It should also be taken into account, that in closely-related groups, e.g. the *Ornithoptera*, *Antimachus* and *Druryia*, which for good reasons are considered highly primitive in many features, there is not the slightest indication of tails. And as to the original groundform of Rhopalocera in general, this can scarcely be supposed to have carried such prominent appendages at its hind-wings.

Though all males of *P. dardanus*, together with some of its (non-mimetic) female forms, can be considered as corresponding to only one type, this type undoubtedly is subject to very wide variation, and the trend of this variability lies in the direction of the pattern of the mimetic females. So we might consider those males which in the extension and the design of their markings come nearest to the females as the least-altered ones, and this view is found to coincide with the general assumption, that absence or restriction of markings is a consequence of their obliteration by the transgression of hues from their original centre over neighbouring areas.

In the male of *P. dardanus* it is the yellow shade which gets the supremacy, and more or less reduces the black markings to total extinction. Consequently racial forms in which the black shows a greater extension, like *meriones*, *tibullus* and *trimeni*, represent the less modified forms of the male type. Comparing these variations with the mimetic females, we see that they agree with them to a higher degree than the more-uniformly yellow males, and that the special features in which this nearer agreement shows itself, are in fact precisely those details of pattern, wherein these females seem to deviate from the assumed specific Dardanus-type, and to simulate their Danaid models.

Let us consider e.g. the narrow black border along the front-margin of the forewing of the male butterfly and those female forms, which bear the masculine type. Some specimens of the typical *Dardanus* show a rather imperceptible thickening in the middle of this rim, proximad to the discoidal nervure. In *tibullus* this thickening is much more striking, in *meriones* and *antinorii* it can touch the back-limit of the discoidal cell, and in *trimeni* it stretches as a black crossbar in an outward and backward direction up to the dark marginal area along the outer wingborder, thereby cutting up the yellow area into an antero-external and a postero-internal part. E. Haase: Untersuchungen über die Mimicry auf Grundlage eines natürlichen Systems der Papilioniden (Bibl. Zool. III, 1893) in his Fig. 4 on page 13, numbers this bar as N°. IV + V. Comparison with the female forms *cenea*, *acene*, *niavina*, *ruspinae*, *trophonius*, *trophonissa*, *hippocoon*,

hippocoonides, clearly proves that in all of them this same oblique dark crossbar is equally present, but that in its distal part, outside the discoidal cell, it becomes broadened by junction with the nearest distal dark marking along the discoidal nervure (HAASE's Terminal-band). In consequence of this junction the bar occupies the proximal part of four successive internervural cells (R_5, M_1, M_2, M_3: HAASE's VR $1 + 2 + 3 + 4$).

By the occurrence of this crossbar the light-hued middle area of the forewing is divided into a smaller apical blotch and a larger more or less triangular field along the hinder (inner) margin, the latter passing without interruption into the light area which fills the proximal part of the hindwing. This division is one of the prominent features on which the similarity with Danaids depends. But it would be quite inadequate to ascribe the occurrence of this bar to secondary deviation from the original specific type under the influence of natural selection in connection with Protective Mimicry. For the same bar occurs in the females of a considerable number of nearly allied species, e.g. *cynorta, homeyeri, jacksoni, ucalegon, auriger, adamastor, agamedes*, whose males, at least part of them, show an uninterrupted chain of light-hued internervural spots, which increase in size from before backward, and on the hindwing blend to the light middle-field. These spots are separated from each other by longitudinal dark striae, caused by the more or less pigmented wing-veins. The anterior light spot in the apical field of the forewing of *P. dardanus* is the first of the series, it occupies the interspace between the roots of nervus radialis 4 and 5 (radial fork) and we get the impression that this position has something to do with its more marked persistence, by means of which it remains visible, when the other spots are effaced either by light or by dark colour-overspreading. Yet this apical spot also is not exempt from reduction or obliteration: in some specimens of all forms of *dardanus*, male as well as female, it may be reduced to a mere speck, or be wholly absent (comp. the figure of the *trophonius*-female on PUNNETT's Pl. VIII).

Nor is the above-named dark cross-bar restricted to *dardanus* and its nearest relatives, it occurs as well in a number of other Papilionids, e.g. *hesperus, pelodurus*, and others.

In numerous other cases the tendency towards the formation of the cross-bar is equally present, but does not lead to such a conspicuous partition between an anterior and a posterior light area. In *epiphorbas* e.g. the forewing is almost entirely black, with the exception of a hooked central green part. The foremost leg of this

hook is formed by the light blotch separating the terminal bar from the third discoidal one, the hindmost leg by three remnants of the above-mentioned chain of light areas in the internervural cells.

Traces of the bar can also be remarked in *theorine*, *latreillanus*, *ausorii*, *phoreas*, *oribazus*, *charopus*, which means, that a tendency towards interruption of the chain of light blotches is manifest in numerous and very different members of the Papilionid tribe. Nor is this tendency restricted to the forms with tripartite wing-design, it occurs as well in richly spotted forms e.g. *cyrnus*, *demodocus*, *rex*, *mimeticus*, *ridleyanus* and even in regularly checked ones as *antimachus*. In the majority of these last-named butterflies the tendency towards interruption of the light chain only shows itself in a reduction of one or two members of this chain to specks, one in *antimachus* and *mimeticus*, two in *rex*.

Applying the above considerations to other details of the pattern, we are always led to the same conclusion. Especially convincing is the careful analysis of the pattern on the underside of the different *dardanus*-forms, and its comparison with that of the upperside. It shows us, that the median dark striae in the internervural cells have much better maintained themselves on the underside, but that their remnants can be more or less retraced on the superior surface, especially on that of the hindwings. Consequently such a condition of this pattern, as is seen on both sides of the hindwings of the *hippocoon*- or *trophonissa*-form, where these striae are sharp and run without interruption through all the cells (thereby agreeing with *zalmoxis* and similar forms) may, as I said before, be considered as primitive. In regard to these striae two remarks may be offered. The first refers to the pattern of the upperside of the male hindwing, on which the submarginal bar presents all degrees of variation, from a broad complete, uninterrupted belt to a few widely separated irregular black markings. In the latter cases the reduction has either led to the persistance of three blotches: an anterior (exterior), middle and posterior (internal) one, or has only left the two extremes. When the middle one is still present, this very often assumes the character of an internervural stria, and thereby betrays its allegiance to the markings on the underside.

The second remark refers to the colour-pattern of a near relative of *dardanus*, viz. *P. cynorta* (alleged forms included, as *norcyta*, *jacksoni fullehorni*, *echerioides*, *cypraeofila* etc.). Here also a similar striking difference exists between male and female, though the latter occurs only in a single form, which shows a mimetic resemblance to *Planema epaea*. The similarity chiefly depends on the presence

of the before-mentioned oblique dark cross-bar in the forepart of the forewing, and on the series of black median striae in the internervural cells of the hindwing. The male differs from the female by the absence of the cross-bar; the medial area of the forewing thereby showing the uninterrupted chain of internervural light spaces, which diminish in size towards the apex. In contrast with *dardanus*, the root-part of the hindwings in *cynorta* is dark, which causes a closer junction between the central chain of light markings on the fore- and on the hindwings. When comparing these dark root-fields with their counter-parts on the underside, they are seen to be present also there, but tinged in a bright orange-brown hue, intersected by a system of darker lines which mark the wing-veins and the internervural striae. As these lines reappear in the distal part of the wing, it is evident that they are interrupted in the middle-area by the white discoloration. So we are justified in assuming that in more original forms both the veins and the striae will run uninterruptedly over the whole surface of the hindwings (on upper- as well as on underside) and we find the affirmation of this assumption in a great many forms of butterflies, belonging to different groups, and counting among them models as well as mimics (e.g. *Planema tellus* and *Pseudacraea terra*, see PUNNETT, Plate IV, Fig. 3 and 8). In the *nireus* and *oribazus*-groups e.g. the upper surface shows a tripartite colour-pattern with light (azure) middle-bar, and black inner and outer region, but only the slightest traces of nervural and internervural striae, while these latter are distinctly marked and in complete array on the underside of many of the appertaining forms (e.g. *nireus*).

When therefore it can be proved for every single detail in the pattern of mimetic forms that it belongs to the stock of generic, familiar or ordinal hereditary features by which the outward appearance of the several members of a group is effected, there is no reason left for ascribing the total effect of the combination of all these details to the influence of Protective Mimicry. Nor can the phenomenon of Polygynomorphism itself be attributed to this cause, it has to be considered as a peculiar complication of sexual difference in general, occurring in certain groups of butterflies, as e.g. Papilionids. That some of the polymorphic females may profit by their accidental likeness to unpalatable forms, is indeed very probable, but this profit can merely be a consequence of the casual similarity, never its cause.

The phenomenon of Polygynomorphism itself should be classed with other cases of Polymorphism, either in connection with sexuality or independent of it, as seasonal, geographical, racial plurality of

type. In the end, it is of the same nature as specific differentiation in general.

So in *Hepialus humuli* the white masculine form has evidently lost the primitive specific livery, which is still preserved by the female and by the Shetland-male.

Though in general my opinions on these subjects disagree with those of HAASE, I feel much satisfaction in making the following quotation from the concluding passage of his "Resumption" (p. 112): "The mimetic transformation was preceded in most cases by atavistic phenomena from the side of the females, which in the beginning reached back to the patterns of the nearest relatives, but as the process proceeded, passed over to those of more distanced forerunners and in this way procured the material for the mimetic adaptation".

So HAASE attributes the uniforms of mimetic females to hereditary influences, instead of considering them as the consequence of secondary deviations from the primitive specific type.

Groningen, Nov. 1920.

Physics. "*On the Equation of State for Arbitrary Temperatures and Volumes. Analogy with Planck's Formula.*" II. By Dr. J. J. van Laar. (Communicated by Prof. H. A. Lorentz).

(Communicated at the meeting of November 27, 1920).

§ 7. Some Notes to § 1—6.

It will be soon two years ago that I wrote the first part of this Article[1]); studies of various kinds prevented me from continuing the subject, and not until now could I take it up again.

Before I proceed to the derivation of the equation of state, based on the found general expression (6) on p. 1194 loc. cit. for the *time*-average of the square of velocity u_t^2, expressed in u_0^2 (in which u_0 represents the velocity with which the considered molecule passes the neutral point in its motion to and fro between two neighbouring molecules), I will add a few remarks to elucidate and complete what was treated before.

1. In the first place a few words about the transition of some "linear" quantities to the corresponding "spatial" quantities.

If we have linear quantities, we can consider all our velocities as the components of the *relative* velocities directed *normally*; as we always imagine a molecule moving rectilinearly to and fro between two molecules *at rest*. We know that $\overline{u_r^2}=2\overline{u^2}$, and that the mean value of the component of $\overline{u_r^2}$, directed normally, in its turn is the third part of this, so that we have (cf. also p. 1195 loc. cit.):

$$\overline{(u^2)_n}=\frac{2}{3}\overline{u^2}.$$

Hence we may write:

$$\frac{1}{2}Nm\,\overline{(u_r^2)_n}=\frac{2}{3}\times\frac{1}{2}Nm\,\overline{u^2},$$

or also, denoting the *time*-average by the index t:

$$\frac{1}{2}Nm\,\{\overline{(u_r^2)_n}\}_t=\frac{2}{3}\times\frac{1}{2}Nm\,\overline{(u^2)_t}.$$

In this $^1/_2\,Nm\,\overline{(u^2)_t}\,(=\,^3/_2\,pv$ in ideal gases$)\,=\,^3/_2\,RT$, so that we may henceforth write:

[1]) These Proc., Vol. XXI, p. 1184.

$$\frac{1}{2} Nm \{\overline{(u_r^2)_n}\}_t = RT,$$

by which the transition in question has been accomplished. In what follows u^2 will, however, always simply be written instead of $\overline{(u_r^2)_n}$, with omission of the indices r and n and of the usual mean-value dash (the time-average is then denoted by u_t^2); the *real* mean velocity square $\overline{u^2}$, if it should occur, being expressed by (u^2). Hence we have:

$$\underline{\frac{1}{2} Nm\, u_t^2 = RT} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

Starting from the relation (cf. equation (a) on p. 1189 loc. cit.)

$$\frac{1}{2} Nm\, u_\sigma^2 = \frac{1}{2} Nm\, u_0^2 + Nf(l-\sigma)^2,$$

in which σ represents *that* distance from the centre of the moving molecule to that of the molecule supposed stationary, towards which it moves, at which the work of the attractive forces reaches its *maximum* value (hence at which the attraction changes into repulsion) — we shall find, after multiplication by $^3/_2$, for the *real* mean squares of velocity:

$$\frac{1}{2} Nm\, (u_\sigma^2) = \frac{1}{2} Nm\, (u_0^2) + \frac{3}{2} Nf(l-\sigma)^2 .$$

In this $\frac{1}{2} Nm\, (u_\sigma^2) = E$ represents the total Energy of the system (the atom-energies within the molecule being left out of consideration). Further $^1/_2\, Nm\, (u_0^2) = L_0$ is the mean kinetic Energy at the neutral point halfway between the two molecules at rest (where the attractive forces neutralise each other), $\underline{^3/_2\, Nf(l-\sigma)^2 = \Delta}$ representing the maximum work of the attractive forces. We have represented this last quantity by E_0 in our first paper, but as this way of representation can easily give rise to misunderstanding, we shall substitute Δ for E_0 in what follows. We have therefore:

$$\underline{E = L_0 + \Delta,} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

in which accordingly $\underline{E = {}^3/_2 \times {}^1/_2\, Nmu_\sigma^2}$, $\underline{L_0 = {}^3/_2 \times {}^1/_2\, Nmu_0^2}$. Hence in the joint neutral points $E = L_0 +$ the total potential energy of the attractive forces; and in the joint points σ in the immediate neighbourhood of the molecules, with which the moving molecule will impinge, E will be $= L_0 +$ the total increment of the kinetic energy in consequence of the attractive forces.

The quantity Δ, therefore, represents the fixed, invariable (potential or kinetic) energy of the attractive forces, which rise or fall

of temperature cannot increase or decrease. Change of temperature can only modify L_0, and consequently also E. Henceforth $E-\Delta$ may always be written for L_0.

The work of the repulsive forces, which become active after the attractive forces in the above-indicated point σ have ceased to act, has been left out of consideration in what precedes, because Δ is entirely unaffected by it. For the diminished kinetic energy is simply converted into a corresponding increase of the potential energy — now of the repulsive forces — which reaches its culminating point when u has become $=0$ (culminating point of the collision). We have, therefore, only to do with the maximum work of the attractive forces.

2. In the first paper it has been shown that the calculation of the *time*-average u_t^2 leads to the relation (equation (6) on p. 1194)

$$u_t^2=\frac{1}{2}u_0^2\frac{\left\{\varphi\sqrt{1+\varphi^2}+\log\left(\varphi+\sqrt{1+\varphi^2}\right)\right\}\sqrt{\dfrac{m}{2f}}+\dfrac{1}{2}\pi(1+\varphi^2)\sqrt{\dfrac{m}{2\varepsilon}}}{\log\left(\varphi+\sqrt{1+\varphi^2}\right)\sqrt{\dfrac{m}{2f}}+\dfrac{1}{2}\pi\sqrt{\dfrac{m}{2\varepsilon}}}.$$

This becomes after division of numerator and denominator by $\log\left(\varphi+\sqrt{1+\varphi^2}\right)\sqrt{\dfrac{m}{2f}}$:

$$u_t^2=\frac{1}{2}u_0^2\frac{\left(1+\dfrac{\varphi\sqrt{1+\varphi^2}}{\log.}\right)+\dfrac{{}^1/_2\,\pi\,(1+\varphi^2)}{\log.}\sqrt{\dfrac{f}{\varepsilon}}}{1+\dfrac{{}^1/_2\,\pi}{\log.}\sqrt{\dfrac{f}{\varepsilon}}}, \quad . \quad (c)$$

in which (loc. cit.) $\varphi=\dfrac{l-\sigma}{u_0}\sqrt{\dfrac{2f}{m}}$. The distance $\sigma-s'$, during which the repulsive forces will act, follows from

$$u_0^2+\frac{2f}{m}(l-\sigma)^2-\frac{2\varepsilon}{m}(\sigma-s')^2=0, \quad \text{or} \quad u_0^2\,(1+\varphi^2)=\frac{2\varepsilon}{m}(\sigma-s')^2$$

at the culminating point of the collision. Hence we have for $l-\sigma$ and $\sigma-s'$:

$$\underline{l-\sigma=u_0\varphi\sqrt{\frac{m}{2f}}}\,; \quad \underline{\sigma-s'=u_0\sqrt{1+\varphi^2}\sqrt{\frac{m}{2\varepsilon}}}\,; \quad \underline{\frac{\sigma-s'}{l-\sigma}=\frac{\sqrt{1+\varphi^2}}{\varphi}\sqrt{\frac{f}{\varepsilon}}}\,, \quad (d)$$

whereas for the times t_1 and t_2 is found:

$$\underline{t_1=\log\left(\varphi+\sqrt{1+\varphi^2}\right)\sqrt{\frac{m}{2f}}}\,; \quad \underline{t_2=\frac{1}{2}\pi\sqrt{\frac{m}{2\varepsilon}}}\,; \quad \underline{\frac{t_2}{t_1}=\frac{{}^1/_2\,\pi}{\log}\sqrt{\frac{f}{\varepsilon}}} \quad . \quad . \quad (e)$$

a. At *high temperatures* where, in consequence of the equation $\varphi = \frac{l-\sigma}{u_0} \sqrt{\frac{2f}{m}}$, φ becomes *small* when u_0 becomes large (supposing $l-\sigma$ always remains comparatively small, which is fulfilled here, because we always consider solid (at most liquid) systems), (c), (d) and (e) with $log(\varphi + \sqrt{1+\varphi^2}) = log(\varphi + 1) = \varphi$ pass into:

$$u_t^2 = \frac{1}{2} u_0^2 \frac{2 + \frac{\frac{1}{2}\pi}{\varphi}\sqrt{\frac{f}{\varepsilon}}}{1 + \frac{\frac{1}{2}\pi}{\varphi}\sqrt{\frac{f}{\varepsilon}}}; \quad \frac{\sigma - s'}{l-\sigma} = \frac{1}{\varphi}\sqrt{\frac{f}{\varepsilon}}; \quad \frac{t_2}{t_1} = \frac{\frac{1}{2}\pi}{\varphi}\sqrt{\frac{f}{\varepsilon}} \ \begin{matrix} high \\ temp.) \end{matrix}, \quad (c_1)$$

so that in the case of weak collisions (in which ε, the constant of the repulsive force, is not *very* much greater than f, the constant of the attractive forces), in consequence of φ in the denominators of the second terms in numerator and denominator of the above fraction for u_t^2, these latter terms will prevail; hence u_t^2 will approach to $\frac{1}{2} u_0^2$ ($c_v = 6$). Whereas in case of *strong* collisions, when ε is supposed *very large* with respect to f, or when φ gradually increases somewhat on *decrease of temperature*, the first terms prevail, so that then u^2 will more and more approach to u_0^2 ($c_v = 3$).

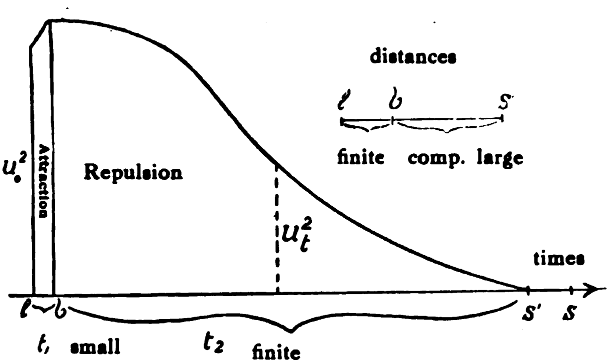
The ratios $(\sigma - s') : (l-\sigma)$ and $t_2 : t_1$ will be *great* for φ small and $f : \varepsilon$ not very much smaller than 1; *smaller* on the other hand for somewhat larger φ, and ε much greater than f.

With regard to $l-\sigma$ and $\sigma - s'$ themselves, it may be observed that according to the supposition $l-\sigma$ always remains *finite*, so that $\sigma - s' = u_0 \sqrt{\frac{m}{2\varepsilon}}$ can become large at increasing temperature and finite ε. But this increase is restricted first of all by this, that u_0 can never become *too* great, because then our suppositions (solid state with small values of $l-\sigma$) would not be fulfilled; and secondly by this that with comparatively large values of u_0, in consequence of which $\sigma - s'$ would become too large, ε will gradually greatly increase, so that the molecules can never approach each other more closely than to a certain minimum distance. Only in case of very strong collisions (ε very large with respect to f) $\sigma - s'$ can approach to 0 at not too large values of u_0.

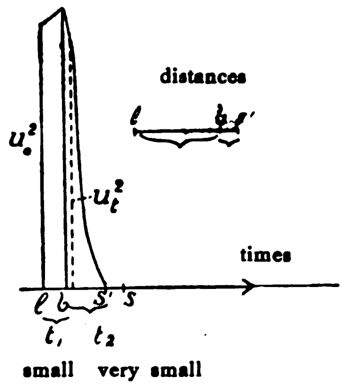
It holds for t_1 and t_2 themselves, that $t_1 = \varphi \sqrt{\frac{m}{2f}}$ will always approach 0 at high temperature, while $t_2 = \frac{1}{2}\pi \sqrt{\frac{m}{2\varepsilon}}$ remains finite — unless ε is very large, in which case t_2 can even become much smaller than t_1.

All these relations are graphically represented by Fig. 1*a* and Fig. 1*b*, in which the values of u are given in function of the *time*.

High temperatures **(u_0 large, φ small).**



Weak **collisions.**
(ϵ/f not very great; $u_t^2 = {}^1/_2\, u_0^2$; $c_v = 6$).
Fig. 1*a*.



Strong **collisions.**
(ϵ/f very great, or T somewhat lower; $u_t^2 = u_0^2$; $c_v = 3$).
Fig. 1*b*.

In the so-called "weak" collisions the velocity of the colliding molecule will not diminish *suddenly*, but gradually. This is among others fulfilled when the attractive force is supposed to change into a repulsive one already before the molecules collide. It may then be further assumed that the repulsive force does not become infinite before the impact itself, so that in general — unless the velocity is infinitely great — the two molecules will never be in absolute contact. Hence there is always between σ and a value s' somewhat greater than s (the distance of the centres at contact) a certain space, in which the decrease of velocity in consequence of the repulsive forces can take place; and there always remains — even at $T = 0$ — some distance, however slight, between the molecules, because of course l cannot become smaller than σ.

It is self-evident that this somewhat modified way of considering the matter is only of a formal nature. Theoretically there is nothing changed when s is displaced to σ, and s' from a point within s to a point outside it; now, however, we need not think the molecule greatly compressed in the weak collisions, as we had to do with the former way of considering the matter.

The two above figures also show clearly why in the case of Fig. 1*a* u_t^2 approaches to ${}^1/_2\, u_0^2$, and in the case of Fig. 1*b* to u^2_0. For as e.g. in the first case the time, during which the repulsive forces act, is so much greater than that under the influence of the attractive forces, the *time-average* will lie in the neighbourhood of

$^1/_2 u_0^2$. In Fig. 1*b* on the other hand the "*action*" (energy $\times$ time) of the repulsive forces will be very much smaller than that of the attractive forces, with this result that now the time-average descends but little below u_0^2.

With decreasing values of u_0 (lower temperatures) the relations of Fig. 1*a* will more and more shift in the direction of Fig. 1*b* in consequence of the continual increase of φ, so that c_v will descend already to a smaller value from the limiting value 6, before the temperature has fallen to such a low value, that u_t^2 is in inverse logarithmic dependence to u_0^2 (see below) — in other words *before the region of quanta proper has been entered.*

b. At *low temperatures* φ will appear to be great; i.e. on the supposition that in $\varphi = \dfrac{l-\sigma}{u_0}\sqrt{\dfrac{2f}{m}}$ the quantity $l-\sigma$ does not approach 0 to the same degree as u_0, but much more slowly, so that $(l-\sigma):u_0$ will approach ∞. It is even probable that $l-\sigma$ does not become $=0$ even at $T=0$, but approaches to a certain small limiting value. This is in agreement with the permanent decrease of the expansibility at very low temperatures, and with the remaining of a certain *finite* zero-point energy $\Delta = {}^1/_2\, Nf(l-\sigma)^2$ at $T=0$.

Our equation (c) now becomes:

$$u_t^2 = \frac{1}{2}u_0^2\,\frac{\dfrac{\varphi^2(1+{}^1/_2\pi\sqrt{f/\varepsilon})}{log\,(2\varphi+{}^1/_{2\varphi})}}{1+\dfrac{{}^1/_2\pi}{log\,2\varphi}\sqrt{\dfrac{f}{\varepsilon}}} = u_0^2\,\frac{\varphi^2}{log\,(4\varphi^2+2)} \times \frac{1+\dfrac{1}{2}\pi\sqrt{\dfrac{f}{\varepsilon}}}{1+\dfrac{{}^1/_2\pi}{log 2\varphi}\sqrt{\dfrac{f}{\varepsilon}}} \;\; (low\ temp.), \quad (c_2)$$

which at *very low* temperatures, at which φ approaches to 0, will become nearer and nearer to

$$u_t^2 = u_0^2\,\frac{\varphi^2}{log\,(2\,\varphi^2+1)} \times \left(1+\frac{1}{2}\pi\sqrt{\frac{f}{\varepsilon}}\right) \;(very\ low\ temp.), \quad . \quad (c'_2)$$

because the finite term $log\,2$ can then also be omitted by the side of $log\,(2\varphi^2+1)$ in $log\,(4\varphi^2+2) = log\,(2\varphi^2+1) - log\,2$. But the factor $1+\frac{1}{2}\pi\sqrt{\dfrac{f}{\varepsilon}}$ can be omitted only when ε is very large with respect to f (strong collisions), which is, however, not very probable in view of what was found at high temperatures — unless at high temperatures φ is so small, that notwithstanding ε is very much greater than f, the quantity $\dfrac{{}^1/_2\pi}{\varphi}\sqrt{\dfrac{f}{\varepsilon}}$ would yet remain comparatively great.

But at all events in the case (c_2) or (c_2') u_t^2 (proportional to the temperature) will be very much greater than u_0^2 (proportional to $L_0 = E - \Delta$). Both — temperature and kinetic energy in the neutral point — approach to 0, but the energy *very much more rapidly* to Δ (the constant zero-point energy of the attractive forces that finally remains) than the temperature to 0.

The relations (d) and (e) now become:

$$\frac{\sigma - s'}{l - \sigma} = \sqrt{\frac{f}{\varepsilon}} \quad ; \quad t_1 = \log 2\varphi \cdot \sqrt{\frac{m}{2f}} \quad ; \quad \frac{t_2}{t_1} = \frac{{}^1/_2\,\pi}{\log 2\varphi}\sqrt{\frac{f}{\varepsilon}},$$

so that for a value of $l - \sigma$ remaining finite, the distance $\sigma - s'$ will not be very *much* smaller than $l - \sigma$, unless again ε is very much larger than f. The time t_1 approaches (logarithmically) to ∞, while at finite t_2 $\left(= \frac{1}{2}\pi \sqrt{\frac{m}{2\varepsilon}}\right)$ the ratio $t_2 : t_1$ will approach logarithmically to 0. These relations are represented by the subjoined figure.

Low Temperatures (u_0 small, φ large).

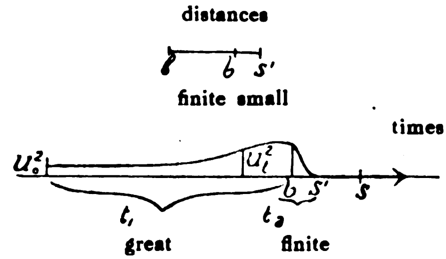


Fig. 2.

As has been said both u_t^2 and u_0^2 approach to 0, and the reason that u_t^2 (i.e. the temperature) does not remain finite at $u_0^2 = 0$ — since there is a *finite* increase of the square of velocity (originally $= 0$) in consequence of the attraction forces — but *likewise* approaches zero, lies in this that the *time* during which this increase takes place, approaches ∞ (though it be logarithmically). In the neutral point the attraction is $= 0$; when the moving point has got somewhat outside the neutral point, there will therefore be only very slowly question of any action of a force (which then increases further linearly with the deviation x, see p. 1188 loc. cit.), hence of acceleration.

3. When we now proceed from u_t^2 to T, and from u_0^2 to $L_0 = E - \Delta$, we have therefore in the case of *high* temperatures from $u_t^2 = {}^1/_2\, u_0^2$:

$$\frac{1}{2} N m u_t^2 = \frac{1}{2} \times \frac{1}{2} N m u_0^2,$$

i. e. (cf. Note 1)

$$RT = \frac{1}{2} \times \frac{2}{3} L_0 = \frac{1}{3} L_0 = \frac{1}{3} (E - \Delta).$$

Hence also

$$E = \Delta + 3\,RT, \quad \ldots \ldots \ldots \quad (f)$$

or

$$c_v = \left(\frac{dE}{dT}\right)_v = 3R = 6 \quad \ldots \ldots \ldots \quad (f')$$

If u_t^2 were $= u_0^2$ instead of $= {}^1/_2\, u_0^2$ (strong collisions, cf. Fig. 1*b*), then E would have become $= \Delta + {}^3/_2 RT$, $c_v = {}^3/_2\, R = 3$.

All this applies to monatomic substances. In the case of multi-atomic (n-atomic) substances it is necessary to take besides the energy of the attractive forces Δ also the atomic energy Δ' within the molecule into consideration, so that E becomes $= L_0 + \Delta + \Delta'$. Now $L_0 = 3RT$, while $\Delta' = 3(n-1)RT$ may be put, when $3(n-1)$ represents the number of supplementary degrees of freedom. We then find $E = \Delta + 3nRT$, i. e. $c_v = 3nR = 6n$ (NEUMANN's law) [1].

At low temperatures we have:

$$\frac{1}{2} Nm\, u_t^2 = \frac{{}^1/_2\, Nm\, u_0^2\, \varphi^2 . \theta}{\log (4\, \varphi^2 + 2)},$$

according to (c_2), when we denote the factor

$$\left(1 + \frac{1}{2} \pi \sqrt{\frac{f}{\varepsilon}}\right) : \left(1 + \frac{{}^1/_2\, \pi}{\log 2\varphi} \sqrt{\frac{f}{\varepsilon}}\right)$$

by θ; hence because $u_0^2 \varphi^2 = (l - \sigma)^2 \dfrac{2f}{m}$, and thus ${}^1/_2\, Nmu_0^2\, \varphi^2 = = Nf (l - \sigma)^2 = {}^2/_3 \Delta$:

[1]) It should be remembered that for gases $E = \Delta + {}^1/_2 R\, (3 + \mu)\, T$ may be put, in which μ also represents the number of supplementary degrees of freedom (see among others BOLTZMANN, Gastheorie II, p. 124—125 and 128). But here μ is simply $= n$ for multi-atomic molecules, so that for mon-atomic molecules n is still $= 0$, for di-atomic molecules however $n = 2$, for tri-atomic ones $n = 3$, etc. Hence when the term Δ, which approaches 0, is neglected, and also the quantity ε introduced by BOLTZMANN, referring to the potential energy of the intramolecular movements, E becomes $= {}^1/_2 RT\, (3 + n)$ for gases, leading to $c_v = {}^1/_2 R\, (3 + n)$, hence (with $c_p - c_v = R$) to $\dfrac{c_p}{c_v} = 1 + \dfrac{2}{3 + n}$. (BOLTZMANN adds the above mentioned correction quantity ε to $3 + n$).

$$RT = \frac{{}^3/_2\,\theta\,\Delta}{log\left(4\dfrac{\Delta}{E-\Delta}+2\right)},$$

because $\varphi^2 = \dfrac{(l-\sigma)^2\,\dfrac{2f}{m}}{u_0^{\,2}} = \dfrac{Nf\,(l-\sigma)^2}{{}^1/_2\,Nm\,u_0^{\,2}} = \dfrac{{}^3/_2\,\Delta}{{}^3/_2\,L_0} = \dfrac{\Delta}{L_0}.$

When we reverse the relation found for RT, omitting $log\,2$ by the side of the so much larger term $log\left(\dfrac{2\Delta}{E-\Delta}+1\right)$, where $E-\Delta = L_0$ is small compared with Δ, and putting also $\theta = 1$ (which is fulfilled for large values of $\varepsilon : f$), we get:

$$E = \Delta + \frac{2\,\Delta}{e^{\frac{{}^3/_2\Delta}{RT}} - 1} \quad \ldots\ldots\ldots \quad (g)$$

As we already remarked in our first paper, it is indeed exceedingly remarkable that (with the exception of a few numerical factors) *exactly the same* relation between E and T appears here as was derived by Planck on the ground of the hypothesis of "quanta" drawn up by him. For this it was only required to take into account the *time averages* in the ordinary dynamic relations, which gives rise especially at low temperatures to a *considerable difference* between $u_t^{\,2}$ (the time average of the value of $u_0^{\,2}$, which has greatly increased under the influence of the attractive forces) and $u_0^{\,2}$, both being very slight and approaching to 0.

From (9) follows with ${}^3/_2\Delta : R = \alpha$:

$$E = \frac{3}{2}\,R\alpha + \frac{3\,R\alpha}{e^{\alpha/T} - 1} \quad ; \quad c_v = \left(\frac{dE}{dT}\right)_v = 3R\,\frac{(\alpha/T)^2\,e^{\alpha/T}}{(e^{\alpha/T} - 1)^2}\ ,\ .\ (g')$$

which exponentially approaches to 0 $\left(\text{viz. to } 3R\,\dfrac{\alpha^2}{T^2}\,e^{-\alpha/T}\right)$, when T approaches to 0.

There is, however, one great difference with Planck's formula. Apart from this that in Planck's work the well-known quantity $Nh\nu$ appears instead of ${}^3/_2\Delta = Nf\,(l-\sigma)^2$, so that $h\nu$ would have to be $\underline{h\nu = f(l-\sigma)^2}$ [1]), our formula (g) is *only* valid for very low temperatures, and (f) *only* for very high temperatures. This is of course only owing to this that (c_1) only ensues from the general formula (c), when φ is supposed to be small, whereas with large values of φ the relation (c_2) results from it. Accordingly our (g) may,

[1]) Cf. what has been said concerning $l-\sigma$ under b) of Note 2.

therefore, not be applied in case of high temperatures, whereas this *may* be done with PLANCK's formula: the latter holds (at small volumes) both for low and for high temperatures.

It is, however, remarkable that *if* (g) were valid for high temperatures (which is not the case according to us), $E = \Delta + 3RT$ is duly obtained as limiting value for E, identical to (f). Our formula, from which (f) ensues for high temperatures and (g) for low temperatures, seems to be more general, and the approach to (f) takes place in a somewhat different way than with PLANCK's formula.

At any rate it will have to be assumed — if φ is to be small at high temperature, and large at low temperature, and if Δ is not to become $= 0$ at $T = 0$ — that with condensed (solid) systems $(l - \sigma)^2$ changes only comparatively slightly; and that it does so in the same degree as the frequency ν. Then PLANCK's quantity h would be related in a definite way with the constant of the attractive forces f (being in its turn again in relation to e^2, when e represents the electric elementary quantum), and in consequence of this also with a_0/v_0 at the absolute zero. There are very strong indications for this: particularly the undeniable connection between the so-called *chemical* constant and also the constant of the *vapour-pressure* on one side, and the quantity $a_0/v_0{}^2$ on the other side, as I demonstrated shortly ago in a Paper in the Recueil des Tr. Ch. of March and May 1920 — while it is known that this chemical constant in its turn is again in relation with h.

I hope to return to this special subject later on.

4. We will now discuss somewhat more fully the nature and the way of acting of the forces assumed by us between the molecules.

In connection with what was already observed above, we might assume that the attractive action of M_1 e.g. rapidly decreases at a certain small distance from M_1, and disappears at a certain very small distance σ, being replaced by a rapidly increasing repulsive force, which for $x = s$, when the moving molecule P would touch the molecule M_1, would become infinitely great. (Cf. further what was already said on this head under a) of Note 2).

Thus no two separate forces are required, nor two separate Virial-parts — an Attractive-Virial part and a Repulsive-Virial part — but only one; which point of view was already set forth by me some twenty years ago. [1]) The difference with the assumption in the first part of this paper lies, therefore, chiefly in this, that then the

[1]) See Arch. Teyler (2) T. VII, 3ième Partie, p. 1—34 (1901): „Sur l'influence des corrections etc." (particularly p. 28 et seq.).

attractive force continued to increase up to σ, after which it sud-

Total force F.

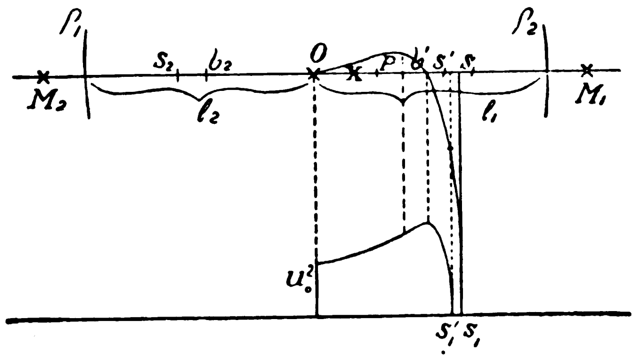


Fig. 3.

denly (hence discontinuously) changed into a repulsive force, with another constant of intensity ε than that of the attractive force f; whereas now we suppose a *continuous* change of force at σ with a single constant f.

Analytically this may be expressed — as far as e.g. the action of a force, exerted on P by M_1 is concerned — by a formula of the form

$$F_1 = f(\rho_1 - l_1 + x)\frac{l_1 - \sigma_1 - x}{l_1 - s_1 - x}, \quad . \quad . \quad . \quad . \quad . \quad (1a)$$

in which $x = OP$, and the indices 1 all refer to distances from M_1, measured towards the left; and this instead of simply $F_1 = f(\rho_1 - (l_1 - x_1))$ as we put formerly (loc. cit. p. 1188 et seq.), i.e. the attractive force proportional to the distance from the moving point P to the boundary of the sphere of attraction ρ_1 (of M_1), so that $F_1 = 0$, when P lies on the boundary of this sphere or outside it. In consequence of this the attraction, after having reached a maximum, again becomes $= 0$ at σ_1 (hence $(x = l_1 - \sigma_1)$, reverses its sign, and again changes into a repulsion, which would become infinite at s_1 $(x = l_1 - s_1)$. From the other molecule M_2 P is subjected to an attraction

$$F_2 = f(\rho_2 - l_2 - x)\frac{l_2 - \sigma_2 + x}{l_2 - s_2 + x}, \quad . \quad . \quad . \quad . \quad . \quad (1b)$$

in which x is again $= OP$, and the indices now refer to distances from M_2, measured to the right.

Hence after some reductions the *total* action exerted on P to the right (see Fig. 3), (with omission of the indices, because $l_1 = l_2 = l$, etc.) is now found to be:

$$F = F_1 - F_2 = f \, . \, 2x\left[1 - \frac{(\rho - s)(\sigma - s)}{(l - s)^2 - x^2}\right], \quad . \quad . \quad . \quad (2)$$

instead of simply $F = F - F_2 = f \, . \, 2x$, as before (p. 1188).

It is self-evident that the *total* attractive force will become $=0$ somewhat earlier, when P moves towards M_1, than at σ_1 (when F_1 acts *alone*) — it does so at σ'_1 or simply σ' — because the attractive force of M_2 acts in opposite sense. In fact the above quantity becomes $=0$, when

$$x_{\sigma'} = \sqrt{(l-s)^2 - (\varrho - s)(\sigma - s)} = \sqrt{(l-\sigma)^2 - (\varrho + \sigma - 2l)(\sigma - s)} = l - \sigma'.$$

As in the case under consideration the molecule P will always be within the spheres of attraction of the two molecules M_1 and M_2, $2l$ is always $< \varrho + s$, hence a fortiori $2l < \varrho + \sigma$ (cf. p. 1187 l.c.). The value of $x_{\sigma'}$ is therefore $< l - \sigma$, i.e. F becomes $=0$ in σ', on the left side of σ_1 [1]).

However, all such functions have the drawback, that the further integrations become impossible to carry out by means of closed forms; for both at high and at low temperatures (u_0 large or small) the term of work ${}^2/_m\int$ in $\sqrt{u^2_0 + \frac{2}{m}\int_0^x F dx}$ can never be considered as permanently small with respect to u_0^2 between the limits $x=0$ and $x=s'$. For in the end (at the culmination point of the collision) the qnantity under the sign of the root becomes $=0$ in both cases (high and low temperature), hence the term of work under consideration of the same order of magnitude as u_0^2. And at low temperatures, which is justly the most important case in our considerations, that term is almost everywhere of the order of magnitude u_0^2 — except in the neighbourhood of the points O and somewhere between σ' and s', where this term becomes $=0$ (Cf. Fig. 3).

For this reason we were obliged in our first paper to consider the attraction and the repulsion *separately*, and to assume, instead of the course of F drawn in Fig. 3, a force which continues to increase in direct ratio to x as far as σ_1, after which it suddenly changes into a repulsive force, which likewise increases linearly as the distance from P to σ_1 (P now thought on the righthand side of σ_1). This renders the integrations easy to carry out, and does not touch the nature of the matter.

If it is thought desirable to avoid the introduction of a so-called "sphere of attraction" — which at the same time offers the advan-

[1]) When the distance l of the molecules becomes *too* small, F will not first become positive on the righthand side of O, become $=0$ in σ', and then negative — but at once negative, i.e. there is already immediately repulsion on the righthand side of O. The same thing applies of course to the left side. We shall return to all these different cases in our next paper on the calculation of the Virial.

tage that the assumption of a transition case (see p. 1186) becomes unnecessary (viz. the case in which the moving molecule is always within the sphere of attraction of M_1, but not always within that of M_2) — a plausible law of attraction must be substituted for $\varrho_1 - l_1 + x$ and $\varrho_2 - l_2 - x$ in the expressions for F_1 and F_2, so that we have e.g.:

$$F_1 = \frac{f}{(l-x)^n} \frac{l-\sigma-x}{l s-x} \quad ; \quad F_2 = \frac{f}{(l+x)^n} \frac{l-\sigma+x}{l-s+x},$$

through which we obtain with *small* values of x:

$$F = \frac{f}{l^n}\left(1+n\frac{x}{l}\right)\frac{l-\sigma}{l-s}\left(1-\frac{\sigma-s}{(l-\sigma)(l-s)}x\right) -$$

$$-\frac{f}{l^n}\left(1-n\frac{l}{x}\right)\frac{l-\sigma}{l-s}\left(1+\frac{\sigma-s}{(l-\sigma)(l-s)}x\right),$$

i.e.

$$F = \frac{f}{l^n}\frac{l-\sigma}{l-s}\left[\frac{n}{l}-\frac{\sigma-s}{(l-\sigma)(l-s)}\right]2x = \frac{f}{l^{n+1}}\frac{l-\sigma}{l-s}\left[n-\frac{l(\sigma-s)}{(l-\sigma)(l-s)}\right]2x,$$

i.e. again proportional to x. This first proportionality and the corresponding quadratic form of the term of work $\int F dx$ continues to exist whatever form is given to the expressions of the action of the force.

According to DEBYE[1]) the exponent n would have the value 9 (for anomalous "Dipol" gases n would be $=7$ on the other hand; cf. the note on p. 183 loc. cit.).

But also the above forms of F_1 and F_2 are in a still greater degree subject to the drawback, that they lead to integrations which it is impossible to carry out in the further calculations.

5. However — without having recourse to the dualistic law of force (one for the attraction and another for the repulsion) which we have chosen for practical reasons — also (2) might be used for the calculation of

$$t = \int_0^{l-s'} \frac{dx}{\sqrt{u_0^2+\omega}} \quad ; \quad u_t^2 = \frac{1}{t}\int_0^{l-s'} \sqrt{u_0^2+\omega_x}\, dx, \quad . \quad . \quad . \quad (3)$$

in which $\omega_x = \frac{2}{m}\int_0^x F dx$; provided one is satisfied with a certain approximation in the logarithmic expression which is then obtained for ω_x. We find namely:

$$\omega_x = \frac{2f}{m}\left[x^2 + (\varrho-s)(\sigma-s)\log\frac{(l-s)^2-x^2}{(l-s)^2}\right], \quad . \quad . \quad . \quad (4)$$

[1]) DEBYE, Die v. D. WAALS'schen Kohäsionskräfte, Physik. Zeitschr. 21, p. 178—187 (1920).

in which x changes in the above integrations from $l-s'$ to 0. And as s' is always $> s$ (only for an infinitely large value of u_0 could s be reached at the culmination of a collision), $l-s'$ is always $< l-s$. Especially at lower temperatures, at which s' remains comparatively far from s, $(l-s'):(l-s)$ can remain considerably smaller than 1 even at the extreme value of x. (If e.g. $l = 1{,}2\,s$, $s' = 1{,}1\,s$, this ratio becomes already $^1/_2$). We may therefore write in approximation for the logarithmic term:

$$log\left(1-\frac{x^2}{(l-s)^2}\right) = -\frac{x^2}{(l-s)^2} - \frac{x^4}{2\,(l-s)^4},$$

so that with

$$\frac{(\varrho-s)\,(\sigma-s)}{(l-s)^2} = a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

the following form is obtained for ω_x:

$$\omega_x = \frac{2f}{m}\,(l-s)^2\left[(1-a)\,\frac{x^2}{(l-s)^2} - \frac{a}{2}\,\frac{x^4}{(l-s)^4}\right]. \quad . \quad . \quad . \quad (4^a)$$

For the form under the sign of the root may therefore be written:

$$u_0^2\left[1+\frac{2f}{m}\frac{(l-s)^2}{u_0^2}\left\{(1-a)\,y^2 - {}^1/_2\,a\,y^4\right\}\right] = u_0^2\,[1+\varphi^2\{(1-a)\,y^2 - {}^1/_2 a y^4\}],$$

when again, as in the first paper, $\dfrac{l-s}{u_0}\sqrt{\dfrac{2f}{m}} = \varphi$ is put, and further y is substituted for $\dfrac{x}{l-s}$. Then:

$$t = \frac{l-s}{u_0}\int_0^{\frac{l-s'}{l-s}}\frac{dy}{\sqrt{1+\varphi^2\{(1-a)\,y^2 - {}^1/_2\,a\,y^4\}}} \quad ; \quad u^2 = (l-s)\,\frac{u_0}{t}\int_0^{\frac{l-s'}{l-s}}\sqrt{ibid}\,.\,dy.$$

For the form under the sign of the root $(1-w_1y^2)\,(1+w_2y^2)$ may be written in this, when

$$w_2 - w_1 = \varphi^2\,(1-a) \quad ; \quad w_1\,w_2 = {}^1/_2\,\varphi^2\,a,$$

so that this form with $y\sqrt{w_1} = z$ passes into $(1-z^2)\left(\quad+\dfrac{w_2}{w_1}\,z^2\right)$, which becomes with $z = cos\,\psi$:

$$sin^2\,\psi\left(1+\frac{w_2}{w_1}\,(1-sin^2\,\psi)\right) = \frac{w_1+w_2}{w_1}\,sin^2\,\psi\left(1-\frac{w_2}{w_1+w_2}\,sin^2\,\psi\right).$$

For dy we have further $\dfrac{dz}{\sqrt{w_1}} = -\dfrac{1}{\sqrt{w_1}}\,sin\,\psi\;d\psi$, so that the above integrals pass into

$$t = -\frac{l-s}{u_0\sqrt{w_1}}\sqrt{\frac{w_1}{w_1+w_2}}\int_{^1/_2\pi}^{0}\frac{d\psi}{\sqrt{1-k^2\,sin^2\,\psi}};$$

$$u_t^2 = -\frac{l-s}{\sqrt{w_1}}\sqrt{\frac{w_1}{w_1+w_2}}\,\frac{u_0}{t}\int\limits_{^1/_2\pi}^{0} \sin^2\psi\sqrt{1-k^2\sin^2\psi}\,d\psi,$$

when $\dfrac{w_2}{w_1+w_2}=k^2$ is put (hence k^2 is always <1). With regard to the limits of the integrals evidently z is also $=0$ for $y=0$, hence $\psi={}^1/_2\pi$. And as at the upper limit $u_0^2+\omega_x$ becomes $=0$ (culmination point of the collision), also $(1-z^2)\left(1+\dfrac{w_2}{w_1}z^2\right)=0$, hence $z=1, \psi=0$. Thus we have, after reversal of the limits, in consequence of which the minus signs drop out:

$$t=\frac{l-s}{u_0}\frac{1}{\sqrt{w_1+w_2}}\int\limits_0^{^1/_2\pi}\frac{d\psi}{\Delta\psi}\,; \qquad u_t^2 = u_0^2\,\frac{w_1+w_2}{w_1}\,\frac{\int\limits_0^{^1/_2\pi}\sin^2\psi\,\Delta\psi\,.\,d\psi}{\int\limits_0^{^1/_2\pi}\dfrac{d\psi}{\Delta\psi}},$$

when for t its value is substituted in the expression for u_t^2. Following LEGENDRE and denoting the complete elliptical integral of the 1st kind, viz. $\int\limits_0^{^1/_2\pi}\dfrac{d\psi}{\Delta\psi}$, by F_1 or singly F, we have also:

$$t=\frac{\varphi}{\sqrt{(w_1+w_2)}}\sqrt{\frac{m}{2f}}\,.\,F\;;\qquad u_t^2=\frac{w_1+w_2}{w_1}\,\frac{u_0^2}{F}\int\limits_0^{^1/_2\pi}\sin^2\psi\,\Delta\psi.d\psi,$$

when $\varphi\sqrt{\dfrac{m}{2f}}$ is substituted for $\dfrac{l-s}{u_0}$ (see above). We have for the modulus k:

$$k^2=\frac{w_2}{w_1+w_2}=\frac{{}^1/_2\,\varphi\left[\varphi(1-\alpha)+\sqrt{\varphi^2(1-\alpha)^2+2\alpha}\right]}{\varphi\sqrt{\varphi^2(1-\alpha)^2+2\alpha}},$$

when we calculate the quantities w_2 and w_1+w_2 from the above expressions for w_2-w_1 and w_1w_2. Hence we may write:

$$k^2=\frac{1}{2}\left[1\pm\frac{1}{\sqrt{1\pm\dfrac{2\alpha}{\varphi^2(1-\alpha)^2}}}\right], \quad . \quad . \quad . \quad . \quad (6)$$

so that at low temperatures (large values of φ) k^2 is always near 1 (provided $\alpha<1$, in which case the $+$ sign is valid).

We must now still reduce the last elliptical integral (the one with

$sin^2\psi$) to that of the 1st and the 2nd kind. According to known formulae of reduction [1]) we have:

$$\int_0^\psi sin^2\psi\,\Delta\psi\,.\,d\psi = \frac{1}{3}\left[\frac{1-k^2}{k^2}\int_0^\psi \frac{d\psi}{\Delta\psi} + \frac{2k^2-1}{k^2}\int_0^\psi \Delta\psi . d\psi - sin\,\psi\, cos\,\psi\,\Delta\psi\right],$$

hence

$$\int_0^{1/2\pi} sin^2\psi\,\Delta\,\psi . d\psi = \frac{1}{3}\left[\frac{1-k^2}{k^2}F + \frac{2k^2-1}{k^2}E\right],$$

when the complete elliptical integral of the second kind, viz. $\int_0^{1/2\pi}\Delta\psi . d\psi$, is represented by E. Hence we find finally:

$$t = \sqrt{\frac{2k^2-1}{1-a}}\sqrt{\frac{m}{2f}}\,F\ ; \qquad u_t^2 = \frac{u_0^2}{3k^2}\left[1 + \frac{2k^2-1}{1-k^2}\frac{E}{F}\right], \quad . \quad (7)$$

because

$$\frac{\varphi}{\sqrt{w_1+w_2}} = \frac{\sqrt{\varphi}}{\sqrt{\varphi^2(1-a)^2+2a}} = \sqrt{\frac{2k^2-1}{1-a}}, \text{ and } \frac{w_1}{w_1+w_2} = 1-k^2.$$

We shall now compare the found formulae (7) with those found before, and again in the two limiting cases: high and low temperatures.

At *high* temperatures (φ small) k^2 approaches $^1/_2$ (if $a < 1$), so that then (7) reduces to

$$t = 0 \quad ; \quad u_t^2 = {}^2/_3\, u_0^2 \ (\textit{high temp.}), \quad . \quad . \quad . \quad . \quad (7^a)$$

instead of $u_t^2 = {}^1/_3 u_0^2$ (weak collisions), as we found before. The

[1]) See among others Durège, Th. der ellipt. Funct., p. 65, formule (29), i. e. (with $\mu = 0, m = 0$)

$$sin\,\psi\, cos\,\psi\,\Delta\,\psi = \int_0^\psi \frac{d\psi}{\Delta\psi} - 2\,(1-k^2)\int_0^\psi \frac{sin^2\psi\, d\psi}{\Delta\,\psi} + 3\,k^2\int_0^\psi \frac{sin^4\psi\, d\psi}{\Delta\,\psi},$$

from which the integral with $sin^4\psi$ can be expressed in both the others, that with $sin^2\psi$ being expressed in F_ψ and E_ψ by the formula (see p. 69)

$$k^2\int_0^\psi \frac{sin^2\psi\, d\psi}{\Delta\,\psi} = F_\psi - E_\psi,$$

as can be easily verified by differentiation, after $\Delta\psi = \sqrt{1-k^2 sin^2\psi}$ has been put everywhere in the denominator. (The integral to be reduced by us then becomes $\int_0^\psi \frac{sin^2\psi - k^2 sin^4\psi}{\Delta\,\psi}d\psi$).

fall of the velocity during the action of the repulsive force — expressed as function of the time — is now less great than in fig. 1^a, so that the descending branch will be much more horizontal. The natural consequence of this is, that the time average gets much nearer to u_0^2. (c_0 would now become $= {}^3/_2 \times {}^3/_2$ $R = 4^1/_2$ instead of 6). However — the above calculation is certainly questionable at high temperatures, because then $log\left(1 - \frac{x^2}{(l-s)^2}\right)$ may certainly not be expanded into a series, as at the culmination point of the impact x would become $= l - s$ $(s' = s)$. The expansion into a series up to x^4 used by us, gives a too great value for ω_x, hence also a too great value for u_t^2. Instead of rather abruptly, the damping of this exceedingly great velocity would take place during a much too long interval — so great even that s' would lie far inside s, which is of course impossible.

At low temperatures (φ great) on the other hand there can be no objection to applying the expansion into a series up to x^4, because then the velocity is so small that it will be reduced to 0 already within a very short interval. Now the modulus k approaches to 1, hence E to $\int\limits_0^{^1/_2\pi} cos\,\psi\, d\psi = (sin\,\psi)_0^{^1/_2\pi}$, i.e. also to 1; but F will approach to

$$\int\limits_0^{^1/_2\pi} \frac{d\psi}{cos\,\psi} = log\,tg\,(45 + \tfrac{1}{2}\psi)_0^{^1/_2\pi} = log\,\infty - log\,1,$$ i.e. to $log\,\infty$. As, however, at the same time $1-k^2$ approaches 0, we must examine what value $(1-k^2)\,F$ assumes in (7), when k^2 is near 1.

According to a well-known theorem [1]) F approaches to $log\frac{4}{\sqrt{1-k^2}}$ in this case.

Hence we get for t and u_t^2, when k approaches 1, from (7):

[1]) Cf. among others LAMB, Treatise on Hydrodynamics, p. 170; CAYLEY, Ellipt. funct., Art. 72; MAXWELL, Elect. and Magn. II, p. 311—316; DURÈGE, p. 190 et seq., particularly p. 213; KIRCHHOFF, Vorl. p. 270; etc. Better than DURÈGE's derivation, which is based on LANDEN's transformation, is KIRCHHOFF's beautiful derivation. The latter is founded on the splitting up of the integral into two parts, viz.

$$\int\limits_0^{^1/_2\pi} = \int\limits_0^{^1/_2\pi-\theta} + \int\limits_{^1/_2\pi-\theta}^{^1/_2\pi},$$ in which θ is a small quantity, which is, however, supposed to be great with respect to $\sqrt{1-k^2}$. But in both derivations only the *limiting* value of F is reached.

It is in my opinion a better method to start from JACOBI's relation

$$t=\frac{1}{\sqrt{1-\alpha}}\log\frac{4}{\sqrt{1-k^2}}\sqrt{\frac{m}{2f}}\quad;\quad u_t^2=\frac{u_0^2}{3\,(1-k^2)\,\log{}^4/\sqrt{1-k^2}},$$

because E approaches 1, and 1 may be omitted by the side of $\frac{2k^2-1}{1-k^2}\frac{E}{F}=\frac{1}{(1-k^2)F}$. Now from (6) follows for large values of $\varphi\ (\alpha<1)$:

$$k^2=\frac{1}{2}\left[1+\left(1-\frac{\alpha}{\varphi^2\,(1-\alpha)^2}\right)\right]=1-\frac{{}^1/_2\,\alpha}{\varphi^2\,(1-\alpha)^2},$$

so that we obtain with $\log\frac{4}{\sqrt{1-k^2}}=\frac{1}{2}\log\frac{16}{1-k^2}$:

$$\left.\begin{aligned}t&=\frac{1}{2\sqrt{(1-\alpha)}}\log\left(\frac{32\,(1-\alpha)^2}{\alpha}\varphi^2\right)\sqrt{\frac{m}{2f}}\\ u_t^2&=\frac{\frac{4}{3}\frac{(1-\alpha)^2}{\alpha}\varphi^2}{\log\left(\frac{32\,(1-\alpha)^2}{\alpha}\varphi^2\right)}u_0^2\end{aligned}\right\}.\ (low\ temp.)\ .\quad .\qquad (7^b)$$

$F=-\frac{\log q'}{\pi}F'$, in which F' refers to the integral with the complementary modulus $k'=\sqrt{1-k^2}$, and q' is one of the auxiliary quantities q and q', introduced by Jacobi. From the relation $\sqrt{k'}=\frac{2\sqrt[4]{q'}(1+q'^2+q'^6+etc.)}{1+2q'+2q'^4+2q'^9+etc.}$ follows first of all $\frac{1}{16}k'^2=q'\left(\frac{1+q'^2+q'^6+\ldots}{1+2q'+2q'^4+\ldots}\right)^4$, from which $q'=\frac{1}{16}k'^2\left(1+\frac{1}{2}k'^2+\frac{21}{64}k'^4+\ldots\right)$. And from this follows:

$$-\frac{1}{2}\log q'=\log\frac{4}{k'}-\left(\frac{1}{4}k'^2+\frac{13}{128}k'^4+\ldots\right).$$

Through expansion into series $F'=\frac{1}{2}\pi\left(1+\frac{1}{4}k'^2+\frac{9}{64}k'^4+\ldots\right)$ is easily derived for $F'=\int_0^{1/2\pi}\frac{d\psi}{\sqrt{1-k'^2\sin^2\psi}}$, so that from $F=\frac{F'}{{}^1/_2\pi}\left(-\frac{1}{2}\log q'\right)$ finally ensues $F=\left(1+\frac{1}{4}k'^2+\frac{9}{64}k'^4\right)\left[\log\frac{4}{k'}-\left(\frac{1}{4}k'^2+\frac{13}{128}k'^4+.\right)\right]$, or approximated $F=\left(1+\frac{1}{4}k'^2\right)\left(\log\frac{4}{k'}-\frac{1}{4}k'^2\right)$, the limiting value of which is evidently $\log\frac{4}{k'}$.

We may still point out that the auxiliary quantity q always remains very small. q' is $=0$ for $k'=0$, but q' is only 0,043 for $k'={}^1/_2\sqrt{2}$ (the same for k and q). It is to this fact that the exceedingly strong convergence of the Jacobi series for elliptical functions is owing.

Again t approaches logarithmically to ∞ and u_t^2 to $\frac{A\varphi^2}{\log B\varphi^2}u_0^2$, just as in (c_2), derived in § 3 with two separate forces.

In order to render a comparison possible, we must now again introduce the maximum work Δ performed by the attractive forces. From (4^a), i.e.

$$\omega_x = \frac{2f}{m}(l-s)^2\left[(1-a)\frac{x^2}{(l-s)^2} - \frac{a}{2}\frac{x^4}{(l-s)^4}\right],$$

follows that this will be maximum, when $\frac{x^2}{(l-s)^2} = \frac{1-a}{a}$, so that

$$(\omega_x)_{max} = \frac{2f}{m}(l-s)^2\frac{(1-a)^2}{2a}.$$

Multiplied by $^1/_2\,Nm$, we get accordingly (Cf. Note 1)

$$Nf\,(l-s)^2\frac{(1-a)^2}{2a} = \frac{2}{3}\Delta.$$

As further $\varphi^2 = \frac{(l-s)^2}{u_0^{\,2}}\frac{2f}{m}$, we get:

$$^1/_2\,Nm\,.\,\frac{32\,(1-a)^2}{a}\varphi^2\,u_0^{\,2} = Nf(l-s)^2\,.\,\frac{32\,(1-a)^2}{a} = 64\times\frac{2}{3}\Delta.$$

Thus we find for $^1/_2\,Nm\,u_t^2 = RT$, because $^1/_2\,Nm\,u_0^{\,2} = {}^2/_3\,L_0 =$
$= {}^2/_3\,(E-\Delta)$:

$$RT = \frac{^1/_{24}\times 64\times {}^2/_3\,\Delta}{\log\dfrac{64\times {}^2/_3\,\Delta}{^2/_3\,L_0}} = \frac{^{16}/_9\,\Delta}{\log\dfrac{64\,\Delta}{E-\Delta}}, \quad . \quad . \quad . \quad . \quad (8)$$

as against $RT' = \frac{^2/_3\,\theta\Delta}{\log\dfrac{4\Delta}{E-\Delta}}$ on the former assumption of two separate forces (see Note 3). The coefficients are different, but the logarithmic relation has remained *entirely the same*.

As however our former assumption leads to better coefficients than the assumption (2) of Note 4, elaborated by us in this paper, and as it does so both for high and for low temperatures, we can in future, by the side of the latter procedure, also base ourselves on the supposition — which is *simpler for the calculations* — of *two separate* forces, in which the repulsive force begins to act at $x = l - \sigma$, after the attractive force has reached its highest point.

I hope that the foregoing Notes go to clear up some of the difficulties that might have presented themselves in the reading of my

first paper.[1]) Now we will proceed with the task which we had set ourselves, and examine the Virial of attraction and of repulsion with a view to the drawing up of the equation of state in case of small volumes, especially at very low temperatures.

(*To be concluded*).

La Tour près Vevry, Autumn 1920.

I gratefully express my thanks again to the VAN 'T HOFF-fund, which has greatly facilitated the execution of this work.

---

[1]) Though *not all* the objections, which some time ago Prof. LORENTZ was so kind as to communicate to me in a letter, may have been removed by this paper, yet I hope that some have been solved. Nobody can be more fully conscious of the great difficulties that are to be surmounted here, than the author himself.

In the autumn of 1919 I had the privilege of having a discussion with Prof. EHRENFEST — to whom we owe the so important theory of the *Adiabatic Invariants* (1916), which theory was later so felicitously continued by BURGERS (1917) and KRUTKOW (1919—1920) — on the contents of my first paper. He advanced, among others, the objection that not *all* the molecules on approach to other molecules would come in collision, after which they would again move away from them, but that some of them would remain for a time in the neighbourhood of them. This is perfectly true, but in case of *gases* we should then have to do with *association*, a case that was purposely left out of consideration by me as an unnecessary complication. But as we have to deal here not with gases, but chiefly with *solid* bodies, where the molecules only move to and fro between the neighbouring ones, this complication is, indeed, quite excluded. Besides already for a long time all the more recent theories of structure have rejected the idea of association in the solid, crystallized state, so that we are justified in leaving it quite out of consideration. But in my opinion there are greater difficulties, of an *entirely different* nature, to which I hope to return later.

Physiology. — "*The function of the Otolithes*". By Prof. R. MAGNUS and A. DE KLEYN.

(Communicated at the meeting of Sept. 25, 1920).

In the course of the last half century an infinite amount of literature has appeared upon the functions of the vestibular organ. From the first, anatomical research rendered it probable that a sharp distinction had to be made between the sensory epithelium in the cristae of the semi-circular canals covered by the so-called cupula and which can move freely in the endolymph, and between the sensory epithelium of the maculae of sacculus and utriculus which, covered by the otolithes with their greater specific weight, appears specially suited to react upon the greater or lesser pressure of the said otolithes.

But, whereas our knowledge of the function of the semi-circular canals is fairly extensive and, moreover, the anatomical data agree fairly well with the clinical and experimental data, this is by no means the case with respect to the knowledge and theories regarding the function of the otolithes.

This is due to various causes. If, as has been supposed in particular by MACH and BREUER, the function of the otolithes is determined by the greater or lesser pressure upon the sensory epithelium beneath, we may expect that the otolithes will have some influence upon those reflexes which come into play by changes in the position of the head, and that their influence will continue as long as this position remains unchanged and the pressure of the otolithes is constant. Or, in other words, that the otolithic reflexes are more particularly tonic reflexes. Up till a few years ago, tonic reflexes of the labyrinth were known only in the form of compensatory positions of the eyeball, and therefore the function of the otolithes had to be studied exclusively from these reflexes. As, however, our knowledge of these positions was still far from complete, and sufficient investigations had not been made, it is obvious that the literature on this subject will contain opinions of a more or less speculative nature. Moreover, the influence of the clinique was inhibitory. While, clinically, the different vestibular reactions upon (rotatory) movements were investigated more and more carefully and began to assume an ever-increasing importance in the diagnostics of the diseases of the labyrinth,

in physiology also attention was concentrated almost exclusively upon this species of labyrinth reflexes. BARANY's attempt to include the compensatory positions of the eyeball in clinical research was not imitated.

Besides by *pressure*, it is a priori very well imaginable that the otolithes might react upon *motion*, if hereby the specifically heavier otolithes by reason of their greater inertia undergo slight displacements with respect to the underlying sensory epithelium. MACH and BREUER, upon purely theoretical grounds, believed this to be the case with the labyrinthine reactions upon progressive movements. Experimental data on this point were lacking, but, as it appeared to be physically impossible that the canals played any part here, the cause had to be ascribed to the otolithes. As will appear below, however, the opinion held by MACH and BREUER, namely that for physical reasons the canals cannot have anything to do with this, is erroneous, while other experimental grounds will be furnished for the theory that in the reactions upon progressive movements it is just the semi-circular canals that play the chief part.

As now during the last ten years various other, hitherto unknown tonic labyrinth reflexes were found at the Pharmacological Institute of Utrecht, it was natural that these new experimental data should be used for the further study of the function of the otolithes. For this purpose the following method was adopted. If the tonic labyrinth reflexes depend upon the greater or lesser pressure of the otolithes upon the underlying sensory epithelium, it may be expected that these reflexes will be at their maximum or minimum at the same moment as this pressure is maximal or minimal. Whether the reflexes will be the strongest under the greatest pressure or under the least, cannot *a priori* be stated. This will depend upon whether the sensory epithelium is excited most strongly by the pressing or the pulling of the otolithes; it might be very well possible that pressure of the otolithes gives rise to a certain reflex, whilst pulling might also excite another reflex action, though a different one.

Therefore, to arrive at as unbiassed a conclusion as possible, all the tonic labyrinth reflexes were first examined quantitatively as exactly as possible, and in particular in what position of the head the maximum and minimum of these reflexes were found.

The tonic reflexes examined were the newly found tonic labyrinth reflexes on the muscles of the body and the so-called labyrinthine "Stellreflexe", while further the already long-known compensatory "positions" of the eyeball were carefully examined quantitatively as to their maxima and minima. This investigation was performed

upon different animals; the compensatory eye "positions" only upon rabbits and guinea-pigs, as in the case of animals with quick voluntary eye-movements, such as cats and dogs, these reflexes are not so well suited for quantitative examination. In the Anatomical Institute, Messrs. DE BURLET and KOSTER kindly determined for us by various methods the position of the otolithes in the rabbit head, and constructed a model from which, in each position of the head, the accompanying position of the otolithes could immediately be seen, so that it was now possible to trace whether with the maxima and minima of the tonic labyrinth reflexes also a typical position of the otolithes was to be met with. This proved to be the case, as this paper will further show.

It was finally desirable to answer the question as to the correctness of the supposition that the tonic labyrinth reflexes are otolith reflexes, and at the same time to investigate whether this is also the case with the reactions upon progressive movement, as it is fairly generally assumed to be according to the theory of MACH and BREUER. For this purpose a method had to be employed whereby the influence of the otolithes was in some way eliminated, while keeping the semi-circular canals intact. The reflexes of the latter should then be present and the otolith reflexes absent. This method of control, as will be seen later, also confirmed the presupposed function of the otolithes.

The result of the above-described investigations, shortly summarized, was as follows:

A. POSITION OF THE OTOLITHES WITH THE DIFFERENT TONIC LABYRINTH REFLEXES IN THE RABBIT.

1. *Tonic labyrinth reflexes upon the muscles of the body. (Extremities and neck)*

a. On the extremities.

When the tonus of the muscles of the extremities is examined in the various positions of the head, it appears that the extensors attain their maximal tonus whenever the head lies symmetrically on the skull, the nose making an angle of 0° — 45° upwards (individual differences). When the tonus of the extensors is minimal, the position of the head differs from the former by 180°.

If the model of the rabbit's skull with its otolithes be now put into the same positions, it will appear that with the aforesaid maximum and minimum, the utriculus otolithes also have a very typical position (the individual variations anatomically found agree

with the variations found experimentally); this is not the case with the sacculus otolithes.

In the position whereby the extensors have their maximal tension, the utriculus otolithes stand horizontally and are hanging on the epithelium in the position of the minimal tonus, the utriculus otolithes also stand horizontally, but press upon the sensory epithelium.

It follows, therefore, that it is not by pressing that the utriculus otolithes exhibit the greatest stimulus, but when hanging.

Just the reverse is the case with the tonus of the flexors; this tonus is strongest when the otolithes press and weakest when they hang.

The influence of the labyrinth upon the tonus of the flexors is, however, much less than that upon the extensors, and can be demonstrated only by special means, so that there is no room for doubt that at all events the strongest stimuli proceed from the hanging otolithes.

Experiments upon rabbits, where the labyrinth on one side had been removed, showed that *one* labyrinth is connected with the muscles of both extremities, so that each utriculus otolith exercises its influence upon the muscles of both extremities.

b. On the neck.

For this the same holds as has been stated for the tonic labyrinth reflexes upon the muscles of the extremities, with but one single fundamental difference. Whereas, from experiments upon rabbits with one labyrinth removed, it appeared that each labyrinth is in connection with the muscles of both extremities, this is not the case with the neck muscles, where the influence is uni-lateral. From which it follows that one utriculus otolith is connected with the cervical muscles of one side of the body only.

Therefore, the fact that both utriculus otolithes lie almost in *one* plane, does *not* imply that the removal of one labyrinth will not be followed by any symptoms, as indeed is seen by the turning of the neck after a uni-lateral labyrinth extirpation.

2. "*Labyrinthstellreflexe*".

By this word those reflexes are meant by which the head is brought back from any abnormal position to the normal again. The centres for these reflexes, like those of the other "Stellreflexe", lie in the mesencephalon.

Examination of the labyrinthine "Stellreflexe" of rabbits after uni-lateral labyrinth extirpation has shown that the "Labyrinth-Stellreflexe" are superimposed upon the turning of the neck above-described and looked upon as utriculus-reaction, in such a way that they try to bring the head into such a lateral position that the intact labyrinth is uppermost. In this position the "Stellreflex" proceeding from the intact labyrinth has its minimum.

Whenever, on the contrary, the intact labyrinth is found underneath, the "Stellreflex" has its maximum.

If the head be placed in any chosen position, the animal will not rest until it has got its head back as far as possible into such a position that the "Stellreflex" proceeding from the intact labyrinth is as small as possible. There is only perfect rest when, as has been said above, the intact labyrinth is uppermost. If the model be examined to see in what position the otolithes now are, it will be seen that the sacculus otolith stands horizontally and presses upon the sensory epithelium.

On the other hand, the greatest unrest is observed when the head is placed into such a position that the intact sacculus otolith hangs from the sensory epithelium.

Thus, in the case of the "Labyrinthstellreflexe" after uni-lateral labyrinth extirpation, it can be proved with certainty that the maximum stimulation proceeds from the otolith when it hangs, and the minimum when it presses.

The "Labyrinth-Stellreflexe" on the head, by means of which, when *both* labyrinths are intact, the head is invariably brought back from an asymmetrical to a symmetrical position, are explained by the coöperation of the stimuli from both labyrinths. The head comes to rest in such a position that the stimuli arising from the two sacculi are equally strong.

From the above it is clear that the maximum positions of the right and left labyrinths differ for the above-described "Stellreflexe" approximately 180°, from which it follows that they must be sacculus reflexes and not utriculus reflexes, since the utriculus otolithes lie almost in one horizontal plane, while the sacculus otolithes form an angle of 120°—140°—150° with each other.

Besides the "Sacculusstellreflexe" just described, by which the head is brought back from asymmetrical to symmetrical position, we must assume that there are other labyrinthic "Stellreflexe", as the head is not merely brought into an undetermined symmetric posture, but always into such a position, that the vertex is above, the lower jaw below, and the mouth rather under the horizontal

plane. Whether, in this case, the sacculi or the utriculi are the cause of these reflexes, cannot yet be stated with certainty.

3. *Tonic labyrinth reflexes upon the eye muscles.*

a. Vertical deviations.

In these tonic labyrinth reflexes one does best to proceed from the symptoms after uni-lateral labyrinth extirpation.

After uni-lateral labyrinth extirpation, the strongest vertical deviation of the eyes will be found when the intact labyrinth is downwards, with the head lying on the cheek. The rectus sup. of the same side and the rectus inf. of the crossed side are then the most strongly contracted. When the intact labyrinth is above, the rectus sup. of the same side and the rectus inf. of the opposite side are least contracted. A tonic influence of the intact labyrinth upon the rectus sup. of the crossed side and the rectus inf. of the same side could not be demonstrated.

With intact labyrinths and a normal position of the head, the stimuli from both labyrinths upon the recti sup. and inf. of both eyes are equally strong, so that the eyes in this case show no vertical deviation.

As the maximum positions of the labyrinths differ about 180° (maximum position of the right labyrinth when lying on the right side, and of the left when lying upon the left side), these reflexes cannot be utriculus reflexes, as then the maximum positions would be the same.

On the other hand, in the maximum and minimum positions of these reflexes the sacculus otolithes have again a very typical position; in the maximum position the sacculus otolith is hanging, whereas in the minimum position it presses.

After a uni-lateral labyrinth extirpation the vertical deviation is maximal with the maximum position of the intact labyrinth, but with the minimum position it is very slight or absent.

At all events it is certain that in the minimum position of the intact labyrinth there is no vertical deviation to the *other* side, from which we may fairly conclude that with the minimum position of the sacculus otolithes the stimulus is really little or none, whereas with the maximum position, on the other hand, it is very strong.

Thus in this case also, as with the "Labyrinthstellreflexe", it can be proved that the stimulus is strongest when the otolith hangs, and weakest or nothing at all when the otolith presses.

b. Rotatory Deviations.

In the rotatory deviations the relations are much more complicated than with the tonic labyrinth reflexes hitherto discussed. A fuller discussion would, however, claim more space than we have at our disposal, for which reason we shall merely mention the results which the different considerations have led to.

The rotatory deviations of the eyes found with the compensatory eye positions may perhaps be explained by the following hypothesis. The foremost sacculus corners have their own innervation, are bent laterally and thereby lie almost in one frontal plane. Each sacculus corner has a functional connection with the obliquus sup. and inf. of both eyes. When the otolith hangs from the sacculus corner, the obliq. sup. of both eyes are the most strongly contracted and the obliq. inf. the most relaxed. When the otolith presses upon the sacculus corner, it is the obliq. inf. which is most contracted and the obliq. sup. most relaxed. As the sacculus corners lie almost in the same plane, no distinct change in the position of the maxima and minima are to be found after uni-lateral extirpation. As each sacculus corner influences the obliq. sup. of both eyes similarly and the obliq. inf. reciprocally, the rotatory deviations remain qualitatively unchanged after uni-lateral extirpation, and are reduced quantitatively to about the half.

Thus, with the rotatory deviations, it is not clear that the hanging otolith exercises the strongest and the pressing otolith the weakest stimuli, but it seems as if stimuli, even though opposed, are caused by hanging as well as by pressing otoliths.

SUMMARY.

By means of a model, constructed in the Anatomical Institute of Utrecht, of the otolithes in their correct position in the rabbit-skull, an examination has been made as to whether the different tonic labyrinth reflexes might be otolith reflexes. Experimentally in all tonic labyrinth reflexes the position of the head has been determined, whereby these reflexes have their maximum and minimum. The following conclusions were arrived at:

1. The tonic labyrinth reflexes upon the muscles of the body proceed from the utriculi. One utriculus macula is connected with the muscles of *both* extremities; but, on the contrary, with the neck (and trunk) muscles of *one* side of the body only.

2. The "Labyrinthstellreflexe" are reflexes of the sacculus. Whether the utriculi also play a part is not certain.

3. Compensatory positions of the eyeball.

a. The vertical deviations are sacculus reflexes and are caused by the main portion of the sacculus macula. One sacculus macula is connected with the rectus sup. of the same side and with the rectus inf. of the crossed side.

b. The rotatory deviations are probably produced in the sacculus corners, which have their own innervation.

Stimuli proceed constantly from the maculae and do not alter as long as the otolithes do not alter their position with respect to the horizontal plane. When the otolith stands horizontally, the strongest or the weakest stimuli are excited.

In the case of the sacculus otolithes it can, with respect to the "Labyrinthstellreflexe" and of the vertical compensatory deviations of the eyes, be proved with certainty, that the strongest stimulus occurs when the otolith is hanging.

For the sacculus corners, on the contrary, things are somewhat more complicated, and the stimuli occur when the otolithes press as well as when they hang.

As our investigation is at present concerned with the rabbit, the above holds good only for this species.

B. Isolated removal of the otolithes in guinea-pigs.

As has appeared from this and former communications, the different labyrinth reflexes may be divided into the following groups.

a. Reflexes upon movement.

1. Reactions and after-reactions of the head and eyes upon rotatory movements.
2. Reactions upon progressive movements.

b. Tonic labyrinth reflexes.

1. Tonic reflexes upon the muscles of the body.
2. "Labyrinthstellreflexe".
3. Compensatory positions of the eyeball.

As regards the reactions and after-reactions upon rotatory movements, these have long been known and are justly regarded by all researchers as reflexes, of the semi-circular canals.

It was demonstrated above that the different tonic labyrinth reflexes are probably otolith reflexes, and also that different otolithes may be held responsible for the different reflexes.

But little experimental research has hitherto been devoted to the reactions upon progressive movements. For some classes of animals, however, we were able to demonstrate certain reactions upon progressive movements dependent on the labyrinths. As the following investigations have been carried out

upon guinea-pigs, the reactions found in these animals may be shortly recorded here.

Lift reactions.

The animal sits in a normal posture upon a plank held horizontally. If the plank be moved vertically upwards, it will be seen that at the commencement of the movement the fore-legs are bent sharply and the head droops.

When the movement of the plank ceases, the fore-legs will be stretched straight out, the fore part of the body is lifted, and frequently too the head is bent backwards. In strong reactions the hind-legs act too, till finally the animal stands upon its four extended legs. The reverse reaction takes place when the movement is downwards. When the movement commences, the extremities, and especially the fore-legs, are extended, and the front part of the body raised. On the cessation of the movement, the fore-legs are bent and the head and front part of the body droop towards the plank.

Muscular vibrations.

The animal is held vertically in the air with the head upwards, the thumb and little finger of the left hand lie round the belly, the fore and third fingers support the fore-legs, with the hind-legs resting upon the palm of the hand, while the thumb and forefinger of the right hand rest upon the neck and shoulders of the animal. The head is in the normal position. One must wait until the animal is perfectly quiet and the fingers of the right hand no longer feel any muscular vibrations in the shoulder and neck muscles. Then, if the animal be moved vertically (up and down), or horizontally (ventrally, dorsally, to right and left), the right hand during these progressive movements will feel a distinct muscular vibration. Closer attention will reveal that here, too, the vibration takes place at the commencement and end of the movement.

The spreading of the toes.

The guinea-pig is held vertically in the air, the right hand under the arm-pits with the palm of the hand turned towards the back of the animal. The toes of the hind feet are then carefully stroked together. If the animal be then moved even very slightly downwards, the toes of the hind feet will immediately spread apart.

The reaction can be demonstrated with most, though not with all animals, and occurs at the beginning of the movement. If the animal be moved in the same way upwards, the toes will also be spread, this occurring some times at the commencement and sometimes at the close of the movement.

All these reactions upon progressive movements are dependent upon the labyrinths, and are not visible after a double labyrinth extirpation. We cannot enter here into details, neither can we discuss the precautions to be observed in the investigation. These will be given in more detail later, when also some other labyrinthic reactions upon progressive movements will be described.

As was mentioned above, Mach and Breuer believed on theoretical considerations that it was physically impossible that the semi-circular canals had anything to do with these reactions, and therefore came to the conclusion that only the otolithes were accountable for these

reactions. These considerations, however, MACH and BREUER based upon the presupposition that the semi-circular canals are a closed system with firm walls and filled with fluid, so that no endolymph currents could be expected in progressive movements. In this they overlooked the fact, however, that the endolymph of the semi-circular canals is connected by the ductus endolymphaticus with the saccus endolymphaticus which lies in the cranial cavity, and also that the perilymph is not surrounded everywhere by a firm wall, but that at two parts there are elastic membranes, viz. the foramen rotundum and the foramen ovale.

From researches made with the help of a model constructed with due regard to the above by Prof. ORNSTEIN and Dr. BURGER, it appeared that under these circumstances with progressive movements endolymph currents might well occur in the endolymph.

Thus it remained an open question whether the reactions upon progressive motions are reflexes of the semi-circular canals or of the otolithes.

It therefore became desirable to find a method whereby either the canals or the otolithes could be isolated. The latter proved to be possible by employing a method of WITTMAACK's, whereby normal guinea-pigs are centrifugated in ether narcosis at a speed of 1000 metres per minute. Previous to the centrifugation, all the animals to be experimented upon were carefully examined as to all labyrinth reflexes in accordance with a certain scheme, and the result recorded. The same examination was repeated after the centrifugation and successively on the following days, so that the complete historia morbi of each animal was obtained in this way. A clinical diagnosis was then made regarding the condition of the canals and the various otolithes, based upon the above-explained views regarding the function of the semi-circular canals and the otolithes. After this the animals were killed, the labyrinths fixed, and later carefully examined microscopically in series-sections. In this way, it was possible to compare the clinical and the anatomical diagnoses with each other.

The results obtained may be summed up as follows; for particulars we refer again to the detailed communication to be published later.

SUMMARY.

1. By centrifugation of normal guinea-pigs, according to WITTMAACK's method, it was found possible to bring about in many of the animals tested a condition in which the labyrinth reflexes upon movement are normal (reactions and after-reactions of head and eyes upon

rotatory movements and reactions upon progressive movements), while all tonic labyrinth reflexes of the position are lacking.

2. If these animals be kept alive for some days, and examined every day as to their labyrinth reflexes, it can be seen that in some animals this condition remains unaltered, while in others the tonic labyrinth reflexes return partially or even wholly.

3. A histological examination of one animal, where a complete recovery had taken place, showed that all the otolithes were in place with the exception of one utriculus otolith, which appeared to be injured. This last, however, was in entire accordance with the phenomena found clinically. The apparatus of the semi-circular canals was likewise intact.

4. In one guinea-pig which, after two days, showed intact labyrinth reactions upon movement (reactions and after-reactions upon rotatory movements and reactions upon progressive movement), but, on the other hand, no tonic labyrinth reflexes, all four otolithes were found, upon histological examination, to be torn off, and were found in other places in the labyrinth, while the canals with the cristae appeared to be intact.

It follows from the above data that:

5. The labyrinth reflexes on movement, and in particular the reactions and after-reactions of head and eyes on rotatory movements, are reflexes of the semi-circular canals, and may be excited even when the otholithes are lacking.

6. The same holds good for reactions upon progressive movements.

7. On the other hand, the tonic labyrinth reflexes (tonic labyrinth reflexes upon the muscles of the body, "Labyrinthstell-reflexe" and compensatory positions of the eyeball) are otolith-reflexes, which cannot be produced after the destruction of the otolith membranes.

8. In many cases, directly after the centrifugation, the function of the otolithes is temporarily suspended owing to the strong mechanical influences upon the specifical heavier otolith membranes, even though the otolith membranes themselves still lie in their right places. The function is only restored after various lengths of time.

9. It thus appears from these and previous data, that the reflexes upon movement (reactions and after-reactions of head and eyes upon rotatory movements and the reactions on progressive movements) are excited by the semi-circular canals, while the otolith apparatus must be held responsible for the tonic labyrinth reflexes (tonic labyrinth reflexes on the muscles of the body, "Labyrinthstellreflexe" and compensatory positions of the eyeball).

10. This naturally does not exclude that the otolith apparatus

can be stimulated also by certain kinds of motion (centrifugal force, inhibition in rapid progressive motions).

Finally some microphotos are reproduced for the further elucidation of the above.

Guinea-pig 2.

After the centrifugation of this animal the following clinical diagnosis was made: All the labyrinth reflexes were present in a normal state. As, before the centrifugation, the animal turned its head to the left when held up by the pelvis with the head downwards, it had to be assumed that the right utriculus had functioned before the centrifugation rather more than the left one. After the centrifugation no further difference in the functioning of the two utriculus otolithes was found, so that it seemed probable that the right utriculus had been injured by the centrifugation. Further it had to be expected that all the otolithes, as also the semi-circular canals, were intact.

The histological examination revealed the following facts:

Semi-circular canals with cristae normal.

Right sacculus (Fig. 2*a*.) quite intact. The photo shows the right sacculus greatly magnified; the sensory epithelium with the intact otolith membrane can be seen.

Left sacculus (Fig. 2*b*.) also quite intact. In the photo the intact otolith membrane can be seen slightly magnified.

Right utriculus (Fig. 2*c*.), as was to be expected, was partly injured. In the photo the otolith membrane can be seen in the region of the greatest lesion, the part of the membrane that is turned towards the semi-circular canals is still lying normally upon the sensory epithelium, the other portion of the membrane has been torn off from the sensory epithelium.

Left utriculus. (Fig. 2*d*.) quite intact. In the photo the sensory epithelium can be seen covered by the normal otolith membrane.

Thus we see complete accordance between the clinical and the pathologic-anatomical diagnoses.

Cavia 8.

In this animal it appeared after the centrifugation that all the labyrinth reflexes upon motion, i.e. the reactions and after-reactions of head and eyes upon rotatory movements and the reactions upon progressive motion, were present in the normal way, whereas the tonic labyrinth reflexes were all lacking (tonic labyrinth reflexes on the muscles of the body, "Labyrinthstellreflexe" and compensatory positions of the eye).

The clinical diagnosis was as follows:

Semi-circular canals with ampullae intact; otolithes all tossed off.

Histologically the following was found:

Right sacculus (Fig. 8*a*.): The otolith membrane proved to be entirely tossed off. In the photo the sensory epithelium can be seen slightly magnified without otolith membrane, while the torn off membrane can be seen in the corner of the sacculus.

Left sacculus (Fig. 8*b*. and 8*c*): In fig. 8*b* the sensory epithelium of the left sacculus can be seen more strongly magnified. Nothing is left of an

otolith membrane; the difference between this and fig. 2*a*., where a normal membrane is present, is clear. In fig. 8*c*. the sensory epithelium of the sacculus, without otolith membrane can also be seen slightly magnified, while the torn off membrane lies in the corner of the sacculus.

Right utriculus (Fig. 8*d*.) In this photo, more strongly magnified, the sensory epithelium of the utriculus can be seen covered with some fragments of the otolith membrane in granular form. (Compare fig. 2*d*. with normal otolith membrane). The torn off membrane itself was found in the posterior vertical semi-circular canal (Fig. 8*e*.).

Left utriculus (Fig. 8*f*.): We see here, slightly magnified, the sensory epithelium of the left utriculus covered with a few granules. The torn off membrane was also in this side found in the posterior vertical canal (fig. 8*g*. with the posterior vertical and with the horizontal canals).

Semi-circular canals with cristae quite intact. (The torn off otolith membrane in the posterior vertical canals).

Thus here again is the most absolute agreement between the clinical and the pathologic-anatomical diagnoses.

Three other cases, not described here, showed the same complete agreement between the clinical and the microscopical investigations.

Pharmacological Institute of the Utrecht University.

Mathematics. — "*On elementary surfaces of the third order*". (Fifth communication). By Dr. B. P. HAALMEIJER. (Communicated by Prof. HENDRIK DE VRIES).

(Communicated at the meeting of November 27, 1920).

Some remarks from professor FELIX KLEIN caused me to go once more through my former communications on this subject [1]), which investigation showed me that several places require correcting and supplementing.

§ 1. For surfaces of the third *degree* with 3 or 7 real lines [2]) there are systems of planes, which, as far as the real part of the section is concerned, have only one line in common with the surface. Only a very artificial interpretation would enable us to bring these surfaces under our definition of F^3 (comm. I. p. 102 and 103). This difficulty disappears when point 2 of that definition is replaced by the following condition (also preferable for other reasons): All plane sections which exist, are elementary curves, amongst which the third order occurs, but no higher order.

With regard to the counting of lines, we prescribe the following:

A line a is considered *triple* (resp. *double*) in a plane α, if every line b ($\neq a$) of α is limiting element of a sequence of lines, situated in one plane and each of which has 3 (resp. 2, but for no sequence 3) points in common with F^3, which points converge towards the point of intersection of a and b.

In all other cases a is considered *single* in α.

The number of times a is counted *on* F^3 we put down as 1, plus 1 for every plane in which a counts double, plus 2 for every plane in which a counts triple.

Theorem: No plane section of F^3 is of the second order.

Such a section would have one of the following forms:

1. Isolated point.
2. Oval.
3. Two different single lines.
4. One double line.

[1]) These Proc. Vol. XX, p. 101—118, 304—321, 736—748 and 1246—1253.

[2]) *Line* stands for *straight line*.

First case. Let A be the point and α the corresponding plane. There exists a plane β with a section of the third order and this plane passes through A as a curve of the third order has at least 1 point in common with every line of its plane. The curve in β has only A in common with α, hence this curve passes at the point A from one side of α to the other. Then however the plane α divides the vicinity of A on F^3 into two parts with only one common limiting point (A), and this contradicts point 1 of the definition of F^3. [1]).

Second case. Let α be the plane of the oval, a a line intersecting the oval at two different points A and B and β a plane through a. Suppose the curve in β is of the third order, then one of the points A and B, for instance A, counts double as point of intersection with a. This is possible in three ways:

The curve in β has a for tangent at A.

A is cusp in β, both branches coming from the same side of α.

A is ordinary double point, two branches coming from each side of α [2]).

The two first possibilities do not agree with theorem 1 p. 311 comm. 2, [3]) for in α A is ordinary point of intersection of the oval and a, and in β we can find lines converging towards a without carrying points of F^3 converging towards A.

Remains the third possibility. The two branches departing from A in α are connected both above and below α. The sector which forms the connection above α cannot have β for tangent plane along a branch in β, for in that case an infinite number of lines in β would belong to F^3. [4]) It follows that this sector crosses β twice and this is impossible as it connects two branches situated on different sides of β.

We conclude that the curve in β cannot be of the third order, hence it must obviously be of the second. The only condition imposed on β was that this plane has two different points in common with the oval in α. On the other hand there exist lines carrying 3 different points of F^3. Let γ be a plane through such a line and intersecting the oval in α at two different points. In γ the curve must be of the

---

[1]) An analogous case was minutely dealt with in comm. 1 p. 104 and 105.

[2]) In case the curve in β consists of a double line through A and a single one through B, we can at once find another plane through a with a curve of the third order and without a double line.

[3]) The demonstration there given made use of point 2 of the definition of F^3 only so far as sections of order higher than the third were excluded.

[4]) This also holds when the curve in β has degenerated.

second order, but at the same time have 3 different points in common with a line and this of course cannot be.

Third case. Let α be the plane, b and c the lines and A their point of intersection. If we exclude the point A, then along the rest of b the sectors of F^3 meet either everywhere from the same side, or always from opposite sides of α[1]). The same holds for c. Now if the sectors meet from the same side of α both along b and c, then an infinite number of lines of α is contained in F^3. If the sectors meet from the same side along b and from opposite sides along c, then, according to our definitions, b counts double in α, hence the section would be of the third order. Remains the possibility that along b as well as c the sectors meet from different sides of α. Let σ be an arbitrary plane, not containing A, a the line of intersection of σ and α and B and C the points of intersection of a and c resp. b. The curve in σ crosses a at B and C and has no further points in common with a. For a curve of the third order this would imply that either B or C is double point and the reasoning used for the *second case* reduces this to a contradiction. Hence the curve in σ is evidently of the second order. On the other hand there exist lines not passing through A and carrying 3 different points of F^3. Thus in a plane through one of these lines and not containing A we have once more constructed an impossibility.

Fourth case. Let α be the plane and a the double line. Along a the sectors of F^3 meet either everywhere from the same side, or always from opposite sides of α. The former is impossible as the space in which we work is supposed to be projective, and the latter would imply that through every line ($\neq a$) of α a plane β passes with a double point on a. This obviously would mean two lines in β. Hence F^3 would contain an infinite number of lines, a possibility excluded throughout.

This completes our demonstration.

From the preceding theorem follows that a plane α with a double line a contains besides a single line b. Let a and b intersect at A. We proceed to show that *along a (point A excluded) the sectors of*

[1]) Every point of b ($\neq A$) is internal point of an interval along which the sectors meet either from the same or from opposite sides of α. Excluding an arbitrarily small open segment of b round A, a finite number of these intervals exist, such that every point of the rest of b is internal to at least one of them (BOREL-theorem). From this the property under consideration follows at once for the entire line b (excluded point A).

F^2 *meet everywhere from the same side of* α. Suppose a carries a point P, such that the branches meeting on a at P are connected on both sides of α. Then there exists a plane through P, not containing a and in which P is double point. The reasoning used for the *second case* (p. 921) again reduces this to a contradiction. Hence every point ($\neq A$) of a is internal to an interval along which the sectors meet from the same side of α. The note at the bottom of the preceding page completes the demonstration.

Along a triple line in α the sectors obviously meet everywhere from opposite sides of α.

§ 2. In this § we intend to give some corrections and additions to comm. 3 (p. 736—748).

p. 740 and 741. The reasoning sub II (p. 740) is incomplete and starting at line 3 (p. 741) to be replaced by the following: Let σ, turning round c, converge towards α, first from one and then from the other side. In both cases the loop of the curve in σ contracts towards A, for if the loops converged towards a finite segment of a, then each of the converging planes would contain 3 branches, with only one common point (A) and breadths larger than some finite constant. These would converge towards a single linesegment in the limiting plane, which circumstance contradicts the assumption that F^2 is a twodimensional continuum. Hence only the principal branches of the curve in σ converge towards a.

We consider a certain plane σ. Let AP, AQ and AR be the branches departing in σ on the same side of c, AQ really being the middle one. The two semilines in which A divides a, are connected on that same side of α by a sector which crosses σ along the branches AP, AQ and AR successively. If σ converges from the one side towards α, the outside branch on the side of AP converges towards a and turning σ in the opposite direction, the branch corresponding to AQ merges in a. In both cases this is a principal branch and as one of the outside branches belongs to the loop, there must somewhere be a change from principal branch to loop branch, and this means a degeneration. This contradicts the assumption that a is the only line of F^2 through A.

p. 743 line 11 says: "If A is the only limiting point, then the contracting ovals would give to A the character of a point of a twodimensional continuum". Here it has been overlooked that A does not belong to the contracting region, defined by the ovals on F^2, but to these ovals themselves. This necessitates further consideration

of the case. In every plane through a ($\neq \varepsilon$) lies an oval touching a at A. Degeneration being excluded, the branches departing from A on the oval, cannot pass from one side of a to the other, unless their plane passes ε. Let the semiplane in which the oval touches a, first turning one way and then the other, converge towards ε. If in both cases the ovals contracted towards A, then they would suffice to give to A the character of a point of a two-dimensional continuum and a sequence of points on a with A for limiting point could not be fitted in anymore.

Considering ε has only a in common with F^3 there remains as only possibility that when their plane turns one way, the ovals contract towards A and in the other case towards the entire line a (from both sides with branches connected via the line at infinity). This means that A is cusp in every plane not containing a and that every other point of a is point of inflexion in every plane not containing a, with all tangents situated in ε. Hence A is *uniplanar point* and the vicinity on F^3 forms a twodimensional continuum. According to our definitions a counts triple in ε. Evidently F^3 contains no further lines and as a counts single in every plane apart from ε, the *total number of lines on F^3 is* 3. The numbers of lines being the principal object of the present investigation, we exclude this case in what follows.

Above we excluded degenerations of the oval in planes through a ($\neq \varepsilon$) on ground of the assumption that no second line of F^3 intersects a. If we only assume that no second line passes through A, then it is possible for the oval to degenerate in two lines of which one coincides with a and the other does not pass through A. Then in a plane turning round a from the starting position ε the following changes take place: At first ovals have a for tangent at A, these ovals extend and in a plane π occurs the generation in a and a line not containing A. After that we find ovals again touching a at A (but now from the other side) and when their plane converges towards ε, these ovals once more contract towards A. It can easily be shown that no further degeneration takes place. A is another kind of *uniplanar point*, with twodimensional vicinity on F^3. In plane π line a counts double. In ε line a counts single and A can be considered as a point-oval. From the foregoing follows directly that F^3 contains no further lines. The second line obviously does not count double in any plane. It can also be easily shown that it does not count triple in more than one plane, but we leave undecided if this can happen in one plane. Hence the *total number of lines on F^3 is*

3 *or* 5 (and we leave undecided whether the case of 5 can occur The occurrence of points as here described is again excluded in what follows.

Comm. 4 p. 1247 line 14 we used some results of comm. 3. These were deduced for points through which passes only 1 line of F^3 and the application on points through which 2 lines pass, causes incompleteness. Besides the note at the bottom of p. 1248 is not quite correct. On these grounds comm. 4 requires alterations. One of these will appear to be the extending of the numbers 3, 7, 15, 27 before mentioned to 3, 5 [1]), 7, 9, 15, 27.

We shall now start on a minute consideration (§§ 3 and 4) of points through which pass 2 or more lines of F^3. This will permit us to deal separately with several singularities. In the last part (§ 5) we intend to point out further changes necessary in comm. 3 and 4.

§ 3. *On points through which pass 2 different lines of F^3, but not 3 different lines situated in one plane.*

Through A pass the lines BD and CE in α (fig. 1). We distinguish 2 cases:

1. Neither line counts double in α.
2. One of the lines counts double in α.

In the first case α contains a third line, not passing through A and along each of the 4 branches departing from A in α, the sectors of F^3 obviously meet from opposite sides of α. It is easily shown that each of these branches is connected with both surrounding ones, the sectors lying alternately above and below α.

In the second case let CE count double. Then along AC the sectors meet from one side of α and along AE from the other (via the point at infinity this is the same side). Along AB and AD the sectors meet everywhere from opposite sides of α. Here also it is easily shown that each of the 4 branches meeting at A in α, is connected with both surrounding ones, 2 successive sectors now lying above and the other 2 below α.

First case. Let AB be connected with AC and AD with AE above α by I and III respectively, and below α AC with AD and AE with AB by II and IV. To begin with we assume that *all plane branches departing from A on F^3 touch α at A.* Then in every plane, containing neither BD nor CE, A is ordinary point with tangent in α. We proceed to show that in a plane β ($\neq \alpha$)

[1]) Further investigation will probably show that the number 5 can be left out.

through BD (or CE) no further branches depart from A. Suppose a further branch leaves A in β, then β contains an oval having BD for tangent at A. Let this oval depart from A above α. Let AF and AH be the branches of this oval and let AF depart on I, then AH is situated on III, for the assumption that both lie on I at once gives a contradiction when we consider the section in a plane ($\neq \alpha$) through RS (fig. 1). Now let β turn round BD, in such a way that the top part moves towards AC, then in every position before α is reached, a branch departs from A on I and we keep ovals, touching BD. From this it follows that in all these planes β the lines through A not coinciding with BD, carry only 1 other point of F^3. Now this is impossible as in every plane not containing BD or CE, A is ordinary point with tangent in α and if this tangent be slightly turned in that plane round A, we obtain 3 different points of intersection with F^3.

A point A as here described we call *normal point of intersection of 2 lines on* F^3.

We shall now consider the alternative, that a plane branch departs from A not having α as tangent plane at A. Let us first assume that a third line of F^3 (not situated in α) passes through A. Let this line depart above α on I and below α on II. Let β denote the plane through this line and BD. If every vicinity of A in β contains points of $III + IV$ then the curve in β is composed of 3 lines through A, which case shall be dealt with later. Remains the possibility that $III + AE + IV$ is situated entirely on one side of β. Then however of the 4 branches departing from A in β, two opposite ones are directly connected and this is impossible, as well when one of the lines in β counts double, as when both count single.

Let a branch departing from A and not having α for tangent plane, be situated on I, then through the line RS of α (fig. 1) planes can be found in which at least 2 branches depart from A on I, but in each of these planes at least 1 branch departs on II and I on IV, hence A is ordinary double point and as RS is not tangent, branches depart on II and IV also, for which α is not tangent plane. The same now follows for III. *Hence if α is turned round a line through A in α ($\neq BD$ or CE) out of its original position, then at first A remains double point.*

Now let α turn round RS in such a way that the back part moves upwards. We proceed to show that there cannot be a last position with double point in A and branches departing on I. Suppose a plane β formed such a last position and let AN and AM be the

branches on I. Let PQ be a line through A in α inside $\angle BAC$ (fig. 1) and γ a plane through PQ, such that the branches AM and

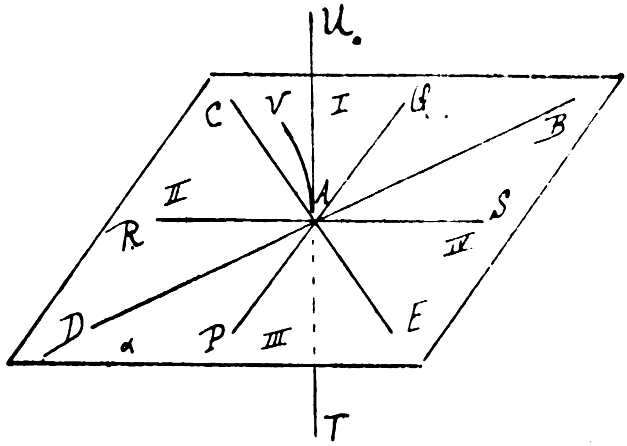


Fig. 1.

AN depart to different sides. In γ a branch departs from A on I in front of β. This branch has the line of intersection of β and γ for tangent, as otherwise β could not be the last plane with double point in A and branches on I. In γ also departs a branch on III, hence A is double point in γ with the line of intersection of γ and β as one of the tangents. The character of the curve in β, however, requires this line to carry a point of F^3 different from A, hence a contradiction is obtained [1]).

Not all planes through RS show a double point at A, for the set of planes with 2 branches on I and the set of those with 2 on III would be separated by a plane with 1 branch on I and III each and these would be lines. In every plane through RS however, 2 branches arrive at A from below α, hence a plane β through RS exists in which A is either ordinary point or cusp. Let us assume the first. Now let β turn round RS, then before α is reached, A has become double point. The assumption that no plane through RS shows a cusp in A leads to an impossibility, for when β turns, there cannot be a first plane with double point in A (as shown above) and that there cannot be a last position with an ordinary point in A is proved by the reasoning of comm. 1 p. 113 and 114.

Hence there exists a plane through RS with a cusp in A. Let β be this plane and TU the cuspidal tangent.

In every plane ($\neq \alpha$) through PQ (fig. 1), not containing TU, 2 branches arrive at A from below on II or IV and also 2 from

[1]) Somewhat analogous reasoning occurs in comm. 1. to which we do not always refer, in order to avoid confusion.

above, namely 1 on I and III each, hence A is always double point. For the plane through PQ and TU remains only the possibility that A is cusp with TU for tangent. From this follows again that A is double point in every plane through RS not containing TU. All this can at once be extended to the following:

In every plane containing neither BD nor CE, A is double point, except in the planes through TU, and in these A is cusp with TU for tangent.

We note that no line through A carries 2 other points of F^3.

There remain to be considered the sections in planes through BD or CE. If such a plane does not pass through TU, then it obviously contains a non-degenerated oval through A. For a plane through TU and one of the lines in α we shall show that the entire section is situated on the line in α. Let us consider the plane β through CE and TU. The restcurve (that is the curve minus CE) cannot contain a line different from CE, for TU has only A in common with F^3 and no third line passes through A. Neither can β contain an oval not passing through A, as no line through A carries 2 other points of F^3. If there is an oval *through* A, then TU is tangent at A. Let the branch AV of this oval (fig. 1) depart on I (above α). Branches leaving from A on I all depart behind the plane through RS and TU. This holds for every line RS inside $\angle BAE$, hence no branch departing from A on I has a semitangent at A situated outside trihedron $AUCB$. The branch AV cannot be isolated on the frontside (that is the side on which D lies) of the plane β (through CE and TU), hence on that side of β, AV is connected with AC. Let γ be a plane through BD, such that AC and AU lie on different sides, then in this plane a branch departs from A on the subsector of I joining AV to AC and the line of intersection of β and γ is tangent to this branch (as no semitangent is situated outside trihedron $AUCB$). This tangent would have only A in common with F^3, but in β it carries a second point of the oval, hence we arrive at a contradiction.

We thus find that the plane through CE and TU contains no points of F^3 not situated on CE and the corresponding thing holds for the plane through BD and TU. Hence all tangents at A are situated in the planes through TU and CE or BD respectively, and on the other hand it is easily shown that every line through A in one of these planes is tangent. Hence *A is biplanar point and in each of the tangent planes the entire section is situated on a single line.* [1])

[1]) A good drawing representing such a point can be found in table II joined to a note of KLEIN, Math. Ann. 6, p. 551.

The above results show that here F^3 contains no line not situated in α. If follows that none of the 3 lines in α counts double in any plane. If none of them ever counts triple either, then the *total number of lines on F^3 is* 3.

Let H and K represent the points of intersection of the third line in α and CE and BD respectively. For each of these points two possibilities exist, namely they can be normal point of intersection of 2 lines or biplanar point of the same type as A. If both are normal points, then the above described character of such points shows that none of the 3 lines in α counts triple in any plane. If on the other hand H is biplanar and K normal, then the line $AH (\equiv CE)$ counts triple in the plane through TU, in which plane then also lies the cuspidal tangent at the biplanar point H. In this case the *total number of lines on F^3 would be* 5 [1]). If lastly, K is biplanar also, then each of the 3 lines in α counts triple in a particular plane and the *total number of lines on F^3 becomes* 9.

Second case. CE counts double in α. Suppose above α AB is connected with AC by I and AC with AD by II. Below α AD with AE and AE with AB by III and IV respectively (fig. 2). To begin with we assume that every branch departing from A has α for tangent plane. Then A is point of inflexion in every plane not containing BD or CE. We consider a plane ($\neq \alpha$) through CE. If in this plane a further branch leaves A, then in this plane an oval touches CE at A. Let this oval depart from A above α and let the branch leaving in the direction AE at first lie on I. Now in a plane ($\neq \alpha$) through the line AL of α (fig. 2), the branches departing on II and IV form a point of inflexion at A, but at least 2 more branches depart on I, hence a contradiction has been obtained.

Let us consider a plane β through BD. If in β another branch leaves A, then an oval touches BD at A. Let this oval depart from A above α. If both branches of this oval at first lie on I, then we get a contradiction as above. Hence one branch AF would have to start on I and the other AH on II. In β now depart from A the branches AB and AD on the line and AF and AH on the oval having this line for tangent. Former results [2]) show that each of these 4 branches is connected with the 2 surrounding ones, alternately on opposite sides of β. Now AB and AD are connected on that side of β where E lies, hence AF and AH also, but then inside every vicinity of A, I and II would be connected by a sector

[1]) We leave undecided whether this case can occur.

[2]) Comm. 3 p. 738 and 739.

not containing AC or AE and this is impossible. This completes the demonstration that in no plane $(\neq \alpha)$ through BD or CE a further branch starts from A.

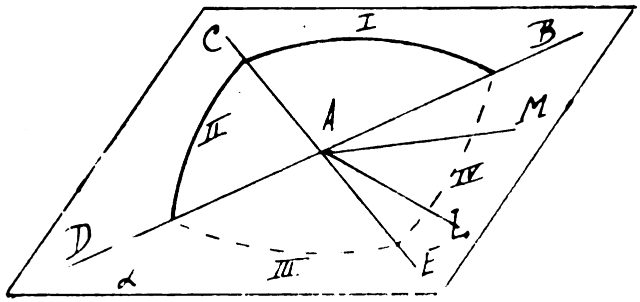


Fig. 2.

Once more we consider a plane $(\neq \alpha)$ through CE. If in this plane a further branch starts from a point $(\neq A)$ of CE, then an oval touches CE at that point, as every point $(\neq A)$ of CE is isolated on one side of α. It can then be fairly easily shown that this point is uniplanar point of the second kind described in § 2. Such points however, we excluded, hence we conclude that in no plane $(\neq \alpha)$ through CE a further branch starts from any point of CE.

Let the planes $\beta_1, \beta_2, \beta_3 \ldots$ all containing BD, converge towards α in such a way that the top parts move towards AC (fig. 2). These planes end up by containing ovals, converging towards CE, intersecting AB and AD, and facing at these points of intersection A with their convex sides. For if these ovals did not so face A, then they would have points in common with every plane through CE, and in such a plane a further branch would depart from some point of CE.

A point A as here described we shall call *normal point of intersection of a double and a single line.*

We now have to consider the case that a branch leaves A, for instance on I, not having α for tangent plane. We distinguish 2 possibilities:

1. Through A passes no line of F^3, not situated in α.

2. Through A passes a line of F^3, not situated in α.

1. Planes through line AL of α (fig. 2) can be found, in which 2 branches depart on I, and 1 branch on II and IV each, and this means that A is double point with AL as one of the tangents. Hence a branch starts from A on I in the direction AL, and it follows that in every plane $(\neq \alpha)$ through the line AM of α (fig. 2) 2 branches leave from A on I. In each of these planes A is double point and we conclude that no line $(\neq CE$ or $BD)$

through A carries 2 other points of F^3. Evidently every line inside $\angle BAE$ can be taken as AM.

Every point ($\neq A$) of CE is isolated on one side of α, hence BD is the only line of F^3 intersecting CE.

In an arbitrary plane ($\neq \alpha$) through AM, A is double point with a tangent in α. Let β be the plane through the other tangent (AH) and CE. The restcurve in β contains no line different from CE, as BD is the only line intersecting CE. Neither can the restcurve in β contain an oval not passing through A, as no line through A carries 2 other points of F^3. An oval in β passing through A is also impossible as the above mentioned tangent AH would also be tangent to this oval, hence this oval would cross CE at a point different from A and this would be irreconcilable with the way in which the sectors meet along CE. Hence β contains no points of F^3, not situated on CE. It follows directly that β is tangent plane at A, hence A is a *biplanar point*.

As β contains no points of F^3 not situated on CE, BD is not intersected by any line except CE. Above we found that CE has no line of intersection except BD, hence CE and BD are the only lines of F^3. It follows at once that neither counts double in any plane ($\neq \alpha$). It can also be easily shown that neither counts triple in any plane. Hence the *total number of lines on F^3 is* 3. In what follows we exclude points as here described, and we leave undecided whether they can occur.

2. Through A passes a line b of F^3, not situated in α. Let this line depart above α on I, then it leaves below α on IV, for if it started on III, we could at once find an infinite number of planes in which 6 branches meet at A (2 on I and III each and 1 on II and IV each) and this would mean an infinite number of degenerations. Besides it is evident that all branches leaving from A on II and III touch α at that point.

We consider the plane β through b and CE. A reasoning analogous to that of p. 926 shows that in β a third line passes through A, again leaving on I and IV. Hence in β 3 different lines pass through A, which case shall be treated in § 4.

§ 4. *On points through which pass* 3 *different lines of F^3, situated in one plane.*

Let A be a point throngh which 3 lines pass, situated in plane α. Six branches start from A in α. Along none of these the sectors meet from the same side of α, for in that case an infinite number

of lines in α would belong to F^3. Two branches, separated by a single one, cannot be directly connected, for if the joining sector were situated above α, then the sector departing above α from the branch in between, could not be fitted in on a twodimensional continuum. All this leaves only 2 possibilities for connecting the 6 branches:

1. Every branch is connected with the 2 surrounding ones, alternately above and below α.

2. A representative case of this possibility is that above α. AC is connected with AF, AD with AE and AH with AB, and below α. AD with AH, AE with AF and AB with AC (fig. 4).

1. Suppose above α AB is connected with AC by I, AD with AE by III, and AF with AH by V, below α AC with AD by II, AE with AF by IV and AH with AB by VI (fig. 3). To

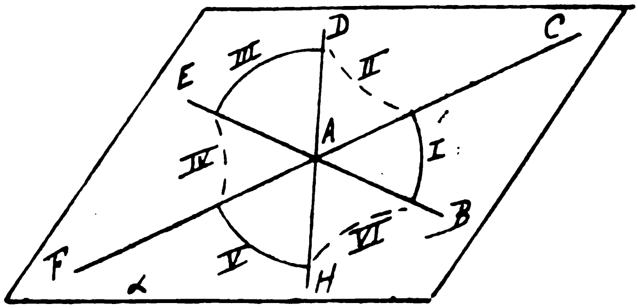


Fig. 3.

begin with we assume that a branch departs from A, for instance on I, not having α for tangent plane. Then a plane through A inside $\angle DAC$ can be found, in which 2 branches start from A on I. Besides one branch on II and V each. The branch leaving on II does not have α for tangent plane. Proceeding in this way we find that a branch leaves on III not touching α and lastly we show that the same holds for IV. Then however an infinite number of planes can be found in which 2 branches start on I and IV each and one on II and V each. This means an infinite number of degenerations, which case we exclude.

Hence all branches starting from A have α for tangent plane. This means that A is point of inflexion in every plane which does not contain one of the lines in α, with all tangents in α. We proceed to show that in no plane through one of the lines in α further branches depart from A. Suppose in a plane β through BE a further branch leaves A, for instance above α. Then in β an oval (with BE for tangent) leaves on I *and* III. Let β turn round BE,

in such a way that the top part moves towards C (fig. 3), then in every position β contains an oval, with BE for tangent at A and departing on I and III. Every line through A in one of these planes has only one further point in common with F^3. This contradicts the above obtained result that in every plane not containing one of the lines in α, A is point of inflexion with tangent in α, for if the tangent be slightly turned round A, we obtain 0 or 2 points of intersection with F^3, different from A.

A point as here described we call *normal point of intersection of* 3 *lines situated in* 1 *plane.*

2. Above α AC is connected with AF (I), AD with AE (III) and AH with AB (V), below α AD with AH (II), AE with AF (IV) and AB with AC (VI) (fig. 4). On none of the sectors III, IV, V, and VI can be situated a branch not having α as tangent plane, for let such a branch start on III, then at once an infinite number of planes can be found in which 2 branches depart on III and one on I, II, IV and VI, each. This makes 6 in all, in other words it would mean degeneration. Hence in every plane

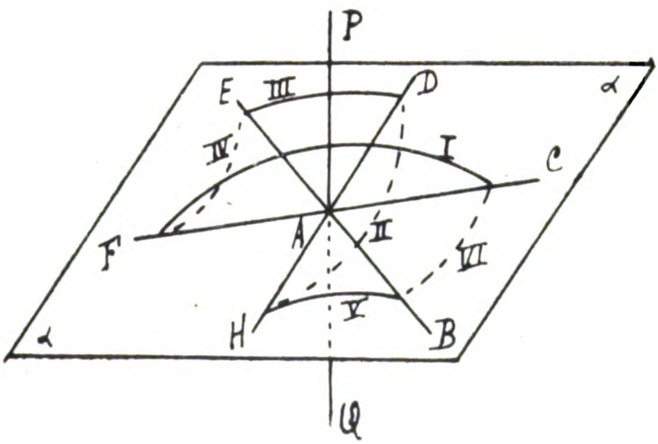


Fig. 4.

through A inside one of the angles FAE or EAD the line of intersection with α is one of the two tangents at the double point A.

We consider an arbitrary plane β through BE. In β a branch leaves A on I and another on II, both situated on the restoval. Let β turning round BE converge towards α, in such a way that the top part moves towards F (fig. 4), then these branches converge towards AF and AD respectively. If β turns the other way the branches converge towards AC and AH respectively. It follows that a position of β exists in which the oval degenerates, hence a line

PQ passes through A, not situated in α, leaving on I (AP) and on II (AQ).

We note that no line through A carries 2 other points of F^3 without belonging entirely to that surface.

Let γ denote an arbitrary plane through PQ *inside* $\angle FAH$. The part of the section in γ which is not situated on PQ, can consist of:

1. Two different lines through A.
2. Nothing.
3. One line through A (possibly to be counted double).
4. An oval touching α at A.

No two planes γ can contain a section falling sub 1, 2 or 3, for then an infinite number of planes through A inside $\angle EAD$ could be found, each containing 2 lines not situated in α and having only A in common with F^3 (namely the lines of intersection with the 2 planes γ) and this is impossible considering that in all those planes A is double point with a tangent in α.

Hence it suffices to consider 4 cases. In the first 3 all planes γ except one, contain a non-degenerated oval and the excepted plane γ shows a section falling sub 1, 2 and 3 respectively. In the fourth case *every* plane γ contains a non-degenerated oval.

First case. In a plane γ through PQ inside $\angle FAH$ (fig. 4) the curve consists of 3 different lines through A. In plane β through PQ and BE no third line passes through A, for then sector I would cross this plane twice and this cannot be as sector I comes from the one side of β (AF) and ends up at the other side (AC). Neither PQ nor BE can count double in β, hence the curve in β consists of 3 lines forming a triangle.

Almost the same reasoning holds for the plane through PQ and DH[1]) and substituting I for II and vice versa, also for the plane through PQ and CF. Besides for PQ can be substituted either of the remaining lines in γ. Hence in all 15 lines on F^3 have been found, namely 6 through A and 9 others.

From the foregoing follows at once that none of the 3 lines in α can be intersected by other lines as those mentioned. Also it can be easily shown that none of the 15 lines counts double or triple in any plane. Hence the *total number is* 15.

Every line through A in α or γ is tangent at that point and on the other hand every tangent is situated in α or γ. Hence A *is*

---

[1]) DH cannot count double in the plane through PQ, for then sector II could not cross γ three times, without passing more than once through the plane o PQ and DH.

biplanar point and each tangent plane has 3 *different lines in common with* F^4 [1]). We found A to be double point in every plane not containing the line of intersection a of α and γ. In every plane ($\neq \alpha$ or γ) through a, A is cusp, for in such a plane a is the only line having no point but A in common with F^4.

Points as here described are excluded in what follows.

Second case. The plane γ has no points in common with F^4, not situated on PQ. This case can be dealt with in almost entirely the same way as the *first case*[2]). Again A is *biplanar point* with α and γ for tangent planes. We find 7 lines, namely 3 in α, PQ and 3 more in the planes through PQ and the first 3. Of all these, except PQ, it can be easily shown that in no plane they count double or triple. For PQ remain the possibilities that this line counts single or triple in γ. Accordingly the *total number of lines on* F^4 *is* 7 *or* 9. In the last case PQ carries a second biplanar point of the same type as A. We leave undecided whether this last case can occur.

Biplanar points as here described are excluded in what follows. They are a cross-type of those immediately preceding and those of p. 928.[3])

Third case. In γ the curve consists of 2 lines through A, one of which counts double. The results of p. 931 show that through the double one passes a plane containing 3 different lines through A. Part of the reasoning given for the *first case* shows that this plane cannot contain one of the lines in α. Neither can it be situated inside $\angle DAE$ or $\angle EAF$, for such planes always contain branches touching α. Hence the above mentioned plane would lie inside $\angle FAH$. Then however the double line is the intersection of 2 planes inside $\angle FAH$, neither of which contains a non-degenerated oval, and this is impossible according to the results of p. 934.

Fourth case. Every plane γ contains a non-degenerated oval with α for tangent plane at A. Obviously these ovals depart either all on I or all on II. Let us assume the latter, then these ovals contract

[1]) A good drawing is to be found in table III of the above mentioned paper of KLEIN.

[2]) That HD does not count double in the plane through PQ appears when we bear in mind that every plane ($\neq \gamma$) through PQ inside $\angle FAH$ contains an oval having α for tangent plane at A. In the planes on the side of γ these ovals depart on I, and in those on the other side of γ they start on II, for otherwise F^3 could not be a twodimensional continuum.

[3]) For cubic surfaces, compare KLEIN, loc. cit, p. 557.

towards A when γ converges towards the plane β through PQ and CF, for the alternative demands the existence of points on PQ. which can be points of inflexion with the tangent in β, and this is impossible as the line CF belongs to F^4. Hence a finite vicinity of A exists, inside which the converging planes contain no points of I not situated on PQ. This means that along AF the sectors I and IV meet from the same side of β, in other words CF counts double in β. From this it follows that CF is not intersected by any line, not passing through A. In the same way as above we find the *total number of lines on* F^4 to be 7 (CF counted double). A is *biplanar point,* such that the line of intersection of the tangent planes lies on the surface, and one of the tangent planes at A is tangent plane all along this line. This case is excluded in what follows and we leave undecided if it can occur.

§ 5 *On surfaces* F^4 *on which the singularities described above do not occur.*

Except points lying on one line or no line at all, the surface can contain what we called *normal points of intersection* of 2 or 3 lines, and of a single and a double line.

In comm. 3 p. 736—744 we proved (theorem 1): *that every point on a line of* F^4 *has a tangent plane, provided this line is not intersected by any other.* Above (§ 2) we already did some supplementing and brought forward two kinds of uniplanar points. Apart from this, the demonstration can, with some small self evident alterations [1]) be used to establish the following theorem: *If a line of* F^4 *counts double in no plane, then every point of this line has a tangent plane provided no second line passes through that point.*

Theorem 2 of p. 744 (comm. 3) holds, with the same demonstration, when *normal points of intersection* are admitted.

We pass on to comm. 4 (p. 1246—1253). As the above described singularities are excluded, theorem 1 (p. 1246) holds and can even be extended to the following: *The lines of* F^4 *passing through one point, lie in one plane.*

Apart from the occurrence of a plane α in which the curve is composed of a double and a single line, the rest of comm. 4 can

[1]) Example of a necessary alteration: p. 739, l. 6 from bottom: If the restcurve consisted of 2 lines, these lines would have a normal point of intersection at B. Hence in β the curve would have b for tangent at B, and this cannot be as A is cusp in β. If the restcurve in α is a double line through B, then a slight turning of b round A in β would replace B by 2 points of intersection and this again is impossible.

remain the same (see however the errata below). In § 2 of comm. 4 occasionally the question arises if none of the lines can count double or triple in any plane. Now the case of a double line will be dealt with below and for a surface F^3 without the singularities mentioned, but with a line which counts triple in a certain plane, the total number of lines is always 3.

We conclude by considering the case that the curve in a plane α consists of a single line a and a double line b. The normal point of intersection A of a and b has α as tangent plane. If α turns round a, then (according to the results of p. 933) a non-degenerated oval appears out of b. The points of intersection of this oval and a start from A to right and left, and at these points the oval faces A with its convex sides (anyway at first). Obviously b cannot be intersected by a line of F^3 not situated in α. Hence all further lines intersect a. If such further lines exist, their number is at least 4, which brings the total up to 7. If there are still more, the oval degenerates in at least 4 planes through a different from α and then the reasoniug on p. 1251 (comm. 4) shows that F^3 contains lines which do not intersect a: a contradiction. Hence for a surface F^3 with a plane section consisting of a double and a single line, *the total number of lines is* 3 *or* 7 (for it is easily shown that no further multiplicity can increase these numbers).

ERRATA.

In comm. 1, p. 102 l. 22 *for*: "straight line and isolated point"; *read*: "straight line and point-oval".

In comm. 2, p. 309 l. 24—34: this part can be left out.

p. 311 l. 12—14: the letter C to be replaced by D.

In comm. 3, p. 740 l. 25 *for*: "Let c be a line through A in α, not being tangent to the oval and not coinciding with a or b"; *read*: "Let c be a line through A in α, not coinciding with a or b (b is tangent to the oval)".

p. 742 l. 7 from bottom, *for*: "It follows that A must be cusp in *every* plane except α"; *read*: "It follows that A must be cusp in *every* plane not containing the line a".

In comm. 4, p. 1251 l. 3 from bottom, *for*: "Now none of these last 4 points can coincide with one of the first, because in that case a line of F^3 would pass through that point and through the point of intersection of b_1 and b'_1"; *read*: "Now none of these last 4 lines of inter-

61

section of c can coincide with one of the first 4, for in that case a line of F^3 would pass through the point of intersection of b_1 and b'_1, without being situated in the plane through these lines".

p. 1253 l. 2 from bottom, *for*: "*cannot occur*"; *read*: "*can only occur*".

Physics. — "*The quadrupole moments of the oxygen and nitrogen molecules*". By Prof. W. H. KEESOM. (Communication N°. 6*a* from the Laboratory of Physics and Physical Chemistry of the Veterinary College at Utrecht). (Communicated by Prof. KAMERLINGH ONNES).

(Communicated at the meeting of November 27, 1920).

§ 1. *Introduction.* In Suppl. N°. 39*a* to the communications from the physical laboratory of Leiden [1]) it has been proved, that, as far as the term with the second virial coefficient, the equation of state of hydrogen can be accounted for in the temperature interval between + 100° C. and — 100° C. (according to Suppl. Nr. 39*c* down to still lower temperatures) by assuming the molecules to act on each other as electric quadrupoles with constant quadrupole moment [2]). Thereto the repulsions which predominate at small distances had to be replaced by the forces that would act when the molecules collided as solid spheres of definite radius. At the same time the quadrupole moment of the hydrogen molecules was determined. This proved a configuration of the two nuclei and the two electrons constituting a H_2-molecule to be very well possible in such a way that it has the demanded quadrupole moment. By these considerations it was evident that the molecular attractions can be explained for homopolar molecules too by the electric forces exerted by the nuclei and the electrons constituting the molecules. To show this was for the moment the principal purpose of those considerations.

A comparison of the quadrupole moment demanded by the equation of state with that of the BOHR-DEBIJE model for the hydrogen molecule was not made because of the many difficulties that arise against this model, especially with respect to the magnetic properties. [3]) Meanwhile BURGERS [4]) has calculated the quadrupole moment of the hydrogen

---

[1]) These Proceedings, vol. 18, p. 636.

[2]) The following Communication will treat the influence on the second virial coefficient of the mobility of the electrons within the molecule, which finds its expression in the dielectric constant.

[3]) Comp. Leiden Comm. N°. 39*a*, p. 15, note 1. At present this difficulty has perhaps a somewhat smaller weight than was thought then. (Added in the translation).

[4]) J. M. BURGERS, Diss. Leiden 1918, p. 186.

molecule according to BOHR-DEBIJE and he found a remarkable agreement with the value derived from the second virial coefficient (Comm. Leiden, Suppl. N°. 39*a*):

from the equation of state [1]) $2{,}03 \times 10^{-26}$ (e. st. e. cm²)
according to BOHR-DEBIJE $2{,}05 \times 10^{-26}$,,

However favourable this result may be for the BOHR-DEBIJE model, still it seems probable that the always increasing objections against this model [2]) will compel us to seek for another. The calculation of the quadrupole moment from the equation of state retains however its value. Abstracted from the incertainty caused by the simplifying assumption on the repulsing forces, it gives namely important data which will have to be taken into consideration in the construction of the definite model of a molecule. In this sense it seemed interesting to calculate the quadrupole moments for other gases too. In § 2 this has been done for oxygen and for nitrogen.

§ 2. In the temperature interval in which the second virial coefficient has been calculated for quadrupole molecules, we possess the data comprised in table I for the second virial coefficients of oxygen and nitrogen [3]).

The index Θ indicates here that the volume v in the equation of state

$$pv = RT \left\{ 1 + \frac{B}{v} + \ldots \right\}$$

is expressed in the theoretical normal volume as a unit [4]).

These data do not admit a control of the change of B for these gases with the temperature compared with that of spherical quadrupole molecules. This would require more values of B especially for higher temperatures. Let us assume for the present that this is the case for the considered temperature interval [5]), then they

[1]) When we attend to the circumstance that the molecules are polarized in their mutual electric fields, this value will still undergo a small alteration (comp. Comm. N°. 6*b* especially § 4).

[2]) Comp Miss H. J. VAN LEEUWEN, these Proc. Vol. XVIII, N°. 7, p. 1071. J. M. BURGERS, these Proc. Vol. XIX, 2. p. 480. A. SOMMERFELD, Atombau und Spektrallinien. Braunschweig 1919, p. 288 and 533. Frl. G. LASKI, Physik. Z S. 20 (1919), p. 550. W. LENZ, Verh. D. physik. Ges. 21 (1919), p. 632.

[3]) These numbers are taken from the calculations by Mr. M. DANIELS, phil. nat. docts, where for the observations of AMAGAT were taken the virial coefficients given by KAMERLINGH ONNES in Comm. Leiden N°. 71.

[4]) Comp. H. KAMERLINGH ONNES und W. H. KEESOM, Die Zustandsgleichung. Math. Enz. V. 10, Leiden Suppl. N°. 23, Einheiten *b*.

[5]) For hydrogen deviations occur in the corresponding interval already.

TABLE I.

Oxygen			Nitrogen		
T	$B_{\Theta} \times 10^3$		T	$B_{\Theta} \times 10^3$	
472.6	+ 0.273	AMAGAT	472.5	+ 0.690	AMAGAT
372.6	— 0.088	"	372.45	+ 0.324	"
293.1	— 0.848	ONNES & HYNDMAN	290.6	— 0.298	LEDUC 1910
288.7	— 0.694	"	289.1	— 0.234	AMAGAT
288.7	— 0.739	AMAGAT	288.0	— 0.316	RAYLEIGH
288.1	— 0.645	LEDUC 1910	273.1	0.372	AMAGAT
284.3	— 0.791	RAYLEIGH			
273.1	— 0.812	ONNES & HYNDMAN			
273.1	— 0.928	AMAGAT			

suffice to determine the quadrupole moment, besides the diameter of the molecules when regarded as spheres.

For this purpose we take the values of B/B_{∞} for different values of $T/T_{inv(\rho=0)}$ for spherical quadrupole molecules from Comm. Leiden, Suppl. N°. 39*a* and *c*.

Applying the method of the logarithmic diagrams (comp. Comm. Leiden, Suppl. N°. 25) we found successively:

For *oxygen*:

for the inversion temperature of the Joule-Kelvin-effect for small densities:

$$T_{inv\,(\rho=0)} = 723 \text{ (450° C)},$$

for the potential energy of the molecules in contact (with the quadrupole axes mutually perpendicular and perpendicular to the line connecting the centres):

$$v = 5{,}71 \times 10^{-14},$$

for the diameter of the molecule:

$$\sigma = 2{,}65 \times 10^{-8},$$

for the quadrupole moment:

$$\mu_2 = 3{,}55 \times 10^{-26} \quad [\text{e. st. e.} \times \text{cm}^2].$$

For *nitrogen:*

$$T_{inv\,(\rho=0)} = 604, \quad (331° \text{ C.})$$
$$v = 4{,}77 \times 10^{-14},$$
$$\sigma = 2{,}98 \times 10^{-8},$$
$$\mu_2 = 3{,}86 \times 10^{-26} \quad [\text{e. st. e.} \times \text{cm}^2].$$

§ 3. In order to form a better judgment of the value found for the quadrupole moment of oxygen we must still investigate whether the magnetic attraction between the paramagnetic oxygen molecules contributes considerably to B. In this case the quadrupole moment ought to be smaller than the value found above with neglect of this attraction. Calculation proves however that this is not the case.

According to WEISS and PICCARD [1]) oxygen should possess 14 magnetons per molecule. This involves a magnetic moment of the O_2-molecule

$$\mu = 2{,}6 \times 10^{-20}.$$

In Comm. Leiden Suppl. n°. 25 § 4 has been shown that $\mu = 9{,}47 \times 10^{-19}$ should be required in order to explain the molecular attraction. The real magnetic moment has only $\frac{1}{36}$ of this value. Taking into consideration, that the statistically remaining molecular attraction is proportional to μ^4, we see that the contribution of the magnetic moment to the molecular attraction does not come into consideration here.

§ 4. We must expressly point out that the calculations of this communication (as well as those for H_2, Leiden Suppl. n°. 39) are based on the supposition that the molecules collide as solid spheres with constant diameter. If this were not the case and if the behaviour at a collision should correspond to a value of σ depending on the temperature, this would become manifest in B by terms also depending on T. The dependency of B on T would then no longer be due to molecular attraction exclusively as in this Comm. For the same reason the values of the quadrupole moments would possibly have to be altered considerably.

1) P. WEISS and A. PICCARD. C. R. 155 (1912), p. 1234.

Physics. — "*The cohesion forces in the theory of* VAN DER WAALS". By Prof. W. H. KEESOM. (Communication N°. 6*b* from the Laboratory of Physics and Physical Chemistry of the Veterinary College at Utrecht). Communicated by Prof. KAMERLINGH ONNES).

(Communicated at the meeting of November 27, 1920).

§ 1. *Introduction.* DEBIJE [1]) has recently shown in an important paper on the cohesion forces in the theory of VAN DER WAALS that these may be explained in this way that one molecule in the field of neighbouring molecules obtains a bipole moment, and that because of this bipole moment it is attracted by the inducing molecule. DEBIJE considered especially those gases the molecules of which possess no spontaneous bipole moment. In the calculation he assumed that the field of the molecules could be treated in first approximation as that of a quadrupole. Evidently neighbouring molecules will influence their mutual direction in such a way that the cases of attraction are more frequent than those of repulsion. In a preliminary orientating calculation the mean mutual attraction of the molecules due to their own quadrupole moment was neglected. In fact, this will be allowed, as was already remarked by DEBIJE, for sufficiently high temperatures. Then this mean attraction vanishes namely as the heat movement hinders the directing influences of the molecules mutually.

On the other hand we have shown (Comm. Leiden Suppl. N°. 39*a*) [2]) that the molecular attraction in hydrogen, at least as far as to the second virial coefficient, may be explained by the circumstance that those molecules possess a quadrupole moment, while a contribution to the attraction due to the mobility of the electrons in the molecules was neglected. In fact, DEBIJE remarks rightly that in the calculation of the molecular attraction we shall have to attend to the attraction of the molecules mutually because of their quadrupole moments as well as to that especially treated by DEBIJE and due to the polarisability of the molecules in an electric field.

In this paper we shall discuss principally the influence of the

[1]) P. DEBIJE, Physik. Z.S. 21 (1920), p. 178.
[2]) These Proceedings, vol. 18, p. 636, See also W. H. KEESOM and Miss C. VAN LEEUWEN, these Proceedings, Vol. XVIII, N°., p. 1568.

molecular attraction on the equation of state, confining ourselves to the second virial coefficient.

Now we may ask which of the two mentioned contributions to the attraction will have the greatest influence on the second virial coefficient at the temperatures for which the measurements on the equation of state were made. It will appear that for these temperatures the influence of the molecular attraction in B is principally due to the spontaneous quadrupole moments of the molecules.

§ 2. *Preliminary orientation.* For shortness sake we shall denote by the name "quadrupole attraction" the contribution to the attraction due to the spontaneous quadrupole moments and by the name "induced attraction" the part due to the forces exerted by the quadrupoles on the bipoles that are induced in the molecules. In the same way we shall speak of "quadrupole terms" for the terms in the second virial coefficient due to the quadrupole attraction and of "induced terms" for those caused by the induced attraction.

Between them this important difference exists, that at high temperatures the quadrupole terms become proportional to T^{-2}, while the induced terms become proportional to T^{-1}.

This comes to the same as saying that the VAN DER WAALS attraction force in the case of quadrupole attraction becomes proportional to T^{-1}, whereas in the case of induced attraction they become constant.

Now it has been shown already (Leiden Suppl. N°. 39*a*. See Fig. 2 there and comp. also Leiden Suppl. N°. 39*c* § 3) that for hydrogen the second virial coefficient behaves more in agreement with the hypothesis of quadrupole attraction than with that $a_w = \text{const.}$, as would be demanded for high temperatures because of the induced attraction. Therefore we may evidently expect that at least for hydrogen, the quadrupole terms prevail.

We may compare the values derived by DEBIJE for the quadrupole moment under the assumption that for high temperatures the induced attraction has only to be considered, with the quadrupole moment found in Leiden Suppl. N°. 39*a* for hydrogen. This comparison is in agreement with the above conclusion. For hydrogen DEBIJE finds (table I l.c.) $3{,}20 \times 10^{-26}$, while in the Leiden Suppl. N°. 39*a* a quadrupole moment of $2{,}03 \times 10^{-26}$ has been shown to give already a quadrupole attraction sufficient to explain the experimentally found equation of state. The first of the quadrupole terms being proportional to the fourth power of the quadrupole moment, the quadrupole moments of DEBIJE would therefore give a quadrupole attraction that would be far too great.

The same conclusion is reached for oxygen and nitrogen, the quadrupole moments of which have been calculated in the preceding paper (Comm. n°. 6*a*) with neglect of the induced attraction:

O_2: DEBIJE: $11{,}2 \times 10^{-26}$, Comm. N°. 6*a*: $3{,}55 \times 10^{-26}$
N_2: $13{,}3 \times$,, $3{,}86$,,

The conclusion that the quadrupole attraction for these gases exceeds the induced attraction is confirmed when in the calculation of the second virial coefficient the induced attraction is also attended to. This has been done in § 3.

§ 3. *The second virial coefficient for spherical polarisable quadrupole molecules.* We suppose the state of polarisation of the molecules (displacements of the electrons from their positions or paths of equilibrium) to be at each moment in correspondence with the field that surrounds the molecule at that moment. Further the molecule to be isotropically polarisable, so that the induced bipole moment has the direction of the electric field E and is equal to

$$_i\mu_1 = aE. \qquad (1)$$

The index 1 indicates here that we have to do with a bipole moment, while both here and further on quantities with the index i are due to the electrostatic induction.

The energy of the induced bipole in the field E (compare DEBIJE l.c.) is:

$$-\frac{1}{2} a E^2. \qquad (2)$$

When the induced bipole is placed at a point P of the field of a zonal quadrupole A with quadrupole moment μ_2 (Fig. 1), in such a way that PA makes the angle θ with the axis of the quadrupole, we find for the energy of the induced bipole

$$-\frac{9}{2} \frac{a\mu_2{}^2}{r^8} \{\sin^4 \theta + 4 \cos^4 \theta\}. \qquad (3)$$

The calculation of the second virial coefficient may be analogous to that of Leiden Suppl. N°. 39*a* § 2. A pair of molecules that may

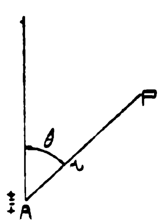


Fig. 1.

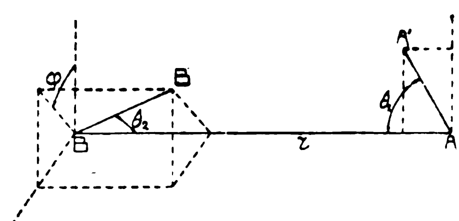


Fig. 2.

be considered as lying in their mutual spheres of action will be characterized again by the distance r of their centres, the angles θ_1 and θ_2 of the quadrupole axes with the line connecting the centres and by the angle φ (see Fig. 2). The energy of this pair will be obtained by adding to the quadrupole term given in the cited paper:

$$ {}_q u_{b1} = \frac{3}{4}\frac{\mu_2^2}{r^5}\{1 - 5\cos^2\theta_1 - 5\cos^2\theta_2 - 15\cos^2\theta_1\cos^2\theta_2 + $$
$$ + 2(4\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos\varphi)^2\} \quad . \quad (4) $$

the induced term:

$$ {}_i u_{b1} = -\frac{9}{8}\frac{a\mu_2^2}{r^8}\{\sin^4\theta_1 + \sin^4\theta_2 + 4\cos^4\theta_1 + 4\cos^4\theta_2\} \quad . \quad (5) $$

Still a term might be added, due to the forces exerted by the two induced bipoles on each other. This term would contain a^2. For the moment we shall however omit terms with a^2.

From Leiden Suppl. N°. 39a we take the notations

$$ v = \frac{3}{4}\frac{\mu_2^2}{\sigma^5}, \quad \sigma = \text{diameter of the molecule}, \quad . \quad . \quad . \quad (6) $$

where for shortness sake v will be written for ${}_q v$, this being the potential energy of the pair of molecules when in contact with the mentioned (l.c.) directions of the quadrupole axes for the case that only the quadrupole attraction is taken into consideration.

Further

$$ {}_q u_{b1} = v\frac{\sigma^5}{r^5}\Psi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7) $$

where

$$ \Psi = A + B\cos\varphi + C\cos 2\varphi \quad . \quad . \quad . \quad . \quad . \quad . \quad (8) $$

when

$$ \left.\begin{aligned} A &= 2(1-3\cos^2\theta_1)(1-3\cos^2\theta_2) \\ B &= 16\sin\theta_1\cos\theta_1\sin\theta_2\cos\theta_2 \\ C &= \sin^2\theta_1\sin^2\theta_2. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (9) $$

We now introduce

$$ X = \sin^4\theta_1 + \sin^4\theta_2 + 4\cos^4\theta_1 + 4\cos^4\theta_2 \quad . \quad . \quad . \quad (10) $$

Then we have

$$ {}_i u_{b1} = -\frac{3}{2}v\frac{a\sigma^5}{r^8}X \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11) $$

The second virial coefficient becomes

$$ B = \frac{1}{2}n\left(\frac{4}{3}\pi\sigma^3 - P'\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12) $$

with

$$P' = \int\limits_{\sigma}^{\infty}\int\limits_{0}^{\pi}\int\limits_{0}^{\pi}\int\limits_{0}^{2\pi}\left(e^{-h\,{}_q u_{b_1} - h\,{}_i u_{b_1}} - 1\right) r^2 \sin\theta_1 \sin\theta_2\, dr\, d\theta_1\, d\theta_2\, d\varphi \quad . \quad (13)$$

Developing into a series of ascending powers of h, with neglect of terms with α^2 etc., we find P' to be split up into:

$$P' = {}_qP' + {}_iP', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

where ${}_qP'$ may be taken from Leiden Suppl. N°. 39*a*. After integration with respect to r we find:

$$P' = \frac{3}{20}\, ahv \int\limits_0^{\pi}\int\limits_0^{\pi}\int\limits_0^{2\pi} \mathrm{X} \left\{ 1 - \frac{1}{2!}\, hv\,\Psi + \frac{1}{3!}(hv)^2\,\Psi^2 .. \right\} \sin\theta_1 \,.\, \sin\theta_2 d\theta_1 d\theta_2 d\varphi. (15)$$

Performing this integration we obtain

$${}_iP' = \frac{16}{5}\,\pi\sigma^3 \,.\, \frac{\alpha}{\sigma^3} \,.\, hv \left\{ 1 + \frac{16}{15}(hv)^2 \;..\right\} \quad . \quad . \quad . \quad . \quad (16)$$

and then [1])

$${}_iB = -\frac{1}{2}\, n \,.\, \frac{4}{3}\,\pi\sigma^3 \,.\, \frac{12}{5}\,\frac{\alpha}{\sigma^3} \,.\, hv \left\{ 1 + \frac{16}{15}(hv)^2 \ldots \right\} \quad . \quad . \quad (17)$$

These terms added to ${}_qB$ of the Leiden Suppl. N°. 39*a* finally give:

$$B = \frac{1}{2}\, n \,.\, \frac{4}{3}\,\pi\sigma^3 \left\{ 1 - 1{,}0667\,(hv)^2 + 0{,}1741\,(hv)^4 \ldots - \right.$$

$$\left. - 2{,}4\,\frac{\alpha}{\sigma^3}\, hv\,[1 + 1{,}067\,(hv)^2 \ldots] \right\} \quad (18)$$

§ 4. *Conclusions.* In the first place we may remark, that for a strong validity of the law of corresponding states the same value of $\frac{\alpha}{\sigma^3}$ would be required for different gases.

With Debije we derive the value of α from the molecular refraction (P_0) for $\lambda = \infty$, while the values of σ are taken from the Leiden Suppl. N°. 39*a* for H_2, from the preceding Communication (N°. 6*a*) for O_2 and for N_2. In this way we obtain:

	P_0	σ	$\frac{\alpha}{\sigma^3}$
hydrogen	2,03	$2{,}32 \times 10^{-8}$	0,0640
oxygen	3,98	2,65 „	0,0842
nitrogen	4,34	2,98 „	0,0646

[1]) The first term of this result corresponds to a value for the van der Waals attraction constant a that perfectly agrees with that given by Debije (l. c. equation (18)).

For hydrogen we have then:

$$B=\frac{1}{2}n\,.\,\frac{4}{3}\pi\sigma^3\left\{1-1{,}0667\,(hv)^2+0{,}1741\,(hv)^6\ldots-\right.$$

$$\left.-\,0{,}1536\,hv-0{,}1638\,(hv)^3\ldots\right\}\ldots\ldots\quad(19)$$

For nitrogen the last row of terms is only slightly different, for oxygen somewhat more.

The values of σ used here being derived from calculations in which the induced attraction has not been attended to, this expression for B may be regarded as a first step only in a series of succeeding approximations. In the Leiden Supplement N°. 39*a*, the value of hv in the JOULE-KELVIN point of inversion has been calculated starting from eqation (19). When we wished to do this here, we should first have to derive still some terms for the "induced part" of B. After this the experimental values of $T_{inv\,(p=0)}$ and B_{inv} would give us corrected values for v and σ. We may expect the alteration of σ in consequence of this correction to be rather small, so that also the change of equation (19) due to it will be not considerable.

In this communication we will however confine ourselves to the following statement: A comparison of the terms in question shows that at least for the mentioned gases, unless the temperature be very high, the "*quadrupole attraction*" has considerably more influence in B than the "induced attraction". For gases as the above the cohesion forces introduced by VAN DER WAALS into the equation of state may therefore be ascribed principally to the forces exerted by the molecules on each other because of their quadrupole moments

Mathematics. — "*Intuitionistische Mengenlehre*" [1]). By Prof. L. E. J. BROUWER.

(Communicated at the meeting of December 18, 1920.)

Im folgenden gebe ich eine referierende Einleitung zu den beiden Teilen der Abhandlung: „*Begründung der Mengenlehre unabhängig vom logischen Satz vom ausgeschlossenen Dritten*", welche ich im November 1917 bzw. Oktober 1918 der Akademie vorgelegt habe.

Seit 1907 habe ich in mehreren Schriften [2]) die beiden folgenden Thesen verteidigt:

I. dass das *Komprehensionsaxiom*, auf Grund dessen alle Dinge, welche eine bestimmte Eigenschaft besitzen, zu einer Menge vereinigt werden (auch in der ihm später von ZERMELO gegebenen beschränkteren Form [3])) zur Begründung der Mengenlehre unzulässig bzw. unbrauchbar sei und nur in einer *konstruktiven* Mengendefinition eine zuverlässige Basis der Mathematik gefunden werden könne;

II. dass das von HILBERT 1900 formulierte *Axiom von der Lösbarkeit jedes Problems* [4]) mit dem *logischen Satz vom ausgeschlossenen Dritten* äquivalent sei, mithin, weil für das genannte Axiom kein

---

[1]) Unter demselben Titel ist ein im wesentlichen gleichlautender Aufsatz im Bd. 28 (1920) des Jahresberichtes der Deutschen Mathematiker-Vereinigung erschienen.

[2]) Vgl. „*Over de grondslagen der wiskunde*", Inauguraldissertation Amsterdam 1907, besonders auch die beigefügten Thesen; „*De onbetrouwbaarheid der logische principes*", Tijdschrift voor wijsbegeerte 2 (1908), abgedruckt in „*Wiskunde, waarheid, werkelijkheid*", Groningen 1919; „*Over de grondslagen der wiskunde*", N. Archief v. Wisk. (2) 8 (1908); Besprechung von MANNOURY, „*Methodologisches und Philosophisches zur Elementarmathematik*", N. Archief v. Wisk. (2) 9 (1910); „*Intuitionisme en formalisme*", Antrittsrede Amsterdam 1912, abgedruckt in „*Wiskunde, waarheid, werkelijkheid*", obengenannt; „*Intuitionism and formalism*", Amer. Bull. 20 (1913); Besprechung von SCHOENFLIES-HAHN, „*Die Entwickelung der Mengenlehre und ihrer Anwendungen*", Jahresber. d. D. M.-V. 23 (1914); „*Addenda en corrigenda over de grondslagen der wiskunde*", Versl. Kon. Akad. v. Wetensch. Amsterdam 25 (1917), abgedruckt in N. Archief v Wisk. (2) 12 (1918).

[3]) Vgl. Math. Ann. 65, S. 263.

[4]) Vgl. z. B. Archiv d. Math. u Phys. (3) 1, S. 52. Nach der hier geäusserten Ansicht HILBERTS entspricht das Axiom einer von jedem Mathematiker geteilten Ueberzeugung. In seinem neulich in Math. Ann. 78 abgedruckten Vortrag „*Axiomatisches Denken*" stellt er jedoch auf S. 412 die Frage nach der Lösbarkeit

zureichender Grund vorliege und die Logik auf der Mathematik beruhe und nicht umgekehrt, der logische Satz vom ausgeschlossenen Dritten ein *unerlaubtes* mathematisches Beweismittel sei, dem kein andrer als ein scholastischer und heuristischer Wert zugesprochen werden könne, so dass Theoreme, bei deren Beweis seine Anwendung nicht umgangen werden kann, jeden mathematischen Inhalt entbehren.

Von der in diesen beiden Thesen kondensierten *intuitionistischen* Auffassung der Mathematik habe ich übrigens in den in Anm. [3]) zitierten Schriften bloss fragmentarische Konsequenzen gezogen, habe auch in meinen gleichzeitigen philosophiefreien mathematischen Arbeiten regelmässig die alten Methoden gebraucht, wobei ich allerdings bestrebt war, nur solche Resultate herzuleiten, von denen ich hoffen konnte, dass sie nach Ausführung eines systematischen Aufbaues der intuitionistischen Mengenlehre, im neuen Lehrgebäude, eventuell in modifizierter Form, einen Platz finden und einen Wert behaupten würden.

Mit einem solchen systematischen Aufbau der intuitionistischen Mengenlehre habe ich erst in der eingangs erwähnten Abhandlung einen Anfang gemacht. Hier möchte ich kurz hinweisen auf einige der am tiefsten einschneidenden, nicht nur formalen, sondern auch inhaltlichen Aenderungen, welche die klassische Mengenlehre dabei erfahren hat.

Die zugrunde gelegte Mengendefinition ist folgende:

Eine Menge ist ein Gesetz, auf Grund dessen, wenn immer wieder ein willkürlicher Ziffernkomplex der Folge 1, 2, 3, 4, 5, ... *gewählt wird, jede dieser Wahlen entweder ein bestimmtes Zeichen oder nichts erzeugt oder aber die Hemmung des Prozesses und die definitive Vernichtung seines Resultates herbeiführt, wobei für jedes* $n > 1$ *nach jeder ungehemmten Folge von* $n-1$ *Wahlen wenigstens ein Ziffernkomplex angegeben werden kann, der, wenn er als n-ter Ziffernkomplex*

eines jeden mathematischen Problems als ein noch zu lösendes Problem. Seine an diese Problemstellung anschliessenden Bemerkungen über die Endlichkeit des vollen algebraischen Invariantensystems wären in meiner Terminologie so zu formulieren, dass aus der Unmöglichkeit der Unendlichkeit einer Menge keineswegs ihre Endlichkeit folgt.

Meiner Ueberzeugung nach sind das Lösbarkeitsaxiom und der Satz vom ausgeschlossenen Dritten beide *falsch* und ist der Glaube an diese Dogmen historisch dadurch verursacht worden, dass man zunächst aus der Mathematik der Teilmengen einer bestimmten endlichen Menge die klassische Logik abstrahiert, sodann dieser Logik eine von der Mathematik unabhängige Existenz a priori zugeschrieben und sie schliesslich auf Grund dieser vermeintlichen Apriorität unberechtigterweise auf die Mathematik der unendlichen Mengen angewandt hat.

gewählt wird, nicht die Hemmung des Prozesses herbeiführt. Jede in dieser Weise von einer in unbegrenzter Fortsetzung begriffenen Wahlfolge erzeugte Zeichenfolge (welche also im allgemeinen einen wesentlich unfertigen Charakter besitzt) heisst ein Element der Menge. Die gemeinsame Entstehungsart der Elemente der Menge M wird ebenfalls kurz als die Menge M bezeichnet.

Auf diesen Mengenbegriff wird sodann die Definition des Begriffes der *mathematischen Spezies,* der den Mengenbegriff als Sonderfall enthält, gegründet.

In der Theorie der *Kardinalzahlen,* welche nunmehr zunächst behandelt wird, tritt vor allem die *Zerlegung des Gleichmächtigkeitsbegriffes* in den Vordergrund. Zwei für die klassische Mengenlehre gleichmächtige Mengen oder Spezies können für die intuitionistische Mengenlehre *gleichmächtig, halbgleichmächtig, äquivalent, von gleichem Umfang, von gleicher Ausdehnung* oder *von gleichem Gewicht* sein[5]). Im Anschluss daran gibt es unter den für die klassische Theorie abzählbaren Mengen oder Spezies für die intuitionistische Mengenlehre *abzählbar unendliche, abzählbare, zählbare, auszählbare, durchzählbare* und *aufzählbare* Mengen bzw. Spezies. Die klassischen Kardinalzahlen a und c bleiben bestehen, dagegen wird das in der klassischen Theorie durch die Menge aller Funktionen einer Variablen gelieferte Beispiel einer Kardinalzahl $> c$ hinfällig.

In der Theorie der *geordneten Mengen* wird für den geordneten Charakter einer Spezies die Existenz der ordnenden Relation nur für je zwei *als verschieden erkannte* Elemente gefordert. Weiter gestaltet sich u.a. die Charakterisierung der Ordinalzahlen ϑ und ζ viel verwickelter, als in der klassischen Theorie; erstere erhält folgende Form:

Jede geordnete Spezies P, welche eine solche abzählbar unendliche, im engern Sinne überall dichte Teilspezies M enthält, dass zwischen je zwei Elementen[6]) *von P Elemente von M liegen, dass die Spezies der vor einem willkürlichen Elemente p von P liegenden Elemente von M eine abtrennbare Teilspezies von M ist, von der entweder kein Element existieren oder wenigstens ein Element bestimmt werden kann, und dass zu jeder der Ordnungseigenschaft entsprechenden Fundamentalreihe von Relationen „nach" oder „nicht nach" zu den Elementen*

[5]) Erst diese Begriffszerlegung hat mir ermöglicht, den Mächtigkeitscharakter, den ich in früheren Schriften nur für gewisse spezielle Mengen zulassen konnte, auf alle Spezies auszudehnen und in dieser Weise gewissermassen die Existenzberechtigung der komprehensiven Auffassung der Spezies wiederherzustellen.

[6]) d. h. zwischen je zwei als verschieden erkannten Elementen.

von M[7]) *ein diese Relationen erfüllendes Element von P bestimmt werden kann, besitzt die Ordinalzahl ϑ.*

In der Theorie der *wohlgeordneten Mengen* müssen allererst die beiden Haupteigenschaften, dass je zwei wohlgeordnete Mengen vergleichbar sind, und dass jede Teilmenge einer wohlgeordneten Menge ein erstes Element besitzt, welche die wichtigsten Beweismittel der klassischen Theorie bilden, preisgegeben werden; demzufolge hat die hier aufzubauende neue konstruktive Theorie mit ihrer Vorgängerin fast gar keine Aehnlichkeit mehr, weder äusserlich noch innerlich. An die Stelle der letzteren Haupteigenschaft bringt sie folgendes Theorem:

Ein Gesetz, welches in einer wohlgeordneten Spezies ein Element bestimmt und jedem schon bestimmten Elemente entweder die Hemmung des Prozesses oder ein ihm vorangehendes Element zuordnet, bestimmt sicher ein Element, dem es die Hemmung des Prozesses zuordnet.

Der Theorie der *ebenen Punktmengen* wird die Menge Q derjenigen Quadrate zugrunde gelegt, von denen ein Eckpunt in bezug auf ein rechtwinkliges Koordinatensystem die Koordinaten $a \,.\, 2^{-n}$ und $b \,.\, 2^{-n}$ und die (den Achsen parallelen) Seiten die Länge 2^{-n} oder 2^{1-n} besitzen. Sodann wird unter einem *Punkte der Ebene* eine unbegrenzt fortgesetzte Folge von Quadraten von Q, deren jedes im Innengebiete des nächstvorangehenden enthalten ist, verstanden.

Auf dieser Grundlage kommen aus der klassischen Theorie der Punktmengen zahlreiche Theoreme in Fortfall.

Vom Cantorschen Haupttheorem bleibt z.B. nur folgende *negative* Teilaussage in Kraft:

Es kann keine abgeschlossene geordnete Punktmenge existieren, deren Mächtigkeit grösser als die abzählbar unendliche ist und von der jeder Punkt einerseits einen nächstfolgenden Punkt aufweist, andrerseits von abzählbarer Ordnung ist bzw. von der Spezies der auf ihn folgenden Punkte einen endlichen Abstand besitzt.

Auch diese Teilaussage muss indes nach einer von der üblichen, auf dem Satz vom ausgeschlossenen Dritten beruhenden, völlig verschiedenen Methode bewiesen werden, z.B. so:

Eine Punktmenge π, deren Mächtigkeit grösser als die abzählbar unendliche ist, lässt sich nur so ordnen, dass endliche Mengen $i_1, i_2, \ldots$ von endlichen Wahlfolgen, welche nicht Abschnitte voneinander sind, der Reihe nach geordnet werden, und zwar in solcher

[7]) d. h. zu jeder der Ordnungseigenschaft entsprechenden unbegrenzten Folge von Relationen „nach" oder „nicht nach" zu den einer Abzählung durch eine Fundamentalreihe unterzogenen Elementen von M.

Weise, dass man für jedes k sicher ist, dass $\Xi\ (i_{k+1}, i_{k+2}, \ldots)$ von jedem Reste jeder unbegrenzt fortgesetzten Wahlfolge einen Abschnitt enthält. Indem wir nun in $\Xi\ (i_{k+1}, i_{k+2}, \ldots)$ nur diejenigen Wahlfolgen behalten, welche keine vorangehende Wahlfolge als Abschnitt enhalten, bestimmen wir eine zählbare Menge von Wahlfolgen j_k. Alsdann können wir an der Hand der fortschreitenden Konstruktion der j_ν bei der Herstellung eines willkürlichen j_ν für jede dazu gehörige Wahlfolge nur für höchstens einen einzigen als Fortsetzung davon erzeugten Punkt (nämlich für denjenigen, der bestimmt wird, indem für $\mu > \nu$ in jedem folgenden j_μ immer wieder die am höchsten geordnete Fortsetzung der schon vorhandenen Wahlfolge zu wählen ist) sicherstellen, dass er in der resultierenden Ordnung von π einen nächstfolgenden Punkt aufweist. Die Kardinalzahl der Spezies der Punkte, für welche diese Sicherkeit zu erlangen ist, kann mithin unmöglich grösser als die abzählbar unendliche sein [8]).

An die Stelle der *positiven* Aussage des CANTORschen Haupttheorems tritt in der intuitionistischen Mengenlehre eine ausführliche *Charakterisierung* derjenigen Punktmengen und Punktspezies, welche die betreffende Eigenschaft besitzen [9]).

---

[8]) Dieser Beweis findet sich schon in den beiden letzten der in Anm. [2]) zitierten Schriften; die daselbst gebrauchte Terminologie stimmt aber noch nicht mit der in meiner Abhandlung eingeführten überein, während in meiner Besprechung des SCHOENFLIESschen Buches die betreffende Stelle überdies einige Schreibfehler enthält (S. *81* ist Z. 3 u. 19 statt „Teilmenge zweiter Art", „nicht-abzählbare Teilmenge zweiter Art", Z. 10 statt „von Gebieten $e_1, e_2, \ldots$," „von einander enthaltenden Gebieten $e_{\alpha_1}, e_{\alpha_2}, \ldots$." und Z. 11 statt „zu $i_1, i_2, \ldots$.", „zu $i_{\alpha_1}, i_{\alpha_2}, \ldots$" zu lesen).

[9]) In meinen in Anm.[2]) zitierten Schriften (die letzte ausgenommen), in denen die Konsequenzen des Intuitionismus sich noch weniger deutlich für mich abgezeichnet hatten, haften der konstruktiven Mengendefinition noch zwei unnötige beschränkende Voraussetzungen an; in meiner jetzigen Terminologie sind nämlich die daselbst betrachteten Punktmengen erstens örtlich individualisiert, und lassen zweitens eine vollständige innere Abbrechung zu. Die Folge davon ist, dass z. B. in meiner Besprechung des SCHOENFLIESschen Buches das Haupttheorem statt als falsch, als selbstverständlich angeführt wird, und dass die daselbst gemachte Unterscheidung zwischen wohlkonstruierten Punktmengen und Punktmengen im allgemeinen (die gleichzeitig gemachte Zusammensetzung der wohlkonstruierten Punktmengen aus solchen erster und solchen zweiter Art, von denen die ersteren einen besonderen Fall der letzteren darstellen, soll als unwesentlich zurückgenommen werden) sich erst nach Fortschaffung der genannten beschränkenden Voraussetzungen mit der jetzigen Unterscheidung zwischen Punktmengen und Punktspezies im wesentlichen deckt.

Zum a. a. O. gegebenen Beispiel einer nicht-wohlkonstruierten Punktmenge ist zu bemerken, dass die daselbst zugrunde gelegte Funktion $f\ (x)$ nicht das volle Kontinuum zum Existenzbereich hat (vgl. meine gleichzeitig vorzulegende Mitteilung über die Dezimalbruchentwickelung der reellen Zahlen), dass Z. 12 statt „rational",

62

Die *inneren Grenzmengen* der klassischen Theorie, d. h. die Durchschnitte von Fundamentalreihen von Bereichen, werden in der intuitionistischen Mengenlehre, weil sie nicht notwendig Mengencharakter besitzen, als *innere Grenzspezies* eingeführt. Dabei bleibt das Theorem der klassischen Mengenlehre, dass der Durchschnitt zweier innerer Grenzspezies wiederum eine innere Grenzspezies ist, bestehen; der analoge Satz für die Vereinigung fällt aber fort, und von der Haupteigenschaft der inneren Grenzmengen der klassischen Theorie, dass zu einer willkürlichen Punktmenge Q eine innere Grenzspezies existiert, welche ausser Q ausschliesslich Grenzpunkte der finalen Kohärenz von Q enthält, bleibt nur der folgende Bestandteil erhalten:

Zu jeder vollständig abbrechbaren Punktmenge π existiert eine innere Grenzspezies, welche mit der Vereinigung von π und einer Teilspezies der Abschliessung der finalen Kohärenz von π örtlich kongruent ist und eine mit π örtlich kongruente Punktmenge als Teilspezies enthält.

Die klassische Definition der *Messbarkeit* erleidet in der intuitionistischen Mengenlehre nur eine geringe Aenderung; die Sicherkeit der Messbarkeit verschwindet aber sowohl für die Bereiche wie für die abgeschlossenen Punktspezies und inneren Grenzspezies, und die Haupteigenschaft des klassischen Messbarkeitsbegriffes, dass die Vereinigung einer abzählbaren Menge messbarer Mengen ohne gemeinsame Punkte messbar und ihr Mass gleich der Summe der Masse ihrer Komponenten ist, wird in der intuitionistischen Mengenlehre folgendermassen formuliert:

Wenn F eine solche Fundamentalreihe von messbaren Punktspezies ist, dass die Inhalte der Vereinigungen ihrer Anfangssegmente eine limitierte Folge i bilden, so ist auch die Vereinigung von F messbar und ihr Inhalt gleich i.

Selbstverständlich erleidet der Begriff des *Punktes der Ebene* eine beträchtliche Verengerung, wenn in der betreffenden Definition statt "unbegrenzt fortgesetzte Folge", "Fundamentalreihe" gelesen wird. Bemerkenswert ist aber, dass das lineare Analogon dieses *engern* Punktbegriffes seinerseits noch erheblich mehr umfasst, als der klassische lineare Punktbegriff, der auf dem *Schnitte* beruht, wie in meiner gleichzeitig vorzulegenden Mitteilung über die Dezimalbruchentwickelung der reellen Zahlen näher erörtert wird.

„durch einen endlichen Dualbruch darstellbar" zu lesen ist, und dass man in der Spezies der endlich definierbaren Punkte der Ebene ein viel einfacheres Beispiel einer nicht-wohlkonstruierten Punktmenge besitzt.

Mathematics. — "*Besitzt jede reelle Zahl eine Dezimalbruchentwickelung?*" [1]) By Prof. L. E. J. BROUWER.

(Communicated at the meeting of December 18, 1920).

§ 1.

Existenzbereich der unendlichen Dezimalbruchentwickelung auf dem Kontinuum.

Verstehen wir in der Menge der endlichen Dualbrüche ≥ 0 und ≤ 1 unter einem Intervalle λ_ν ein zwei Dualbrüche $\frac{a}{2^\nu}$ und $\frac{a+2}{2^\nu}$ als Endelemente besitzendes geschlossenes Intervall, unter einem *Punkte des Kontinuums* eine in unbegrenzter Fortsetzung begriffene Folge von Intervallen λ, deren jedes im Innern des nächstvorangehenden enthalten ist [2]), unter x einen variablen Punkt des Kontinuums, unter $F_n(x)$ einen n-stelligen Dezimalbruch mit der Eigenschaft, dass jeder links von ihm liegende Punkt des Kontinuums links von einem Intervalle von x liegt, während $F_n(x) + 10^{-n}$ rechts von einem Intervalle von x liegt, unter $F(x)$ die eindeutige unendliche Dezimalbruchentwickelung von x, so besitzt $F_n(x)$ die (übrigens allen unstetigen Funktionen gemeinsame) Eigenschaft, dass ihr Existenzbereich G_n *nicht* mit dem Kontinuum zusammenfallen [3]) kann. Der Existenzbereich $G = \mathfrak{D}(G_1, G_2, \ldots)$ von $F(x)$ kann also erst recht nicht mit dem Kontinuum zusammenfallen, obgleich er sich (ebenso wie der Existenzbereich der regelmässigen Kettenbruchentwickelung von x) dem Kontinuum so eng anschmiegt, dass er mit

---

[1]) Ueber den Inhalt dieser Abhandlung wurde am 22. September 1920 auf der Naturforscherversammlung in Bad Nauheim ein referierender Vortrag gehalten.

[2]) Vgl. meine in Bd. XII der Verhandelingen der Koninklijke Akademie van Wetenschappen te Amsterdam (Eerste Sectie) erschienene Abhandlung: „*Begründung der Mengenlehre unabhängig vom logischen Satz vom ausgeschlossenen Dritten*", 2. Teil, S. 3, 4. Wie daselbst S. 4 Fussnote [1]) hervorgehoben und durch die vorliegende Arbeit klar ins Licht gestellt wird, sind die beiden S. 9 des 1. Teiles benutzten Begriffe der „reellen Zahl" bedeutend enger als der hier definierte Begriff des Punktes des Kontinuums. In einem ganz andern, aus dem Zusammenhang ersichtlichen Sinne wird der Ausdruck „reelle Zahl" der Expressivität wegen in der Ueberschrift und im Schlussparagraphen der vorliegenden Arbeit gebraucht.

[3]) a. a. O., 2. Teil, S. 5.

demselben einerseits örtlich übereinstimmt [1]), andrerseits inhaltsgleich [2]) ist [3]).

Die Definition des Punktes des Kontinuums erleidet indessen eine erhebliche Einschränkung, wenn wir in derselben statt „in unbegrenzter Fortsetzung begriffene Folge", „Fundamentalreihe" [4]) lesen. Zweck der folgenden Paragraphen ist, klarzustellen, inwiefern für diese *Punkte des Kontinuums im engern Sinne*, die unendliche Dezimalbruchentwickelung existiert.

§ 2.

Die Ergänzungselemente der abzählbar unendlichen, überall dicht geordneten Mengen.

Es sei eine abzählbar unendliche, im engern Sinne überall dicht geordnete [5]) Menge H gegeben. Es seien $g_1, g_2, g_3, \ldots$ die nach irgend einem, H als abzählbar unendliche Menge charakterisierenden, Abzählungsgesetze γ numerierten Elemente von H und es sei $\mathfrak{S}(g_1, g_2, \ldots g_\nu) = s_\nu$ gesetzt. Unter einem i_ν bzw. j_ν verstehen wir ein (eventuell aus einem einzigen Elemente bestehendes) geschlossenes Intervall [6]) von H, deren Endelemente zu s_ν gehören, deren Inneres aber höchstens ein bzw. kein einziges Element von s_ν enthält.

Unter einem *Ausfüllungselemente* r von H verstehen wir *erstens* eine jedenfalls ein Element besitzende Spezies von in unbegrenzter Fortsetzung begriffenen Folgen $F_\alpha, F_{\alpha+1}, F_{\alpha+2}, \ldots$ (α eine für r bestimmte positive ganze Zahl), wo jedes F_ν ein i_ν und jedes $F_{\alpha+\nu+1}$ in $F_{\alpha+\nu}$ enthalten ist, während F_ν für jedes ν zu einer für r bestimmten Spezies S_ν gehört, von der je zwei Elemente ein Element von s_ν gemeinsam haben; *zweitens* eine jedenfalls ein Element besitzende Spezies von in unbegrenzter Fortsetzung begriffenen Folgen $\zeta_1, \zeta_2, \zeta_3, \ldots$ von je ein bestimmbares Element besitzenden abtrennbaren Teil-

---

[1]) a. a. O., 2. Teil, S. 6.

[2]) a. a. O., 2. Teil, S. 29, 30.

[3]) Natürlich kann auch der Existenzbereich einer mittels einer Funktion der unendlichen Dezimalbruchentwickelung von x erklärten Funktion von x nicht über G hinausgehen. Z. B. hat die im Jahresber. d. D. M.-V. 23, S. 80 von mir definierte Funktion $f(x)$ genau G zum Existenzbereich. Während aber die Funktion $F(x)$ des Textes in der auf dem Kontinuum überall dichten Punktmenge G *gleichmässig stetig* ist und sich *auf Grund dieser Eigenschaft* zu einer *auf dem vollen Kontinuum existierenden* Funktion $\varphi(x) = x$ erweitern lässt, ist für $f(x)$ jede Erweiterung auf das volle Kontinuum ausgeschlossen.

[4]) Vgl. „*Begründung der Mengenlehre usw.*", 1. Teil, S. 14.

[5]) a. a. O., 1. Teil, S. 16.

[6]) a. a. O., 1. Teil, S. 13.

mengen [1]) von H, wenn in jeder Folge jedes $\zeta_{\nu+1}$ in ζ_ν enthalten ist und eine Fundamentalreihe $n_1, n_2, n_3, \ldots (n_{\nu+1} \geq n_\nu)$ von ganzen positiven Zahlen und ein Ausfüllungselement erster Art r_1 von H bestimmt sind mit der Eigenschaft, dass zu jedem Elemente $\zeta_1, \zeta_2, \zeta_3, \ldots$ von r ein Element $\varepsilon_\alpha, \varepsilon_{\alpha+1}, \varepsilon_{\alpha+2}, \ldots$ von r_1 existiert, so dass ζ_{n_ν} zu $\varepsilon_{\alpha+\nu}$ gehört.

Unter einem *Ergänzungselemente nullter Ordnung* oder kurz einem *Ergänzungselemente* r von H verstehen wir *erstens* eine Fundamentalreihe $\varepsilon_\alpha, \varepsilon_{\alpha+1}, \varepsilon_{\alpha+2}, \ldots$ (α eine für r bestimmte positive ganze Zahl), wo jedes ε_ν ein i_ν und jedes $\varepsilon_{\alpha+\nu+1}$ in $\varepsilon_{\alpha+\nu}$ enthalten ist; *zweitens* eine jedenfalls ein Element besitzende Spezies von in unbegrenzter Fortsetzung begriffenen Folgen $\zeta_1, \zeta_2, \zeta_3, \ldots$ von je ein bestimmbares Element besitzenden abtrennbaren Teilmengen von H, wenn in jeder Folge jedes $\zeta_{\nu+1}$ in ζ_ν enthalten ist und eine Fundamentalreihe $n_1, n_2, n_3, \ldots (n_{\nu+1} \geq n_\nu)$ von ganzen positiven Zahlen und ein Ergänzungselement erster Art $\varepsilon_\alpha, \varepsilon_{\alpha+1}, \varepsilon_{\alpha+2}, \ldots$ von H bestimmt sind, so dass jedes ζ_{n_ν} von r zu $\varepsilon_{\alpha+\nu}$ gehört. [2])

Wenn ${}_1r$ und ${}_2r$ Ausfüllungselemente von H sind und jedes ${}_1\varepsilon_\mu$ mit jedem ${}_2\varepsilon_\nu$ ein gemeinsames Element besitzt, so sagen wir, dass ${}_1r$ und ${}_2r$ *in H zusammenfallen.* Ein mit einem Ergänzungselemente von H in H zusammenfallendes Ausfüllungselement von H *wird gleichfalls als Ergänzungselement von H bezeichnet.*

Wenn das Element g von H zu jedem ε_ν des Ausfüllungselementes r von H gehört, so sagen wir, dass r und g *in H zusammenfallen.*

Wenn ${}_1r$ und ${}_2r$ Ausfüllungselemente von H sind und man ein ${}_1\varepsilon_\mu$ und ein ${}_2\varepsilon_\nu$ ohne gemeinsame Elemente angeben kann, so sagen wir, dass ${}_1r$ und ${}_2r$ *in H örtlich verschieden sind.*

Wenn man ein ε_ν des Ausfüllungselementes r von H angeben kann, zu dem das Element g von H nicht gehört, so sagen wir, dass r und g *in H örtlich verschieden sind.*

Das Ergänzungselement bzw. Ausfüllungselement r von H heisst ein *Ergänzungselement erster Ordnung* von H, wenn für jedes Element g von H entweder die Relation $g \geq r$ (d. h. jedes rechts von g gelegene Element von H liegt rechts von einem bestimmbaren ε_ν von r), oder die Relation $g \leq r$ (d. h. jedes links von g gelegene Element von H liegt links von einem bestimmbaren ε_ν von r) hergeleitet werden kann, oder, was auf dasselbe hinauskommt, wenn r mit einem Ergänzungselemente r' von H, von dem jedes ε'_ν ein j_ν ist, zusammenfällt.

[1]) a. a. O., 1. Teil, S. 4.

[2]) Ob der Begriff des Ausfüllungselementes sich auf den des Ergänzungselementes zurückführen lässt, bleibe hier dahingestellt.

Die Ergänzungselemente erster Ordnung von H *entsprechen den Dedekindschen Schnitten* von H.

Das Ergänzungselement erster Ordnung r von H heisst ein *Ergänzungselement zweiter Ordnung* von H, wenn für jedes Element g von H die Relation $g \leq r$ entweder hergeleitet, oder ad absurdum geführt werden kann, oder, was auf dasselbe hinauskommt, wenn r' sich so wählen lässt, dass kein μ mit der Eigenschaft, dass die rechten Endelemente von F'_ν und F'_μ für jedes $\nu > \mu$ identisch sind, existieren kann.

Das Ergänzungselement zweiter Ordnung r von H heisst ein *Ergänzungselement dritter Ordnung* von H, wenn für jedes Element g von H entweder die Relation $g > r$ (d. h. man kann ein links von g gelegenes r_ν von r bestimmen), oder die Relation $g \leq r$ hergeleitet werden kann, oder, was auf dasselbe hinauskommt, wenn r' sich so wählen lässt, dass zu jedem F'_μ ein solches F'_ν bestimmt werden kann, dessen rechtes Endelement links vom rechten Endelemente von F'_μ gelegen ist.

Ein Ergänzungselement dritter Ordnung von H heisst ein *Ergänzungselement vierter Ordnung* von H, wenn für jedes Element g von H entweder die Relation $g > r$, oder die Relation $g = r$ (d. h. g und r fallen in H zusammen), oder schliesslich $g < r$ (d. h. man kann ein rechts von g gelegenes F_ν von r bestimmen) hergeleitet werden kann, oder, was auf dasselbe hinauskommt, wenn r' sich so wählen lässt, dass zu jedem F'_μ ein solches $\nu > \mu$ bestimmt werden kann, dass die beiden Endelemente von F'_ν von den beiden Endelementen von F'_μ verschieden sind.

Die vorstehenden Definitionen der Ausfüllungselemente sowie der Ergänzungselemente nullter, erster, zweiter, dritter und vierter Ordnung von H sind für gegebene ordnende Relationen in H offenbar unabhängig vom Abzählungsgesetze γ.

Sei M eine endliche Menge oder eine Fundamentalreihe von Ergänzungselementen vierter Ordnung von H_ν, deren je zwei in H_ν örtlich verschieden sind und deren jedes von jedem Elemente von H_ν in H_ν örtlich verschieden ist. Die Vereinigung von M und H_ν bildet eine abzählbar unendliche, im engern Sinne überall dicht geordnete Menge $H_{\nu+1}$. Jedes Ergänzungselement von H_ν ist gleichzeitig Ergänzungselement von $H_{\nu+1}$ und jedes Ergänzungselement h-ter Ordnung von $H_{\nu+1}$ fällt in $H_{\nu+1}$ zusammen mit einem Ergänzungselemente h-ter Ordnung von H_ν.

Die vorstehende Beziehung besteht sowohl zwischen der geordneten Menge der endlichen Dualbrüche H_0 und der geordneten Menge der

endlichen Dezimalbrüche H_1, wie zwischen H_1 und der geordneten Menge der rationalen Zahlen H_2.

§ 3.

Ergänzungselemente, Dezimalbruchentwickelungen und Kettenbruchentwickelungen.

Ein Ergänzungselement erster Ordnung von H lässt in H die *Ortsbestimmung erster Ordnung* zu, welche sich, wenn H als die Menge der endlichen Dezimalbrüche gelesen wird, als die *mehrdeutige unendliche Dezimalbruchentwickelung* herausstellt. Umgekehrt ist jedes Ausfüllungselement von H, das in H die Ortsbestimmung erster Ordnung zulässt, ein Ergänzungselement erster Ordnung von H.

Die Ortsbestimmung erster Ordnung in H kann für in H zusammenfallende Ergänzungselemente von H verschieden ausfallen.

Ein Ergänzungselement zweiter Ordnung von H lässt in H die *Ortsbestimmung zweiter Ordnung* zu, welche sich, wenn H als die Menge der endlichen Dezimalbrüche gelesen wird, als die *eindeutige unendliche Dezimalbruchentwickelung* (für welche die Existenz einer letzten von 9 verschiedenen Ziffer ausgeschlossen ist) herausstellt. Umgekehrt ist jedes Ausfüllungselement von H, das in H die Ortsbestimmung zweiter Ordnung zulässt, ein Ergänzungselement zweiter Ordnung von H.

Zwei Ergänzungselemente von H, für welche die Ortsbestimmung zweiter Ordnung in H verschieden ausfällt, können in H nicht zusammenfallen.

Ein Ergänzungselement dritter Ordnung von H lässt in H die *Ortsbestimmung dritter Ordnung* zu, welche sich, wenn H als die Menge der rationalen Zahlen gelesen wird, als die *unendliche reduziert-regelmässige Kettenbruchentwickelung* herausstellt. Umgekehrt ist jedes Ausfüllungselement von H, das in H die Ortsbestimmung dritter Ordnung zulässt, ein Ergänzungselement dritter Ordnung von H.

Zwei Ergänzungselemente von H, für welche die Ortsbestimmung dritter Ordnung in H verschieden ausfällt, sind in H örtlich verschieden.

Ein Ergänzungselement vierter Ordnung von H lässt in H die *Ortsbestimmung vierter Ordnung* zu, welche sich, wenn H als die Menge der rationalen Zahlen gelesen wird, als die *eindeutige regelmässige Kettenbruchentwickelung* (welche eventuell endlich ausfallen kann) herausstellt. Umgekehrt ist jedes Ausfüllungselement von H, das in H die Ortsbestimmung vierter Ordnung zulässt, ein Ergänzungselement vierter Ordnung von H.

Zwei Ergänzungselemente von H, für welche die Ortsbestimmung vierter Ordnung in H verschieden ausfällt, sind in H örtlich verschieden.

§ 4.

Existenz der Dezimalbruchentwickelung reeller algebraischer Zahlen.

Seien r_1 und r_2 beliebige reelle algebraische Zahlen, d. h. je einer algebraischen Gleichung mit ganzen rationalen Koeffizienten genügende Ausfüllungselemente der von den rationalen Zahlen gebildeten geordneten Menge H_2. Alsdann kann man eine algebraische Gleichung $F(x) \equiv a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1}x + a_n = 0$ mit ganzen rationalen Koeffizienten und nicht verschwindender Diskriminante D bestimmen, der sowohl r_1 wie r_2 genügt. Seien $w_1, w_2, \ldots w_n$ die (mit jedem beliebigen Grade der Genauigkeit approximierbaren) Wurzeln von $F(x) = 0$, so können w_r und w_s für $r \neq s$ nicht in H_2 zusammenfallen. Sei ϱ eine rationale Zahl, welche die Moduln aller Wurzeln von $F(x) = 0$ übersteigt, und $b = 2\varrho$, so ist

$$| w_r - w_s | < b \qquad (r \neq s).$$

Weil aber

$$\prod_{\mu \neq \nu} (w_\mu - w_\nu)^2 = \frac{D}{a_0^{2n-2}},$$

so ist andrerseits

$$| w_r - w_s |^2 > \frac{D}{a_0^{2n-2}\, b^{n^2-n-2}},$$

so dass wir mittels hinreichend genauer Approximierung von r_1 und r_2 entweder Sicherheit erlangen, dass r_1 und r_2 mit derselben Wurzel w_α zusammenfallen, oder ein r_1 und r_2 trennendes rationales Intervall bestimmen können. Indem wir dieses Resultat zunächst spezialisieren für den Fall, dass r_2 eine rationale Zahl ist, ersehen wir mühelos, dass r_1 in H_2 entweder mit einem Elemente von H_2 zusammenfällt oder von jedem Elemente von H_2 örtlich verschieden ist, so dass r_1 *sich als Ergänzungselement vierter Ordnung von H_2* erweist, mithin *sowohl in einen eindeutigen unendlichen Dezimalbruch, wie in einen eindeutigen regelmässigen Kettenbruch entwickelt werden kann.*

Setzen wir nun weiter voraus, dass weder r_1 noch r_2 mit einem Elemente von H_2 zusammenfällt, so *fallen sie entweder in H_2 zusammen, oder sind in H_2 örtlich verschieden.*

Hieraus folgern wir, dass die Spezies der reellen algebraischen Zahlen eine abzählbar unendliche, im engern Sinne überall dicht geordnete Menge H_3 bildet, welche zu H_2 die am Schluss von § 2 erklärte Beziehung eines $H_{\nu+1}$ zu einem entsprechenden H_ν besitzt.

§ 5.

Existenz der Dezimalbruchentwickelung von π.

Seien a und b ganze positive Zahlen und $a < b$. Wir verstehen unter K_0 den unbedingt konvergenten [1]) unendlichen Kettenbruch

$$\left[\frac{a}{b}, -\frac{a^2}{(2\nu+1)\,b}\right]_1^\infty$$

und unter K_m den unbedingt konvergenten unendlichen Kettenbruch

$$\left[\frac{a^2}{(2m+1)\,b}, -\frac{a^2}{(2\nu+1)\,b}\right]_{m+1}^\infty.$$

Alsdann gelten die Beziehungen

$$tg\,\frac{a}{b} = K_0 = \frac{a}{b - K_1}$$

$$K_m = \frac{a^2}{(2m+1)\,b - K_{m+1}} \qquad (m \geq 1)$$

Seien $x_0, x_1, x_2, \ldots$ reelle Variablen, welche durch die Beziehungen

$$\left.\begin{aligned} x_0 &= \frac{a}{b - x_1} \\ x_m &= \frac{a^2}{(2m+1)\,b - x_{m+1}} \qquad (m \geq 1) \end{aligned}\right\} \quad \ldots \quad (\dagger)$$

verbunden sind, und x'_a eine rationale Zahl zwischen 0 und 1, also $< \frac{1}{2}b$. Mittels $(\dagger)$ leiten wir aus x'_a weitere rationale Zahlen $x'_{a-1}, x'_{a-2}, \ldots. x'_1, x'_0$ und $x'_{a+1}, x'_{a+2}, \ldots.$ her. Von diesen fallen $x'_{a-1}, x'_{a-2}, \ldots. x'_1, x'_0$ alle positiv aus, während $x'_{a-1}, x'_{a-2}, \ldots x'_1$ alle $< \frac{1}{2}b$ und $x'_0 < \frac{2a}{b}$ wird. Weiter kann man ein kleinstes $r > a$ bestimmen mit der Eigenschaft, dass $x_r' \leq 0$ oder ≥ 1 wird [2]).

Sei α eine (für das weitere hinreichend klein gewählte) positive rationale Zahl und η_a ein solches geschlossenes rationales Wertintervall von x_a, dass sowohl η_a, wie die auf Grund von $(\dagger)$ entsprechenden Wertintervalle $\eta_{a+1}, \eta_{a+2}, \ldots \eta_r$ von $x_{a+1}, x_{a+2}, \ldots x_r$ rechts vom Werte 0 und links vom Werte 1 liegen, während, wenn wir noch die auf Grund von $(\dagger)$ entsprechenden Wertintervalle von $x_{a-1}, x_{a-2}, \ldots. x_0$ mit $\eta_{a-1}, \eta_{a-2}, \ldots \eta_0$ bezeichnen, jedes K_ν für $0 \leq \nu \leq r$ in η_ν enthalten ist und eine Entfernung $> 2\alpha$ von den Endwerten von η_ν besitzt. Alsdann können wir eine solche ganze nichtnegative Zahl $s \leq r$ bestimmen, dass $x'_0, x'_1, \ldots x'_{s-1}$ der Reihe nach in $\eta_0, \eta_1, \ldots \eta_{s-1}$ enthalten sind, während x'_s eine Entfernung $> \alpha$ von K_s besitzt.

[1]) Vgl. Pringsheim, Münchener Berichte 28 (1898), S. 299 fgg.

[2]) a. a. O., S. 318.

Sei β' eine solche positive rationale Zahl, dass für jedes zu η_s gehörige x_s die Ungleichung

$$\frac{dx_s}{dx_s} > \beta'$$

gilt, so besitzt x'_s eine Entfernung $> \alpha\beta'$ von K_0.

Sei x_s'' eine solche rationale Zahl, dass die auf Grund von (†) entsprechende Zahl $x_a'' \leq 0$ oder ≥ 1 ausfällt. Alsdann können wir eine solche ganze nichtnegative Zahl $t \leq a$ bestimmen, dass $x''_s, \ldots x''_{t-1}$ der Reihe nach in $\eta_s, \ldots \eta_{t-1}$ enthalten sind, während x_t'' eine Entfernung $> a$ von K_t besitzt.

Sei β'' eine solche positive rationale Zahl, dass für jedes zu η_0 gehörige x_s die Ungleichung

$$\frac{dx_0}{dx_t} > \beta''$$

gilt, so besitzt x_s'' eine Entfernung $> a\beta''$ von K_0.

Zu einer beliebigen positiven rationalen Zahl $i_1 < 1$ und einer beliebigen positiven rationalen Zahl i kann man mithin eine solche positive rationale Zahl $i_2 < 1$ bestimmen, dass

$$| i - tg\, i_1 | > i_2.$$

Insbesondere kann man zu einer beliebigen positiven rationalen Zahl $i_1 < 1$ eine solche positive rationale Zahl $i_2 < 1$ bestimmen, dass

$$| 1 - tg\, i_1 | > i_2,$$

mithin auch (weil im zwischen den Werten 0 und 2 enthaltenen Wertegebiet von y die Ungleichung

$$\frac{d\, arctg\, y}{dy} \geq \frac{1}{5}$$

besteht)

$$\left| \frac{\pi}{4} - i_1 \right| > \frac{i_2}{5},$$

so dass *die Zahl π sich als Ergänzungselement vierter Ordnung von H_2 erweist*[1]), *mithin sich sowohl in einen eindeutigen unendlichen Dezimalbruch wie in einen eindeutigen regelmässigen Kettenbruch entwickeln lässt.*

Die Entwickelungen dieses und des vorangehenden Paragraphen bieten Beispiele der Charakterisierung von Ergänzungselementen bzw. Ausfüllungselementen r von H als Ergänzungselemente vierter

[1]) Die gleiche Eigenschaft der Zahl e ist eine unmittelbare Folge der regelmässigen Kettenbruchentwickelung

$$\frac{e-1}{2} = \left[\frac{1}{1}, \frac{1}{2+4\nu}\right]_1^\infty.$$

Ordnung von H mittels *positiver Rationalitätsbeweise in H* (die ein Element von H bestimmen, mit dem r zusammenfällt) oder *positiver Irrationalitätsbeweise in H* (die r als von jedem Elemente von H örtlich verschieden erkennen lassen). Hierzu ist zu bemerken, dass sich aus einem *negativen Rationalitäts- bzw. Irrationalitätsbeweise in H* (der die Annahme, dass r von jedem Elemente von H örtlich verschieden wäre bzw. mit einem Elemente von H zusammenfiele, ad absurdum führt) nicht einmal folgern lässt, dass r Ergänzungselement erster Ordnung von H ist. Eben deshalb haben wir in diesem § den LAMBERTschen negativen Irrationalitätsbeweis von π einer passenden Umarbeitung unterzogen und in die obige positive Form gebracht. Die weiteren klassischen Beweise desselben Satzes lassen sich übrigens in analoger Weise ergänzen.

§ 6.

Reelle Zahlen, welche keine Dezimalbruchentwickelung besitzen.

Sei c_n die n-te Ziffer der unendlichen Dezimalbruchentwickelung von π. Wir werden sagen, dass n sich im *ersten Falle* befindet, wenn $c_n, c_{n+1}, \ldots . c_{n+4}$ alle gleich sind, im *zweiten Falle*, wenn $c_n, c_{n+1}, \ldots . c_{n+9}$ alle verschieden sind, und im *dritten Falle*, wenn weder der erste, noch der zweite Fall vorliegt.

Wir definieren ein Ergänzungselement r der geordneten Menge der endlichen Dezimalbrüche H_1 mittels der unendlichen Reihe

$$\sum_{n=1}^{\infty} a_n \,.\, 10^{-n-1},$$

wo $a_n = 0$, wenn n sich im ersten Falle befindet, $a_n = 10$, wenn n sich im zweiten Falle befindet, sonst $a_n = 9$.

Dieses Ergänzungselement würde erst dann ein Ergänzungselement erster Ordnung von H_1 darstellen, m. a. W. eine unendliche Dezimalbruchentwickelung zulassen, wenn man eine Methode besässe, für jedes beliebige im dritten Falle befindliche n, *entweder* die Existenz eines im zweiten Falle befindlichen $m > n$ mit der Eigenschaft, dass jede zwischen n und m liegende ganze Zahl sich im dritten Falle befände, ad absurdum zu führen, *oder* die Existenz eines im ersten Falle befindlichen $m > n$ mit der Eigenschaft, dass jede zwischen n und m liegende ganze Zahl sich im dritten Falle befände, ad absurdum zu führen.

Wir definieren weiter ein Ergänzungselement erster Ordnung r von H_1 mittels der unendlichen Reihe

$$\sum_{n=1}^{\infty} a_n \,.\, 10^{-n-1},$$

wo jedes a_n entweder gleich 0 oder gleich 9 ist, während $a_1 = 9$ und a_{n+1} dann und nur dann von a_n verschieden ist, wenn n sich im zweiten Falle befindet.

Dieses Ergänzungselement erster Ordnung würde erst dann ein Ergänzungselement zweiter Ordnung von H_1 darstellen, m. a. W. die im § 3 definierte eindeutige unendliche Dezimalbruchentwickelung zulassen, wenn man eine Methode besässe, für jedes ganze positive n mit der Eigenschaft, dass entweder keine oder eine gerade Anzahl von ganzen positiven Zahlen $\leq n$ sich im zweiten Falle befindet, *entweder* die Existenz *oder* die Abwesenheit eines im zweiten Falle befindlichen $m > n$ ad absurdum zu führen.

Ein Ergänzungselement dritter Ordnung von H_1 würde dasselbe Ergänzungselement erst dann darstellen, wenn man eine Methode besässe, für jedes ganze positive n mit der Eigenschaft, dass entweder keine oder eine gerade Anzahl von ganzen positiven Zahlen $\leq n$ sich im zweiten Falle befindet, *entweder* die Existenz eines im zweiten Falle befindlichen $m > n$ ad absurdum zu führen, *oder* ein im zweiten Falle befindliches $m > n$ anzugeben.

Wir definieren schliesslich ein Ergänzungselement dritter Ordnung r von H_1 mittels der unendlichen Reihe

$$\sum_{n=1}^{\infty} a_n \,.\, 10^{-n-1},$$

wo $a_n = 9$, wenn n sich im zweiten Falle befindet, sonst $a_n = 0$.

Dieses Ergänzungselement dritter Ordnung würde erst dann ein Ergänzungselement vierter Ordnung von H_1 darstellen, wenn man eine Methode besässe, für jedes ganze positive n, *entweder* die Existenz eines im zweiten Falle befindlichen $m > n$ ad absurdum zu führen, *oder* ein im zweiten Falle befindliches $m > n$ anzugeben.

Sämtliche Beispiele dieses § fallen übrigens in H_1 zusammen mit Ergänzungselementen vierter Ordnung der geordneten Menge der endlichen Dualbrüche H_2.

Für Beispiele reeller Zahlen ohne Dezimalbruchentwickelung besteht bei der Weiterentwickelung der Mathematik stets die Möglichkeit, dass sie einmal hinfällig werden; dann aber können sie immer durch solche, welche ihre Gültigkeit behalten haben, ersetzt werden.

ERRATUM.

Proceedings, Vol. XX (2), p. 1156, l. 13 from the bottom: *read* in the formula: „— 0.00003219 T^4" *for* „— 0.00005102 T^4".

KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN
TE AMSTERDAM.

PROCEEDINGS

VOLUME XXIII

N°. 7.

President: Prof. H. A. LORENTZ.

Secretary: Prof. L. BOLK.

(Translated from: "Verslag van de gewone vergaderingen der Wis- en Natuurkundige Afdeeling," Vols. XXVIII and XXIX).

CONTENTS.

Chemistry. — "*The Electromotive Behaviour of Aluminium*". II.[1]). By Prof. A. SMITS and G. J. DE GRUIJTER. (Communicated by Prof. P. ZEEMAN).

(Communicated at the meeting of Nov. 27, 1920).

With a view to obtaining a better insight into the electromotive behaviour of aluminium and its alloys with mercury, the melting-point diagram was first determined. It was found that, as follows from the subjoined T,X-figure 1, no compound occurs in the system

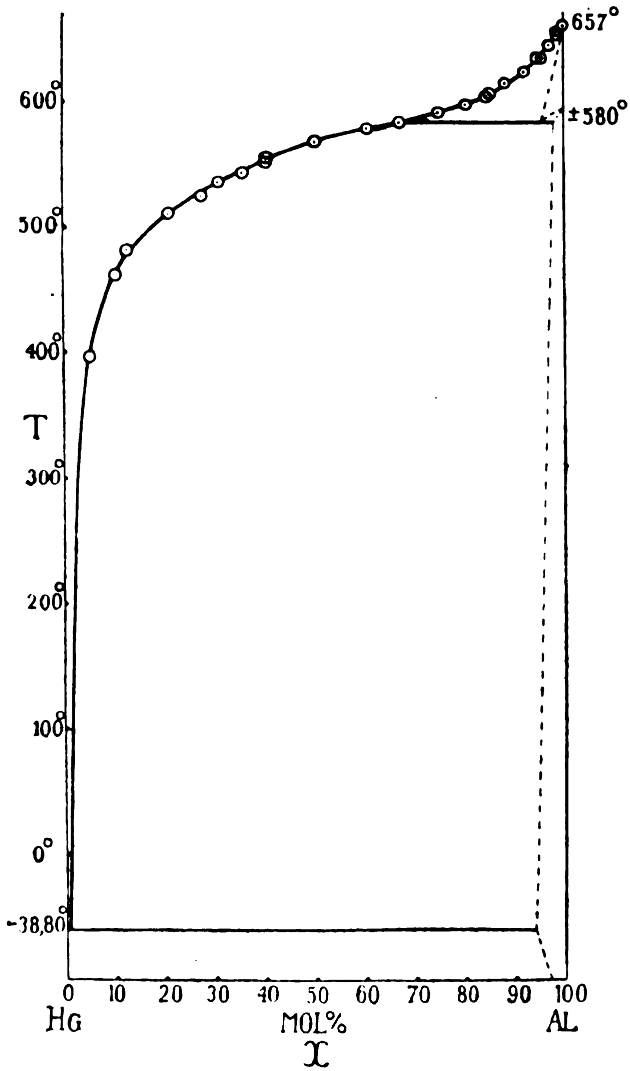
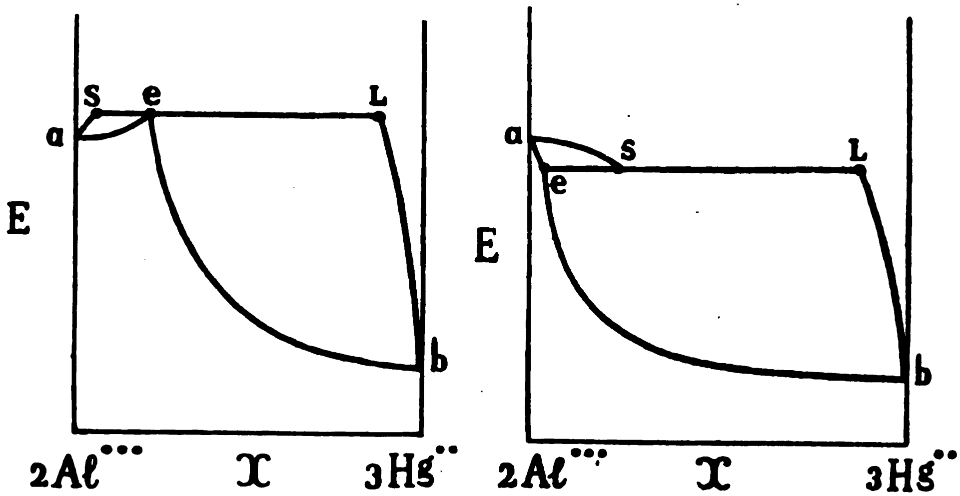


Fig. 1.

[1]) First communication, These Proc. XXII, N°. 9 and 10, p. 876.

Hg-Al, and that the melting-point line of aluminium consists of two branches, in consequence of the occurrence of a transition point lying at about 585°.

The determination of the melting-point diagram served by way of orientation, and now that no compound appeared to exist, it was certain that the EX-figure corresponding to the ordinary temperature, must belong to the type I or II in Fig. 2.



Type 1. Fig. 2. Type 2.

Assuming, as has been done before, that GIBBS's paradox is also applicable to the components of a mixed crystal phase present in diluted state, the formula [1]):

$$E = -\frac{0.058}{\nu F}\log\frac{L_M}{(M_L^{\nu\cdot})} - 2,8.$$

was found for the exp. electric potential.

This formula already shows that when we first determine the potential of Aluminium immersed in a non-aqueous solution of an Al-salt, and then in an equivalent solution of an Al-salt + a Hg-salt, the direction of the potential change will depend on which quantity has decreased more, L_M or $(M_L^{\nu\cdot})$ [2]).

If $(Al_L^{\cdot\cdot\cdot})$ has decreased more, the potential will be more negative, if on the other hand (L_{Al}) has decreased more, the potential will have become less negative.

It is evident that it is also possible to follow the opposite course;

---

[1]) These Proc. XXI, N°. 4, p. 562.

[2]) L_m is now not a constant quantity, but decreases with greater mercury content.

electrodes with different quantities of mercury may be made beforehand, and then they may be immersed in a non-aqueous solution of an aluminium salt, to determine the electric potential.

Without entering here further into the method of experimenting, we will state already now that the coexisting electrolyte is always relatively richer in aluminium than the mixed crystal, so that it could be deduced with certainty from this that the E,X-figure of the system Al-Hg belongs to the second type.

But what is remarkable is that though in virtue of the concentration of the coexisting phases it would be expected that the potential of the Aluminium in an Al-salt solution becomes less negative on addition of a little of a mercury salt, just the reverse takes place, and even to a very considerable degree.

This exceedingly remarkable phenomenon shows that, as was already pre-supposed before, the mercury dissolved in aluminium is a catalyst for the internal conversions in the aluminium. The metal magnesium, which is being examined by Mr. BECK, behaves in an analogous way, but the effects are less.

In a following communication we shall enter more deeply into the interesting phenomenon mentioned here.

Laboratory for General and Inorganic Chemistry of the University.

Amsterdam, June 1920.

Chemistry. — "*The Existence of Hydrates in Aqueous Solutions*". By Prof. A. SMITS, L. V. D. LANDE, and P. BOUMAN. (Communicated by Prof. P. ZEEMAN).

(Communicated at the meeting of December 18, 1920).

Since the research "On Retrogressive Melting-Point Lines" [1]), in which an indirect proof was given for the presence of hydrated Na_2SO_4 molecules in the aqueous solution, attemps have now and then be made to find other methods, which might be able to give an answer to the question whether formation of hydrates takes place in the aqueous solution, also in cases of melting-point lines with normal courses.

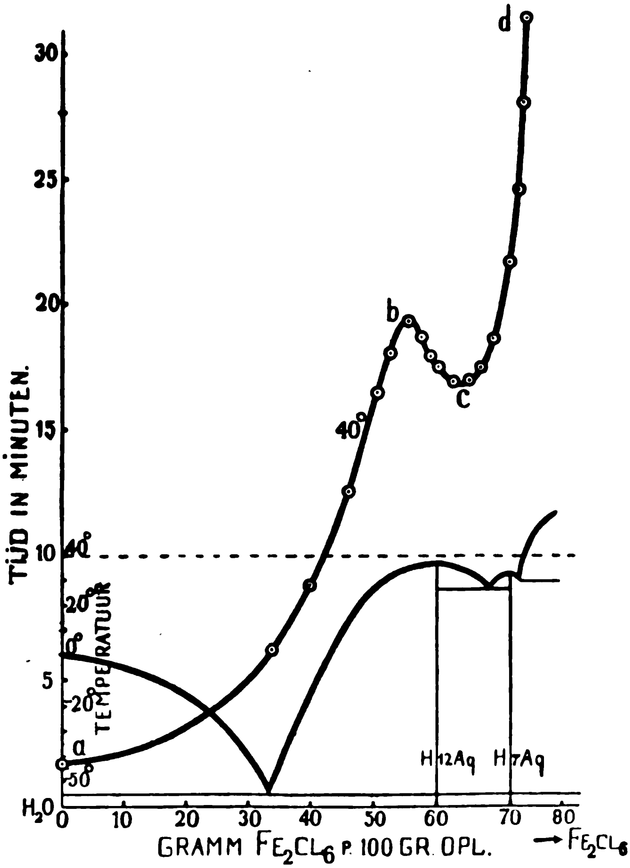


Fig. 1.

[1]) SMITS, These Proc. Vol. 14 p. 170 (1911).

Thus at constant temperature different properties of aqueous solutions of hydrate-forming substances, as e.g. spec. gravity, surface tension, refraction etc. were studied as function of the concentration, in which curves were obtained, which on the whole taught little or nothing of importance. Also the determination of the viscosity for a few systems yielded at first but inconclusive results; in the conviction, however, that nevertheless this method promised most on the whole for the end we had in view, the investigation was continued with the favourable result that in a few cases curves were obtained which very convincingly pleaded for the existence of hydrates in the aqueous solutions.

The system $H_2O—FeCl_3$ was chosen, of which part of the melting-point figure, as it was studied by BAKHUIS ROOZEBOOM, has been reproduced in Fig. 1.

The plan was to carry out the experiment at the temperature of 40°, because then, passing very close over the top of the compound

conc. in weight % Fe_2Cl_6	(Times of outflow in minutes).
0	1.70
33.50	6.15
39.58	8.70
46.13	12.50
50.19	16.45
52.57	18.05
55.39	19.32
57.45	18.66
58.69	17.92
60.18	17.50
62.44	16.90
65.09	16.95
66.84	17.50
68.45	18.60
71.06	21.73
72.88	24.63
73.83	28.17
74.12	31.80 (This solution is supersaturate).

Fe_2Cl_6 12 aq. (melting point 37°), and over that of $FeCl_6$ 7 aq. (melting-point 32.5°) there is the greatest chance that in this neighbourhood the homogeneous solution will contain considerable hydrate concentrations. For at higher temperatures the hydrates will dissociate more strongly as a rule, hence their concentrations will decrease.

The times of outflow found at 40° are recorded in the table of the preceding page.

When these results are graphically represented, as has been done in Fig. 1, we get a curve of viscosity exhibiting a very pronounced maximum and minimum, lying on the left and the right of the concentration of the hydrate with 12 aq.

This peculiar shape of the viscosity curve must in my opinion be interpreted in the following way.

When no hydrates were formed in the solution, the viscosity of the solution with the Fe_2Cl_6-concentration would increase in an ever greater degree, and in the end very high values would occur, because the viscosity of supercooled liquid Fe_2Cl_6 at 40° will be exceedingly great.

At *b* a decrease of the viscosity is now found here, which in my opinion must be ascribed to the increasing hydrate concentration. When only the hydrate Fe_2Cl_6 12 aq conld exist in solution, it was to be expected that the minimum would lie near the concentration of this hydrate.

At 40°, however, we pass not only over the top of the melting-point line of Fe_2Cl_6 12 H_2O, but also, though not at such a small distance, over that of the melting point line of Fe_2Cl_6 7 H_2O.

At the descent from *b* to *c* an appreciable increase of the concentration of the hydrate with 7 H_2O will, therefore, also take place, and when this gives also rise to a decrease in viscosity, the result will be that the viscosity curve, which must finally ascend again in consequence of the increase of the Fe_2Cl_6 concentration, presents a minimum, lying on the righthand side of the concentration of the hydrate with 12 aq.

After this result had been obtained, and the plan had been formed to examine also the system HO_2—SO_3, because this seemed to be particularly suitable for this purpose, it appeared that KNIETSCH [1]), who studied this system from different points of view, also gives a viscosity line which presents a close resemblance with the curve discussed above and of which up to now no notice had been taken.

[1]) Ber. 34, 4102 (1901).

This circumstance brought, however, no change in our plan, as it was our purpose to study the influence of the temperature on the shape of the curve of viscosity in the system $H_2O—SO_3$.

It is seen from the subjoined $T—X$ figure of the system $H_2O—SO_3$, which is not yet quite completed, that at 15° the tops of the compounds $H_2SO_4.H_2O$ and H_2SO_4 are passed at temperature distances of 6.47°, resp. 4.65°.

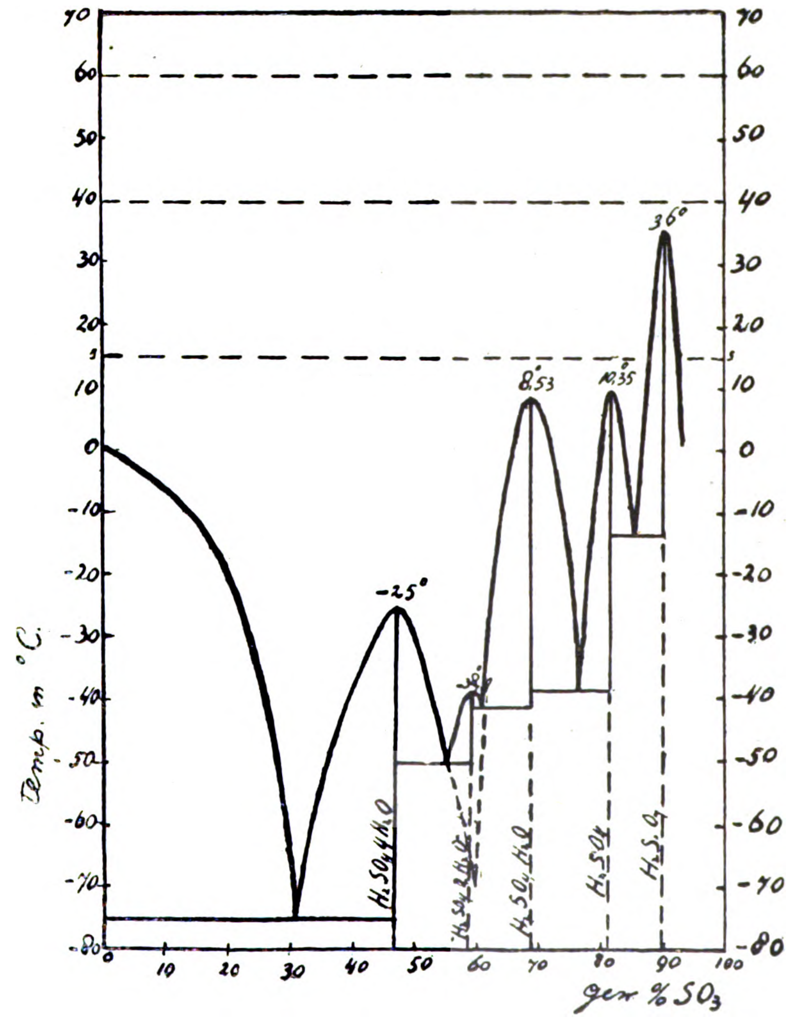


Fig. 2.

The compounds H_2SO_4 . 2 H_2O and H_2SO_4 . 4 H_2O are, indeed, also passed, but the melting-points of these compounds lie so far below 15° (—38.9° resp. —25°) that it is not to be expected that the

curve of viscosity will still give any information about these hydrates at 15°.

Also with a view to this the concentrations were examined between 61 and 83 weight % SO_3.

The investigation was carried out at three different temperatures, viz. 15°, 40°, and 60° with the following result:

weight % SO_3	Time of outflow at 15°	Time of outflow at 40°	Time of outflow at 60°
88,84	5 min. 27 1/5 sec.	2 min. 21 1/5 sec.	1 min. 20 4/5 sec.
82,13		2 „ 15 3/5 „	1 „ 18 4/5 „
81,72	4 „ 6 1/5 „		
81,55	4 „ 4 2/5 „	2 „ 10 2/5 „	1 „ 18 1/5 „
81,42	3 „ 56 4/5 „		
80,53	3 „ 56 4/5 „	2 „ 2/5 „	1 „ 15 3/5 „
77,55		1 „ 41 3/5 „	1 „ 7 2/5 „
76,27	4 „ 1 2/5 „		
75,72		1 „ 51 2/5 „	1 „ 7 4/5 „
75,33	4 „ 8 3/5 „		
75,19	4 „ 14 2/5 „		
75,03		1 „ 56 „	1 „ 10 1/5 „
73,96		1 „ 57 1/5 „	1 „ 10 3/5 „
72,69		1 „ 57 1/5 „	1 „ 10 4/5 „
72,08	4 „ 42 2/5 „		
69,72		2 „ 1 2/5 „	1 „ 10 2/5 „
69,60	4 „ 35 „		
64,78		1 „ 43 3/5 „	1 „ 2 3/5 „
63,76	3 „ 27 3/5 „		
63,28		1 „ 36 „	59 1/5 „

Fig. 3 represents these results graphically. It shows that the curve of viscosity at 15° really has the same shape as that of the system H_2O—Fe_2Cl_6 at 40°. The curves found at 40° and 60° show further that the peculiar character of the shape of the curve of viscosity

becomes less and less pronounced at higher temperature, which can be accounted for by the increasing dissociation of the hydrates on increase of temperature.

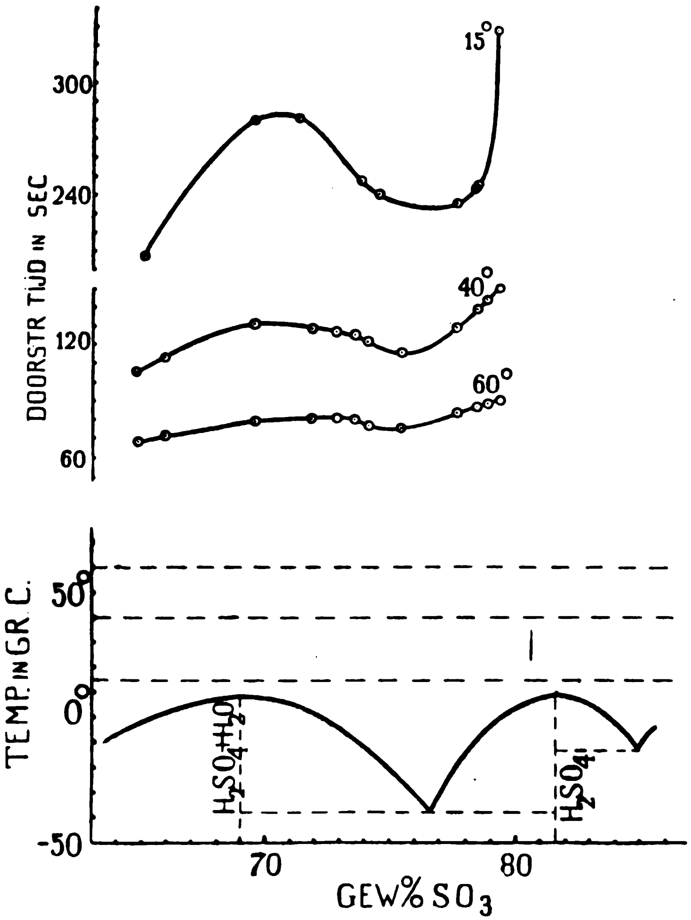


Fig. 3.

Laboratory of General and Inorganic Chemistry of the University.

Amsterdam, December 1920.

Chemistry. — "*The Electromotive Behaviour of Magnesium*". I. By Prof. A. SMITS and R. PH. BECK. (Communicated by Prof. P. ZEEMAN.)

(Communicated at the meeting of Dec. 18, 1920).

As was stated before magnesium closely resembles aluminium in its electromotive behaviour. Also the pure magnesium is as a rule a state disturbed in a noble direction, which reaches internal equilibrium by the absorption of small quantities of mercury; but in this case with magnesium these phenomena are weaker than with aluminium.

Our purpose was to examine the electromotive behaviour of mixtures of magnesium and mercury of different concentration in order to be able to set forth still more clearly the particular influence of small quantities of mercury.

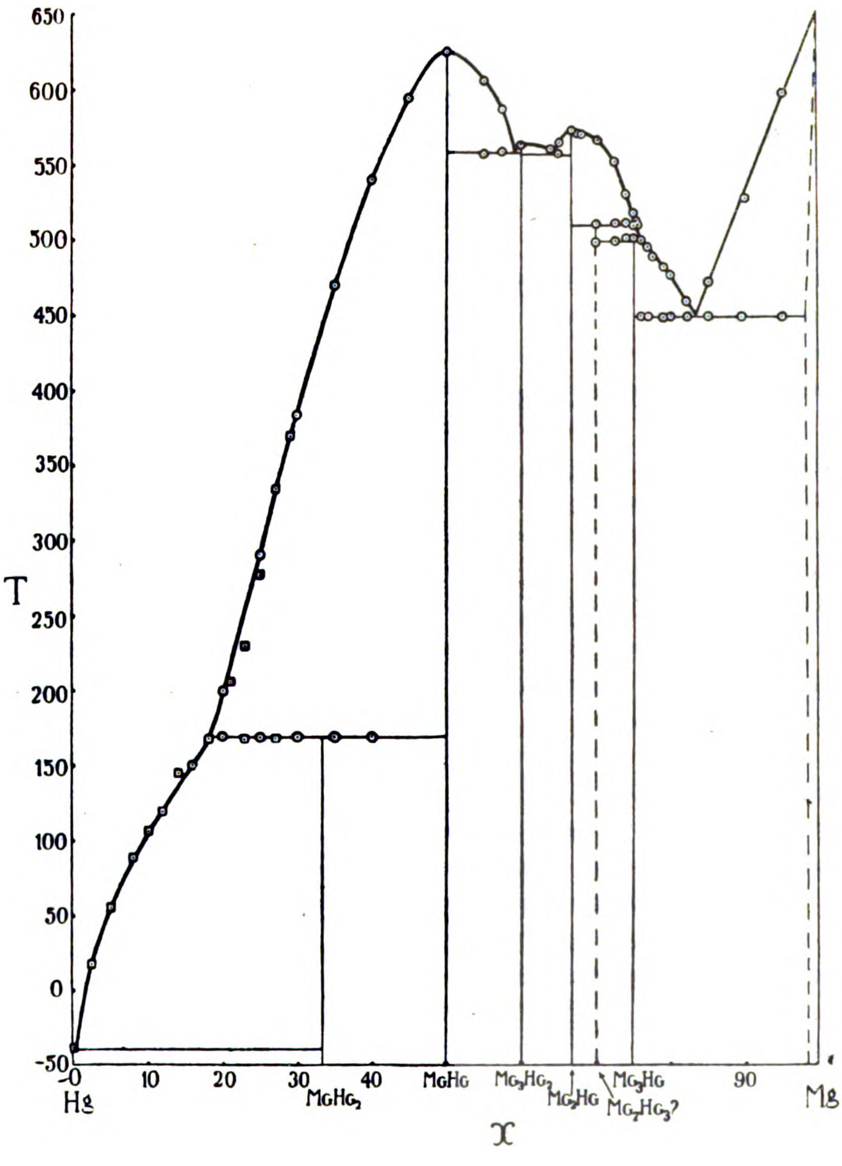
Before proceeding to this research, it was desirable to determine the melting-point diagram of the system Mg—Hg, which investigation was attended with several difficulties, which we will, however, not discuss here. The result to which this research led, is represented in the adjoined T,X-figure. It must be pointed out here that this system had already been examined on the Hg-side by L. CAMBI and G. SPERONI[1]), but that this research had been discontinued at the very point where the difficulties set in, and the system becomes most interesting.

We see from our diagram, which represents the situation of the melting-point lines under the varying vapour pressure, that the system Mg—Hg is very complicated, and contains several compounds.

In order to ascertain what corrections must be applied in the concentration, in connection with the mercury in the vapour phase, the determination of the vapour tension of the different mixtures was undertaken, in which use was made of a glass spring indicator. This investigation, which is now being continued with the mixtures rich in mercury, however, yielded the result that for mixtures from 0 to 50 at. °/₀ Hg the vapour tensions are very small, even up to the final temperatures of fusion.

[1]) Atti della R. Accad. dei Lincei **24**, 734 (1915).

After the T,X-diagram found had given something of a general insight, the investigation of the electromotive behaviour of Mg—Hg-



mixtures was started, the results of which will be communicated in a following publication.

Laboratory of General and Inorganic Chemistry of the University.

Amsterdam, December 13 1920.

Chemistry. — "*A Thermo-electrical Differential Method for the Determination of Transition Points of Metals at Comparatively Low Temperatures*". By Prof. A. SMITS and J. SPUYMAN. (Communicated by Prof. P. ZEEMAN).

(Communicated at the meeting of December 18, 1920).

This paper should be considered as a continuation of the publication "The Thermo-electrical Determination of Transition Points I" [1]). In the latter paper it was already stated that the very favourable results obtained by us on application of the thermo-electrical method in the investigation of the combination, iron-tin, and copper-tin, induced us to examine also other important metals in the same way, starting with copper. Our purpose was to examine in the first place whether copper shows a point of transition in the neighbourhood of 70°. The arrangement which we used at first for this purpose is represented in outline in the subjoined figure.

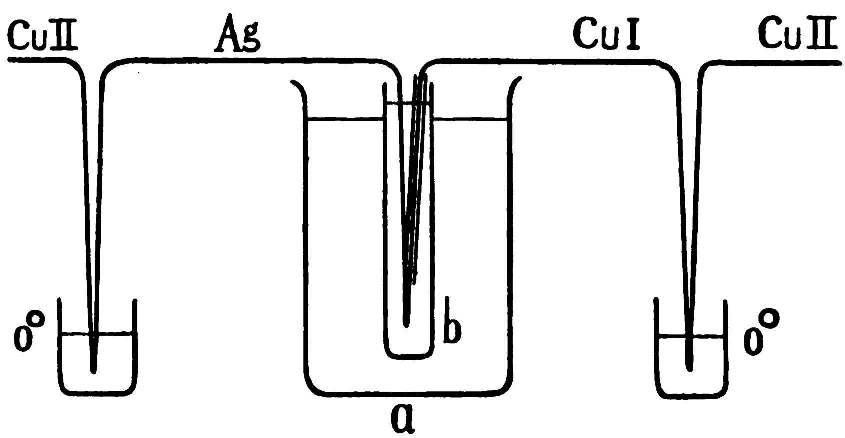


Fig. 1.

The combination pure silver-pure copper (Cu I), of which the copper wire is passed through a glass capillary, was placed in a wider tube *b*, filled with an electrolyte, with which silver and copper were in electromotive equilibrium. For this purpose we took a solution of copper-sulphate, because the electrolyte, which is in

[1]) These Proc. Vol. XXIII, 5, p. 687.

electromotive equilibrium with copper and silver, contains but exceedingly little silver.

As was shown before [1]) there then exists between the copper and the silver wire still a potential difference equal to the Volta-effect. Now the tube *b* was placed in a thermostat *a* and the two soldering places Ag—CuII and Cu I—Cu II in melting ice. After the thermostat had been kept constant at a definite temperature for some time, the electromotive force of the circuit was measured. This measurement, ranging over a temperature interval of from 40°—80°, gave no indications of a discontinuity in the neighbourhood of 60°, after which it was resolved to apply a more accurate measurement, which we found in a method, which we shall call the *differential method.* The arrangement of this differential method is given in outline below in Fig. 2.

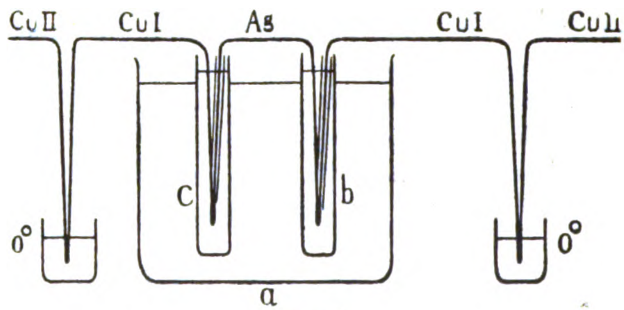


Fig. 2.

The two extremities of the silver wire, which are soldered to wires of pure copper, are now *both* in the thermostat *a*, but the capillary *b* is filled with a solution of copper sulphate, and *c* contains anhydrous paraffin oil. Both solutions are under a nitrogen atmosphere [2]).

The consideration that led to this arrangement, is as follows. So long as no transition point is reached on change of temperature, the electromotive force of the circuit will be very small, hence it will change but exceedingly little with the temperature in the thermostat; but when a transition temperature of silver or copper is passed, it is probable that the thread of the *differential thermoelement,* which is in electromotive equilibrium with the electrolyte, is

[1]) These Proc. Vol. XXI, 3, p. 386.

[2]) In reality the tubes *b* and *c*, which contain the metal wires immersed in a liquid, have been first exhausted of air, then filled with nitrogen, and finally fused to.

sooner transformed than the thread of the same combination which is immersed in paraffin oil [1]).

If this is actually the case, the electromotive force of the differential thermo-element will undergo a rather sudden change, and the initial point of this change will correspond with the transition point of one of the metals of the element. On application of this simple and very sensitive differential method it was found that the electromotive force was practically zero throughout the temperature range that we examined, for it was less than 0,001 milli Volt. We did not consider the investigation as finished then, and we once more examined the same differential thermo-element Cu—Ag—Cu, taking care that one pair of wires was not only immersed in a solution of $CuSO_4$, but in such a way that the etched copper wire, the soldering place, and a small piece of the silver wire were in contact with powdery copper.

This procedure in our experiment, however, did not bring about the slightest change in the results, for now too the electromotive force of the circuit between 40° and 80° remained certainly smaller than 0,001 milli-Volt.

We will still mention here that the times of observation have been taken very long here on purpose, and amount to 2×24 hours. Notwithstanding this prolonged heating at temperatures above and below those at which dilatometrically indications were found for a transition point, the electromotive force of the differential thermo-element appeared to be smaller than 0,001 milli-Volt. In the first place this result shows that both the silver wire and the copper wire were very homogeneous, and in the second place that neither the silver wire nor the etched copper wire, though they were in contact with a solution of $CuSO_4$ and with fine copper powder over a length of 20 cm., showed appreciable transformation.

The differential method discussed here is now being applied to the other important metals. In the following paper also the theory will be discussed.

Laboratory of General and Inorganic Chemistry of the University.

Amsterdam, December 13, 1920.

[1]) For COHEN found that contact with an electrolyte has an accelerating effect on the transformation of one metal modification into another. This accelerating action must probably be ascribed to this, that when the stable modification has appeared only in one point, this gives rise to local currents, which greatly promote the transformation.

Physics. — "*Discontinuities in the Magnetisation*" II. By Dr. BALTH. VAN DER POL Jr. (Communicated by Prof. H. A. LORENTZ).

(Communicated at the meeting of Nov. 27, 1920).

In a previous paper[1]) under the same title, some preliminary experiments were described about the discontinuities in the magnetic induction which occur under certain circumstances in ferro magnetic bodies, when the magnetic force is varied quite continuously. The simplest method to observe these discontinuities is to connect a coil surrounding the ferromagnetic substance to be investigated to a telephone receiver (through a triode amplifier or even without one) in which the discontinuities in the induction can be observed, if one brings a permanent magnet slowly near the coil. In our previous paper the attention has already been called to the fact that these discontinuities can also be made visible galvanometrically and that the phenomenon is especially conspicuous with nickelsteel.

Some further investigations on these discontinuities form the contents of the present paper.

In order to determine on which part of a magnetic cycle the discontinuities occur, some hysteresis curves were obtained by means of an improvised magnetometer. Figure 1 relates to the above mentioned nickelsteel which has the same thermal expansion coefficient as glass. The cylindrical wire used has a diameter of 1.98 millimetres

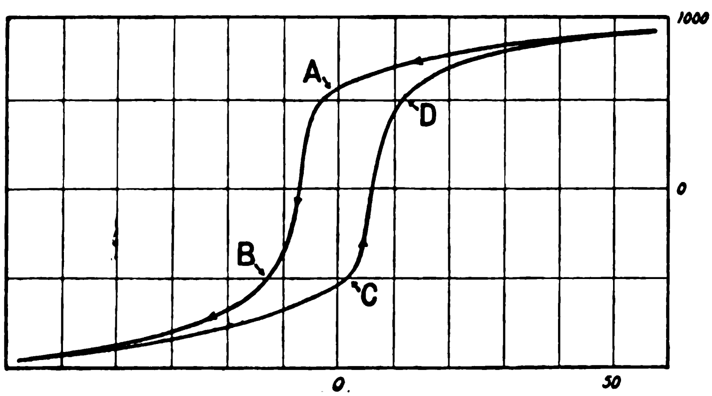


Fig. 1.

1) These Proceedings Vol. XXIII, N^0^. 4, page 637 (1920).

and is 343 millimetres long. Figure 2 shows the hysteresis curve of a soft iron wire (diametre 1.82 millimetres, length 349 millimetres).

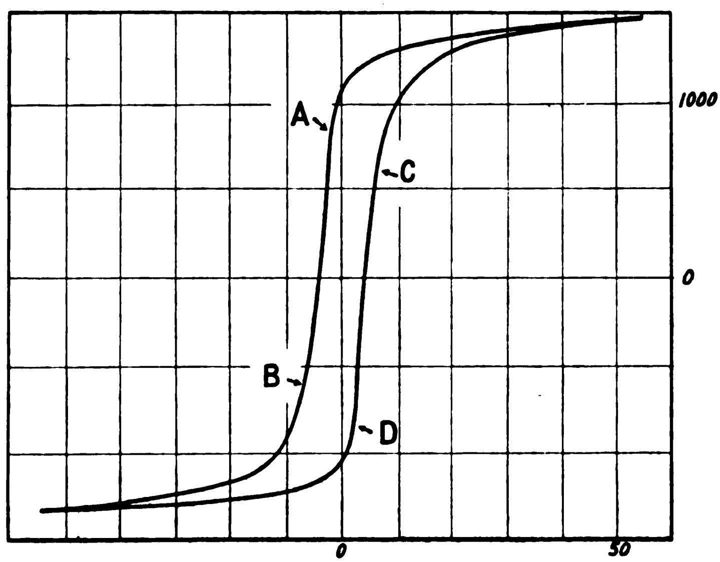


Fig. 2.

In order to obtain (for the determination of the part where the discontinuities occur) a very gradual increase of the magnetising current, two copper plates in series with the magnetising solenoid were placed next to each other in a glass jar. With the aid of a siphon arrangement this jar was gradually filled with a copper-sulphate solution, which caused a decrease of the resistance and a rather continuous increase of the current through it. After the *Ni Fe* or *Fe* wire was put in the magnetising solenoid together with a second induction solenoid closely surrounding the wire, we determined with the aid of a telephone connected to the induction solenoid through a three stage triode amplifier, the point on the hysteresis curve where the discontinuities for the first time occur and the spot where they disappear again.

It appeared that the discontinuities in the magnetic induction only occur on the steep parts of the hysteresis curves, viz. in figure 1 and 2 on the parts A—B and C—D. The part from saturation to where $H = 0$, and where therefore only the remanent magnetisation is left, is, at any rate as far as can be observed, quite continuous. However, as soon as H changes its sign and therefore the existing magnetisation is reduced, the discontinuities begin to occur. They are further audible over a certain distance, which however does not reach quite to saturation. The rustling noise in the telephones, as

soon as it appears for the first time at A and C (fig. 1) or D (fig. 2) is still very weak at these points. Thereupon its intensity increases and at B and D it gradually disappears again.

The electrolytic current control mentioned above, is not sufficiently continuous for galvanometric observations. By using the following method, however, we could obtain a very continuous current variation, though the total magnitude of it was limited.

The magnetising coil M [fig. 3] is put in series with a precision

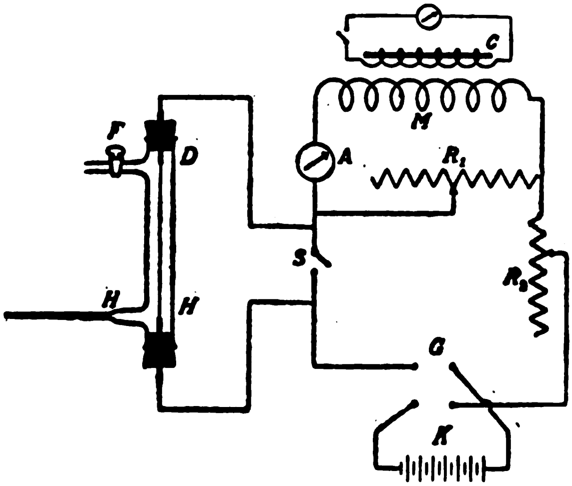


Fig. 3.

ammeter A, rheostat R_2, key S, commutator G, and battery K. Moreover M is shunted by the variable resistance R_1, while S is put in parallel to a platinum wire DH, the latter being placed in a glass tube. This tube has two connections. The one at D goes through the tap F to a Gaedepump the other at H to a long capillary glass tube. After S is closed, we can, by variation of R_1 and R_2, and with the aid of the commutator G submit the ferromagnetic wire C a few times to a complete magnetic cycle. For the sake of clearness the wire C has been drawn in fig. 3 outside the coil M, though actually of course its place is inside. Thereupon, after the pump has been working for some time, the current through M is made zero and the key S is opened. With the aid of the resistances the platinum wire DH is brought to a dull red heat and the magnetising current is brought to a value which brings the ferro magnetic substance on the steep part of the hysteresis curve. After the tap has been closed, the pressure of the air in DH slowly increases; due to the leakage through the capillary tube, the platinum wire gets colder, and the magnetising current through M increases very

BALTH. VAN DER POL Jr.: "Discontinuities in the Magnetisation."

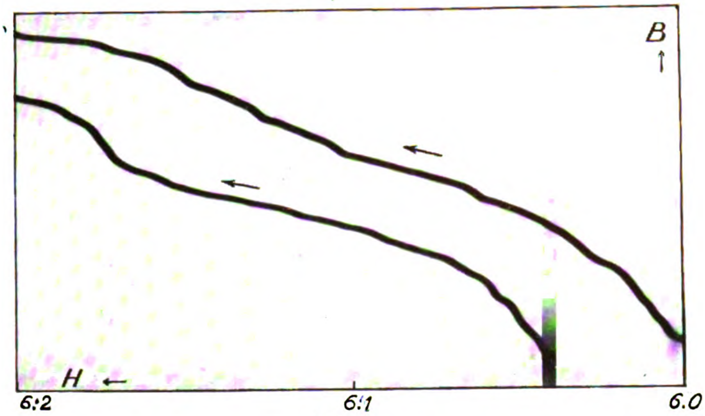


Fig. 4.

Fig. 5.

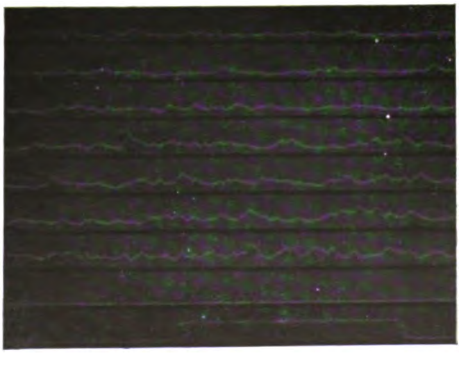


Fig. 6.

continuously. By using this arrangement the magnetising current could in one minute very gradually be increased e.g. from 0.200 Amp. to 0.230 Amp.

The discontinuities which occur in the magnetic induction in the *NiFe* wire, the magnetic field being controlled by the thermal method, are shown in fig. 4, which gives two photographic reproductions on the same plate of the deflection of an ordinary SIEMENS and HALSKE moving coil galvanometer. This galvanometer was connected (without series resistance or triode amplifier) to a long secondary coil of 16500 turns (566 Ohm), which narrowly surrounded the *NiFe* wire over its total length. In 6 seconds the field increased from 6.0 to 6.2 Gauss, it having been brought before a few times alternately to $\pm$ 36 Gauss. As the galvanometer was very damped, we can consider the curves of fig. 4 to approximately represent a small part of the hysteresis curve. (See appendix). The discontinuities in the induction are very well visible; the upper curve e.g. shows 6 sudden increases of the B.

Obviously the biggest discontinuities only are registered by a moving coil galvanometer. However, the phenomenon can be followed more in detail by using an EINTHOVEN string galvanometer. Various photograms directly on bromide paper were obtained by using the latter galvanometer, the bromide paper having been mounted on a revolving cylinder, every point of which described a screw curve with a pitch of 1 centimetre.

Figures 5 and 6 are parts of larger photograms. Figure 5 is obtained with the above mentioned nickelsteel wire and figure 6 with a soft iron wire. In both cases the 566 ohm induction solenoid surrounded the ferromagneticum over its total length, and in both cases the sensitivity of the galvanometer was the same, viz. such, that a deflection of 1,0 centimetre (the distances between the subsequent pitches) corresponds to $0{,}18.10^{-6}$ Amp. The width of either photogram corresponds to one second. The photos were taken after (in both cases) the material was brought on the steep part of the hysteresis curve, where the discontinuities are most frequent. The continuous increase of the field was obtained with the thermal current control of fig. 3 such, that the field, when using nickelsteel, increased in 37 seconds from 6.0 to 6.7 Gauss and, when using iron, in 60 seconds from 4,8 to 5,5 Gauss. In the photograms the time runs from left to right and the bottom part was described first. The time in which the string reaches its new equilibrium position when a constant current is made and broken, is shown at the bottom of the photograms.

The considerable difference between the discontinuities in the iron and nickelsteel is very well marked in these figures. When using nickelsteel the clapses are much stronger; the photograms therefore confirm the observations made before with triode amplifier and telephone. Apart from giving the intensity, the photograms also enable us to count approximately the number of discontinuities.

Thus for nickelsteel 1554 clapses were found for a change of field from 6.0 to 6.7 Gauss, and for iron 1546 clapses for a change of field from 4.8 to 5.5 Gauss. Moreover the region on the hysteresis curve where the discontinuities occur (see fig. 1 and 2) being known, as well as the fact that the number of clapses at A and C in the figures mentioned does not reach its maximum value at once, we can approximately estimate the number of clapses during a complete reversal of the magnesitation. Thus we found:

for nickelsteel wire (dia. 1.98 mm., length 343 mm.) 5000 clapses,
soft iron wire (dia. 1,82 mm., length 349 mm.) 6500 clapses.

We did not examine more in detail the dependency of the number of discontinuities upon the diameter and the length of the ferromagnetic wire. One should be inclined to expect this number to be proportional to the volume or, if long iron filaments are magnetised as a whole, to the diameter; but on the other hand eddy currents will occur in the outer layers, if an ironcrystal in the inner part is suddenly remagnetised, which eddy currents may be expected to reduce the suddenness of the currents in the induction solenoid, and also the rustling noise in the telephone receiver. Very likely, owing to these eddy currents, jumps of most different values appear on the photograms, though possibly one definite intensity of magnetic moment of the crystal is prevalent.

Finally, the assumption that, under certain circumstances, the magnetic moments of long filaments suddenly change as a whole (see l.c.) was found confirmed by the following experiment.

The nickelsteel wire was brought into the magnetising solenoid, and the current was controlled thermally in the way described above. The nickelsteel core was surrounded by two small flat coils of 3 mm. axial length, on either of these coils 2600 turns of enamelled copper wire of 0,05 mm. had been wound (the inner diameter of the coil being 3,4 mm., the outer one 13.0 mm.). The distance between the two coils could be given any arbitrary value. Either coil was connected to one of two identical moving coil galvanometers. Both the galvanometer spots had the form of a line inclined at 45° to the horizon. They were projected on the same scale and formed together a cross. [X]

When now the coils were put immediately near each other, the jumps in the galvanometer deflections occurred simultaneously, and the galvanometer spots, in the form of a cross, moved as a whole. With distances between the coils bigger than 10 cm., the average deflections of the galvanometer were the same, but the jumps were quite incoherent. However, even up to a distance between the coils of 7 centimeters, the bigger jumps occurred quite simultaneously, which shows, that in the nickelsteel used, crystals exist, or at any rate groups of crystals, which have to be considered as a magnetic unity, having a length of 7 cm. Possibly these long crystals are formed during the drawing process to which the wire is submitted while being manufactured. An effort also to submit annealed or electrolytic iron to this test did not succeed, as the discontinuities in iron are too small to be detected with a moving coil galvanometer (a double-string galvanometer was not at our disposal).

Appendix. Interpretation of the galvanometer curves.

If in the galvanometer circuit with a total resistance r an E.M.F. $\frac{dN}{dt}$ is induced (by varying the flux N through the induction solenoid), we have the following equations:

$$\left.\begin{aligned} ir + L\frac{di}{dt} + K\frac{d\theta}{dt} &= \frac{dN}{dt} \\ m\frac{d^2\theta}{dt^2} + f\frac{d\theta}{dt} + \mu\,\theta &= Ki \end{aligned}\right|$$

where θ is the deflection of the system (moving coil or string), μ the elastic force or couple, f a friction constant, m the mass or moment of inertia of the system, and $-K\frac{d\theta}{dt}$ the E. M. F. induced for a change of deflection $\frac{d\theta}{dt}$.

Neglecting the selfinductance L, the elimination of i yields the following equation for the deflection

$$m\frac{d^2\theta}{dt^2} + \left(f + \frac{K^2}{r}\right)\frac{d\theta}{dt} + \mu\theta = \frac{K}{r}\frac{dN}{dt} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The term $\frac{K^2}{r}$ represents in the usual way the electromagnetic damping of the galvanometer.

If we put (1) in the form

$$\frac{d^2\theta}{dt^2} + 2a\frac{d\theta}{dt} + \omega^2\theta = \frac{dN'}{dt}$$

where

$$N' = \frac{K}{mr} N$$

$$2a = \frac{f + \frac{K^2}{r}}{m}$$

$$\omega^2 = \frac{\mu}{m}$$

we find, with the method of variation of the constants and with the initial conditions $\left(\text{for } t = 0,\ \theta = \frac{d\theta}{dt} = 0\right)$:

$$\theta = \frac{1}{(x_1 - x_2)}\left[e^{x_1 t}\int_0^t e^{-x_1 t}\frac{dN'}{dt}\,dt - e^{x_2 t}\int_0^t e^{-x_2 t}\frac{dN'}{dt}\,dt\right] \quad . \quad . \quad . \quad (2)$$

where x_1 and x_2 are the (negative) roots of

$$x^2 + 2a\,x + \omega^2 = 0.$$

If we now suppose that at the time $t = 0$, $N = 0$, we have

$$e^{x_1 t}\int_0^t e^{-x_1 t}\frac{dN'}{dt}\,dt = N_t' + x_1\,e^{x_1 t}\int_0^t N'\,e^{-x_1 t}\,dt$$

Hence (2) becomes

$$\theta = \frac{1}{(x_1 - x_2)}\left[x_1\,e^{x_1 t}\int_0^t N'\,e^{-x_1 t}\,dt - x_2\,e^{x_2 t}\int_0^t N'\,e^{-x_2 t}\,dt\right]$$

Further, if we suppose the flux N' to increase with jumps, so that the discontinuous increase at the time $t = 0$ is $\Delta_0 N'$, at the time $t = t_1$ is $\Delta_1 N'$ etc. and if we consider N' to remain constant during the intervals between the discontinuities, we may in the following way replace the integrals by summations

$$x_1 e^{x_1 t}\int_0^t N' e^{-x_1 t}\,dt =$$

$$-\Delta_0 N'(1 - e^{x_1 t}) - \Delta_1 N'(1 - e^{x_1 (t - t_1)}) - \Delta_2 N'(1 - e^{x_1 (t - t_2)}) \ldots .$$

The deflection θ can now be written as:

$$\theta = \frac{1}{(x_1 - x_2)}\sum_n \Delta_n N'\{e^{x_1 (t - t_n)} - e^{x_2 (t - t_n)}\} \quad . \quad . \quad . \quad . \quad (3)$$

but with the restriction, that up to the time $t = t_k$ the first k terms only of the series are to be taken. Hence, with each discontinuity the number of terms of the series expression for θ is increased by one.

For the velocity $\frac{d\theta}{dt}$ we get

$$\frac{d\theta}{dt} = \frac{1}{(x_1 - x_2)} \sum_n \Delta_n N' \{ x_1 e^{x_1 (t-t_n)} - x_2 e^{x_2 (t-t_n)} \}$$

Immediately after the moment t_k the velocity $\frac{d\theta}{dt}$ is therefore increased by the amount

$$\frac{1}{(x_1 - x_2)} \Delta_k N' (x_1 - x_2) = \Delta_k N'$$

Therefore, the effect of the k'^{th} discontinuity is, that the velocity existing just before it occurred, is increased by the amount

$$\Delta_k N' = \frac{K}{rm} \Delta_k N$$

Hence every discontinuity in the magnetic induction causes a proportional increase of the velocity of the galvanometer system.

Finally let us consider what becomes of the deflection θ if, as was the case with the moving coil galvanometer, the galvanometer is greatly damped and therefore

$$\omega^2 \ll \alpha^2.$$

In this case a deflection, once obtained, decreases only very slowly (as with the Grassot-fluxmeter).

Now the roots x_1 and x_2 are approximately

$$x_{1,2} = -\alpha \pm \alpha \sqrt{1 - \frac{\omega^2}{\alpha^2}} \doteq -\alpha \pm \alpha \left(1 - \frac{1}{2}\frac{\omega^2}{\alpha^2}\right)$$

hence

$$x_1 \doteq -\frac{1}{2}\frac{\omega^2}{\alpha}$$

$$x_2 \doteq -2\alpha$$

The momentary deflection now becomes

$$\theta \doteq \frac{1}{2\alpha} \sum_n \Delta_n N' \left\{ e^{-\frac{1}{2}\frac{\omega^2}{\alpha}(t-t_n)} - e^{-2\alpha (t-t_n)} \right\}$$

If the experiment does not last too long, we have for all values to be considered

$$e^{-\frac{1}{2}\frac{\omega^2}{\alpha}(t-t_n)} \doteq 1.$$

The other terms

$$- e^{-2\alpha(t-t_n)}$$

every time disappear very soon after each discontinuity and produce a small bend at the upper part of each jump. See e.g. fig. 4. *The more the galvanometer is damped,* the greater therefore α, the smaller these bends are and *the quicker the galvanometer is able to follow the sudden changes of the flux.*

If we disregard these small bends the deflection of the greatly damped galvanometer is simply given by

$$\theta = \frac{1}{2\alpha} \Sigma \Delta_n N' = \frac{N'}{2\alpha}$$

and the galvanometer deflections are at any moment proportional to the flux going through the induction solenoid [1]). If the H therefore increases proportionally to the time, the deflection of a greatly damped galvanometer accurately describes a hysteresis curve. This extreme case can be better approximated with a moving coil galvanometer than with a string instrument. The curve described by the image of the string galvanometer representing the solution of (1) from which the function N has to be found back, does not lead to such a simple interpretation.

Haarlem. *Physical Laboratory,* TEYLER'S *Institute.*

[1]) This relation is obtained at once from (1) by neglecting the first and third term of the left hand member.

Physics. — "*Note on the paramagnetism of solids.*" By Prof. P. Ehrenfest.

(Communicated at the meeting of December 18, 1920).

1. The object of this note is to show that the validity of the Curie-Langevin law for the susceptibility of *solid* paramagnetic substances may be arrived at on a different theoretical basis from that discussed by Weiss[1]) Stern[2]) and Lenz[3]) (comp. section 3 of this paper), namely on the following assumption: the atoms or molecules of a solid paramagnetic substance contain electrons circulating in definite, practically fixed orbits of lowest quantum-number (which we shall call "rest-orbits"), so that, in the absence of an external magnetic field, there is no perceptible difference in the energy of the two opposite (right and left) directions of circulation. It will be further assumed, that with a magnetic field H at a given temperature T the statistical distribution between right and left which corresponds to H and T[4]) will automatically establish itself. On these assumptions the Curie-Langevin law will be found to hold c.f. § 4), and, in the case of a crystal symmetry or of a crystalline powder of any crystalline structure, even with the correct numerical factor 3 in the denominator.

2. Langevin's theory explains the fact, that paramagnetic gases

---

[1]) P. Weiss, C. R. **156** (1913) 1674.

[2]) O. Stern. Z. f. Phys. **1** (1920) 147.

[3]) W. Lenz. Phys. Zschr. **21** (1920) 613.

[4]) If the motions of the electrons are not submitted to any limitations by means of quantum-conditions and if the law of thermal statistics is applied to all the degrees of freedom, the body is found to be *unmagnetic*; comp. H. A. Lorentz. Vortr. kinet. Theorie der Mat. u. Elektr. p. 188. (Teubner 1914); H. J. van Leeuwen, Vraagstukken uit de elektronentheorie van het magnetisme p. 54. [Dissertation Leiden 1919]. Accordingly in his theory of paramagnetism Langevin assumes, that the ampère currents, which represent the elementary magnets, are not subjected to thermal statistics in their cyclic co-ordinates. But when, as in our case, quantum conditions are introduced into the statistic scheme, the possibility of paramagnetism returns, even if not a single degree of freedom remains outside the scheme. To this circumstance my attention was drawn by Prof. N. Bohr in a conversation (1919).

and solutions of paramagnetic salts obey CURIE's law; in these cases, taking account of the smallness of

$$a = \frac{\mu H}{rT} \quad \ldots \ldots \ldots \ldots (1)$$

it gives at the limit the CURIE-LANGEVIN relation

$$\chi = \frac{N\mu^2}{3rT} \quad \ldots \ldots \ldots \ldots (2)$$

where χ = molair susceptibility; N = AVOGADRO's number $r = \frac{R}{N} =$ BOLTZMANN's constant, μ = magnetic moment of one molecule. Experiment shows that this relation also holds for certain solid paramagnetic substances. In particular KAMERLINGH ONNES has found that gadolinium sulphate follows the law exactly down to the lowest (Helium) temperatures. This is not what one would expect in view of the assumption underlying the derivation of the formula, namely that the orientation of the elementary magnets depends exclusively on: (1) the thermal motions, (2) the directing influence of the external field H. In solid (crystalline) bodies there must be additional forces of a different nature which also play a part in the orientation of the elementary magnets.

3. It follows from the papers by WEISS, STERN, and LENZ, that in the case of forces of that kind which depend on a crystalline structure the same relation is obtained, if only it is assumed that:

1. The potential energy Φ of the forces which try to keep the axis of an elementary magnet parallel to a definite direction R is centrally symmetrical, i. e. it is equal for any two opposite orientations of the elementary magnets.

2. When the temperature or the field changes, the statistical distribution of the orientations, which according to BOLTZMANN corresponds to the new values of T and H, and to Φ, actually establishes itself; this involves, that there is no retardation in the necessary reversal of the elementary magnets (false equilibrium).

4. Let us now consider a solid body containing electrons whose "rest-orbits" satisfy the conditions mentioned in § 1. For a particular electron let the magnetic moment of its orbit be μ and its projection on the direction of the field $\pm \mu \cos \vartheta_0$, according to whether it circulates to the right or left [1]). At the temperatures considered we

[1]) Strictly speaking μ itself depends on the orientation of the orbit relatively to the field H, since the velocity of the electron is affected by it. In a magnetic field it is not simply the *mechanical* momentum of the electron that is to be

can leave out of account the possibility of the electron jumping to an orbit of higher quantum-number. Now the times during which the electron moves to the right and to the left are in the ratio [1])

$$e^{a\cos\vartheta_0} : e^{-a\cos\vartheta_0} \qquad \left(a = \frac{\mu H}{rT}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and the time-average of the projection of its magnetic moment on the direction of H is therefore given by

$$\frac{\mu\cos\vartheta_0\, e^{a\cos\vartheta_0} - \mu\cos\vartheta_0\, e^{-a\cos\vartheta_0}}{e^{a\cos\vartheta_0} + e^{-a\cos\vartheta_0}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Since a is small we may put

$$e^{\pm a\cos\vartheta_0} \cong 1 \pm a\cos\vartheta_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

hence (4) becomes

$$\cos^2\vartheta_0 \frac{\mu^2 H}{rT} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and the susceptibility χ for N such electrons will be

$$\chi = \frac{N\mu^2}{rT}\overline{\cos^2\vartheta_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

the mean being taken over the possible orientations of the "rest-orbits" of the electrons. For a crystal of cubic symmetry or a powder of arbitrary crystalline structure we have obviously

$$\overline{\cos\vartheta_0} = \frac{1}{3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

whereby equation (2) is arrived at.

5. *Additional remarks.*

1. According to the above theory the Röntgen-reflection of a crystal would not be changed by magnetisation. By a very sensitive

---

taken equal to $\frac{h}{2\pi}$ or a multiple of it, but the *electro-kinetic* momentum of the electron (reduced to mechanical units) has to be added. However, even with a very high value of H, this term is small compared to the other. We may therefore neglect this *dia*magnetic action, depending on induction, just as LANGEVIN did in his fundamental theory.

[1]) In the power of e we must put the quantity which remains constant during a "collision". For an electron, which in a constant field H changes from a right-hand to a left-hand motion, this quantity is *not* the sum of the mechanical and electro-kinetic energies, but a kind of "ROUTH-*Function*" has to be taken. (Comp. Dissertation by H. J. VAN LEEUWEN. Leiden 1919, p.p. 11, 18, 52—54, an extract of which is soon to appear in the Journal de Physique). A simple calcution on this basis, with the approximation referred to in the previous footnote, gives the ratio (3).

null-method COMPTON and ROGNLEY [1]) have actually established the absence of any such effect for the (ferro-magnetic) crystals of magnetite.

2. In the theory of LANGEVIN-WEISS the rotational movement of the elementary magnets gives its own contribution to the kinetic energy and therefore to the specific heat, whereas in our theory the corresponding term does not occur. At first sight this appears strange, since the form of equation (2) seems to point to equipartition. But a similar result will always be obtained, in cases, where the lowest quantum-motions which are possible, possess a very small difference of energy $\Delta\varepsilon$ with respect to each other (in our case $2\mu H\cos\vartheta$ with a field H). In those cases there is always a range of temperatures T, where T is small enough for the higher quantum-orbits to be disregarded, and at the same time large enough with respect to $\Delta\varepsilon$.

3. Although the transition between the right- and left-hand motions requires an amount of energy small as compared to rT, still it may require the coincidence of favourable circumstances to bring about the corresponding reversal of the motion (moment of momentum). Since we are dealing with a quantic process, it is probably difficult to treat this question quantitatively. In general we may expect, that the corresponding retardations in the establishment of the magnetisation would show themselves most easily at very low temperatures and rapidly alternating fields [2]). (For light-vibrations χ is always $= 0$). They would for instance give rise to a kind of hysteresis and a corresponding development of heat, when gadolinium-sulphate is periodically magnetized in opposite directions.

Leiden. *Physical Department.*

---

[1]) A. H. COMPTON and O. ROGNLEY. Is the atom the ultimate magnetic particle? Phys. Review. **16** (1920) 464.

[2]) The possibility of this retardation was pointed out by LENZ in his address at Nauheim (l.c. p. 615) from the point of view of the sudden reversals of the magnetic atoms. Previously to this in the beginning of July 1920 the question was discussed by Prof. KAMERLINGH ONNES and me, both from the point of view of WEISS' theory and of the assumptions ol this paper, together with the possibility of testing it experimentally.

Palaeontology. — "*The identity of the genera Poloniella and Kloedenella.*" By Miss J. E. VAN VEEN. (Communicated by Prof. J. W. MOLL.)

(Communicated at the meeting of December 18, 1920).

In the year 1896 a treatise appeared by Prof. Dr. GÜRICH about the Palaeozoïcum of the Polish middle mountain range. In this treatise the author instituted the new genus *Poloniella* (4, p. 388) for a few carapaces and valves of formerly unknown Ostracoda, originating from the middle devonian Ostracoda marl of Dombrowa near Kielce. These remains he united into one species viz. *Poloniella devonica.*

Some twelve years afterwards the two American palaeonotologists Dr. ULRICH and Dr. BASSLER supplied a contribution to the knowledge of the *Beyrichiidae*. On this occasion the new genus *Kloedenella* (8, p. 317) was founded also, under which group they intended to bring together eight species at the least.

In 1914 Prof. Dr. BONNEMA (2, p. 1087; 3, p. 1105) was able to amplify the characteristics of the genus *Kloedenella* as given by ULRICH and BASSLER as a result of his investigation into the nature of the Ostracod, which Dr. AUREL KRAUSE formerly described under the name *Beyrichia hieroglyphica.*

In comparing what the above mentioned authors have said about the genera *Poloniella* and *Kloedenella*, it is obvious that the latter are identical. It should however be observed that what BONNEMA takes to be the anterior part of the carapace — and rightly in my opinion — is considered the posterior part by the others. As a natural result the valve, which is the left one, according to BONNEMA, is called the right one by the others.

Thus BONNEMA found as the most characteristic feature of the genus *Kloedenella* that the right valve before the straight part of the hinge line has a notch in which a process of the left valve fits. (fig. 3).

In the genus *Poloniella* a similar connection of the valves seems to be present. GÜRICH does not mention this fact emphatically, but as he writes: "Ganz am vorderen Ende jedoch tritt der linke Saum wieder zurück und auf der hinteren Kantenhälfte springt der rechte Saum sogar stark über", I should conclude from this that it occurs here also.

Besides BONNEMA had found that in the genus *Kloedenella* the right valve overlaps the left one at the hinge line, whereas the opposite is the case with the free edges. In accordance herewith, GÜRICH writes ".... greift am Schlossrande die linke Klappe in einer gradlinigen Leiste vorspringend über den entsprechenden Rand

der rechten Klappe" and "Längs des Bauch-, Vorder- und Hinterrandes greift die rechte Klappe über...."

The identity of the two genera, however, appears much more clearly from the figures GÜRICH gives of his *Poloniella devonica* and BONNEMA of *Kloedenella hieroglyphica* and which are partly copied on the accompanying plate. If we compare Fig. 1 of *Poloniella devonica* with Fig. 2 of *Kloedenella hieroglyphica* and Fig. 7 of the former with Fig. 8 of the latter, it appears that also of the former the right valve has undoubtedly a notch in which a process of the left valve fits. The fact that GÜRICH represents complete carapaces of *Poloniella devonica* and loose valves of the other Ostracoda, originating from the same locality, renders it also probable that in *Poloniella* the connection of valves is present, which is characteristic of the genus *Kloedenella*.

At the same time it is easy to see that the furrows on the lateral sides of the carapaces of the Ostracoda correspond, when we only assume that in *Poloniella devonica* the anterior and the posterior furrows are joined at the ventral side, so that we cannot distinguish here the two small furrows that are present in *Kloedenella hieroglyphica*.

If we compare the figures 7 and 9 of *Poloniella devonica* which were given by GÜRICH, it strikes us immediately that the carapaces illustrated are very different in thickness. This is easily explained by assuming that the first comes of a male and the second of a female individual, as has occurred in many other Ostracoda. (1, p. 79; 7, p. 66). The carapace of the female is taken to be thicker than that of the male as a result of the stronger development of the genital apparatus.

The same phenomenon appears also in *Kloedenella hieroglyphica*. Among the material of this Ostracod, which is to be found in the Mineralogical-geological Institute at Groningen, occur two kinds of carapaces viz. thick ones which I think originating from females (Fig. 10) and less thick ones originating from males (Fig. 8).

Thus we can see that in both genera *Poloniella* and *Kloedenella* sexual dimorphism appears in the same manner.

I, therefore, do not doubt the identity of the genera *Poloniella* and *Kloedenella*. The former being founded before the latter, the genus *Kloedenella* must be abandoned.

The criteria of the genus *Poloniella* are: carapace elongate and small; the length usually less than 1½ mm.; the thickness of the carapace of the male individuals practically everywhere the same; in the female much larger especially at the posterior end. At the anterior and posterior ends the carapace is equal in height with the

males; with the females the posterior part is higher. The dorsal edge straight; the ventral one convex or somewhat concave. The anterior edge equally curved and passing almost unperceptibly into the dorsal edge, as together with it a very obtuse angle is formed; the posterior edge less curved and forming almost a right angle with the dorsal edge. Valves unequal; the right one on the anterior part with a half circular notch, in which a process of the left valve fits. Owing to this peculiar connection of the valves complete carapaces have generally been preserved. The sharp hinge line of the left valve lies in a furrow on the hinge line of the right valve, the latter being higher than the left one along the hinge line. The sharp free edges of the right valve lie in a furrow on the free edges of the left, so that with the free edges the left valve overlaps the right one. The surface of the carapaces is different. On the anterior part of each valve two more or less vertical furrows are found that are separated by a narrow lobe. Also on the posterior part a furrow may occur which can be linked to the anterior furrow below. For the rest the surface is generally smooth and without ornamental markings.

Remains of these Ostracoda have been found in upper silurian, devonian and probably also in carboniferous strata of the temperate zones of the Northern Hemisphere.

In the foregoing we have seen that *Poloniella devonica* can easily be derived from the upper silurian *Poloniella hieroglyphica* by assuming that the two small furrows which are found below the middle of the three larger ones, are joined together and with the anterior and posterior furrow, through the disappearance of the intermediate lobes.

Poloniella hieroplyphica is sure to have found its origin in a species of this genus which resembled to a degree the older but yet upper silurian *Poloniella Hallii* JONES sp. (Fig. 12) (5, p. 15). Here the two small furrows are wanting, but the three larger ones are already well developed. The occurrence of valves with one small furrow in *Poloniella hieroglyphica* points to this fact also (Fig. 6).

The forms resembling *Poloniella Hallii* can be easily derived from the type represented by *Poloniella pennsylvanica* (Fig. 13) (6 p. 341) which occurs in under-devonian deposits and where no more than two vertical furrows are present.

Finally I give my best thanks to Prof. Dr. J. H. BONNEMA for kindly putting the material of *Poloniella hieroglyphica* at my disposal, and to Miss A. J. POTT, who has been so obliging as to make the necessary drawings.

Mineral.-Geol. Inst. University at Groningen.

LITERATURE.

1. BONNEMA, J. H. „*Beitrag zur Kenntnis der Ostracoden der Kuckers'schen Schicht (C_2)*".
   Mitt. a. d. Mineral.-Geol. Institut d. R. Univ. zu Groningen, Bd. II, Heft 1., S. 1—84. Taf. I—VIII; 1909.
2. BONNEMA, J. H. „*Bijdrage tot de kennis van het geslacht Kloedenella*, ULRICH en BASSLER".
   Kon. Akad. v. Wet. te Amsterdam. Versl. v. d. gew. Verg. d. Wis- en Nat. Afd. v. 28 Maart 1914. Deel XXII blz. 1087—1092.
3. BONNEMA, J. H. „*Contribution to the knowledge of the genus Kloedenella* ULRICH and BASSLER."
   These Proceedings Amsterd., Meeting of 24 April 1914. Vol. XVI, p. 1105—1109.
4. GÜRICH, G. *Das Palaeozoïcum im Polnischen Mittelgebirge.*
   Verhandl. der Russisch-Kaiserl. Mineral. Gesellsch. zu St. Petersburg. 2e Serie, 32ter Band. 1896.
5. JONES, T. R. *On some Palaeozoic Ostracoda from North America, Wales and Ireland.*
   Quarterly journal of the geol. society of London. Vol. XLVI p. 15. Pl. IV Nov. 6. 1889.
6. JONES, T. R. The American Geologist, Vol. IV., N°. 6, 1889. p. 337—342.
7. MOBERG, J. C. och GRÖNWALL, K. A. *Om Fyledalens Gotlandium.*
   Lunds Universitets Årsskrift, N. F. Afdl. 2. Bd. 5. N°. 1.
   Kongl. fysiografiska Sällskapets. N. F. Bd. 10. N°. 1.
8. ULRICH, E. O. and BASSLER, R. S. "*New American Palaeozoic Ostracoda. Preliminary revision of the Beyrichiidae with descriptions of new genera.*"
   Proc. of the U. S. Nat. Mus., Vol. XXXV, p. 227—340, with pl. XXXVII—XLIV, 1908.

EXPLANATION OF THE PLATE.

Fig. 1. Carapace of *Poloniella devonica* G. GÜRICH seen from the right side. (After GÜRICH).

Fig. 2. Carapace of *Poloniella hieroglyphica* A. KRAUSE sp. seen from the right side 40 ×. (After BONNEMA).

Fig. 3. Right valve of *Poloniella hieroglyphica* A. KRAUSE sp. 40 ×.

Fig. 4. Carapace of *Poloniella devonica* G. GÜRICH seen from the left side. (After GÜRICH).

Fig. 5. Carapace of *Poloniella hieroglyphica* A. KRAUSE sp. seen from the left side. 40 ×. (After BONNEMA).

Fig. 6. Left valve of *Poloniella hieroglyphica* A. KRAUSE sp. 40 ×.

Fig. 7. Carapace of *Poloniella devonica* G. GÜRICH of a male individual seen from the dorsal side. (After GÜRICH).

Fig. 8. Carapace of *Poloniella hieroglyphica* A. KRAUSE sp. of a male individual seen from the dorsal side. 40 ×. (After BONNEMA).

Fig. 9. Carapace of *Poloniella devonica* G. GÜRICH of a female individual seen from the ventral side. (After GÜRICH).

Fig. 10. Carapace of *Poloniella hieroglyphica* A. KRAUSE sp. of a female individual seen from the dorsal side. 40 ×.

Fig. 11. Transverse section at the height of the muscle impression of a carapace of *Poloniella hieroglyphica* A. KRAUSE sp. seen from the posterior end. 35 ×. (After BONNEMA).

Fig. 12. Left valve of *Poloniella Hallii* JONES. sp. 15 ×. (After JONES).

Fig. 13. Carapace of *Poloniella pennsylvanica* JONES sp. seen from the right side, from the anterior end and from the ventral edge. 15 × (After JONES).

J. E. VAN VEEN: "The identity of the genera Poloniella and Kloedenella".


Fig. 1.

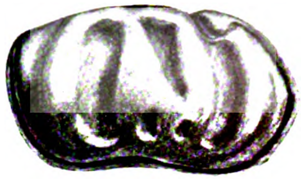


Fig. 2.

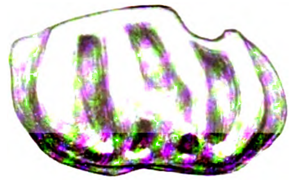


Fig. 3.


Fig. 4.

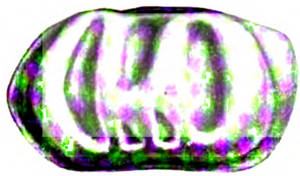


Fig. 5.


Fig. 6.


Fig. 7.


Fig. 8.


Fig. 9.


Fig. 10.


Fig. 11.


Fig. 12.


Fig. 13.

Geology. — "*On Manganese Nodules in Mesozoic Deep-sea deposits of Dutch Timor*". By Prof. G. A. F. MOLENGRAAFF, with a preliminary communication on "*Fossils of Cretaceous Age in those Deposits*". By Dr. L. F. DE BEAUFORT.

(Communicated at the meeting of November 27, 1920).

Deep-sea deposits, which resemble in nearly every respect the recent deep-sea oozes have in the latter three decades been observed [1]) in many islands of the East-Indian Archipelago [2]), notably in the islands Borneo, Rotti, and Timor. In Borneo they are of mesozoic, probably of pre-cretaceous age, in Rotti partly of jurassic, and in Timor, as had been accepted until now of triassic and of jurassic age. Red clay-shale here and there containing radiolaria, being the equivalent of the recent red clay, as well as chert and hornstone with radiolaria, so-called radiolarites, being the equivalent of the recent radiolaria-ooze, have been found and take up a foremost place among the rocks composing the soil of these islands. Manganese nodules are not wanting in the mesozoic deep-sea deposits and I have succeeded in proving [3]) that they enclose numerous radiolaria, and thus have been formed by the precipitation of manganese in an ooze containing radiolaria. The nodules of manganese, which had been found prior to those described in this paper, differ from those of the recent deep-sea deposits in two respects. They do not present, at least not distinctly, a concentric structure, and they do not include other fossils besides radiolaria. Recent manganese nodules from the deep-sea, on the contrary, have as a rule a concentric arrangement and not seldom the nuclei around which they are grown, consist of fossil remains, as e. g. teeth of sharks. Shark's teeth devoid of any coating of manganese were frequently brought up in great quantities by the Challenger-expedition from great depths in the red clay, showing that in such cases these teeth were lying loose on the bottom of the sea.

---

[1]) They are deposited in the deepest parts of the mesozoic Tethys-geosyncline and considering their character of deep-sea deposits, comparatively close to the land.

[2]) See References 1, 2, 7, 8, 9, 10, 11, 12.

[3]) G. A. F. MOLENGRAAFF, Ref. 9, pp. 426 and 427.

The late Prof. Dr. H. G. JONKER [1]), who, in the year 1916, made palaeontological explorations in the island of Timor, was fortunate enough to make a discovery, which decidedly increases the knowledge of fossil deep-sea deposits.

In the bed of a little brook, discharging itself into the Noil Tobee, also called Noil Toninu, on its right bank he found near its mouth a good exposure of beds of red deep-sea clay, containing, besides numerous nodules of manganese also teeth of *Elasmabranchii*, especially of sharks. The Noil Tobee is a small river rising about $4^1/_2$ km. E. N. E. of Niki-Niki in the district of Amanuban in Central Timor and joins the Noil Bunu at 3 km. to the North of its source, a little below the Fatu Toninu. The Noil Bunu flows into the Noil Noni [2]), and this again into the Noil Benain, which river discharges into the Timor sea, not far from Besikama. The spot in the bed of the brooklet, where JONKER has found this red clay, is situated about 480 m. above the sealevel.

JONKER had a clear vertical section dug out and entirely freed from debris which had been transported by the brooklet. A sketch taken from his diary is reproduced here without any alteration.

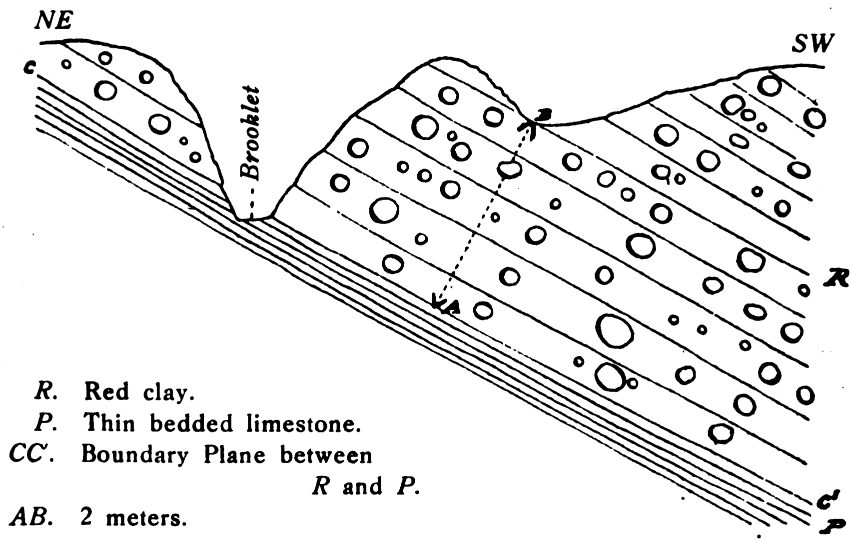


Fig. 1. Section in the Noil Tobee.

---

[1]) Prof. JONKER by his sudden death on 19 Jan. 1917, was prevented from preparing any of the results of his explorations for publication. His collections are stored now in the geological and palaeontological museum of the Technical High School at Delft. I have been able to make use of his diary in preparing this article.

[2]) This portion of the Noil Noni is sometimes considered to form a part of the Noil Benain.

A clay ranging in colour from yellow to red and brown, and containing nodules of manganese overlies, apparently, quite conformably a thin bedded limestone, which contains badly preserved shales of Aviculidae (Halobia). The bedded limestone is well stratified, the clay, on the contrary, is not distinctly stratified except in a portion of the section, where brown and yellowish-red clay are found to alternate. In the section the beds of the clay are apparently undisturbed, but the polished slickensides, which traverse the clay, testify to its having been exposed to considerable mountain-pressure. In consequence of the large number of slickensides the clay is crumbly and it is impossible to obtain a good-sized specimen without joints. The largest entire specimen brought by JONKER, measures $8 \times 6 \times 2$ cm. It is represented on Pl. I fig. 2. The position of the limestone and the clay is the same; both have a strike N. 35° W. and a dip 42° towards S.W. The yellow clay which prevails in the lower portion of the section is about 40 cm. thick and is followed by red, and chocolate-brown clay rather more than 3 metres thick; with it the section terminates against the surface soil.

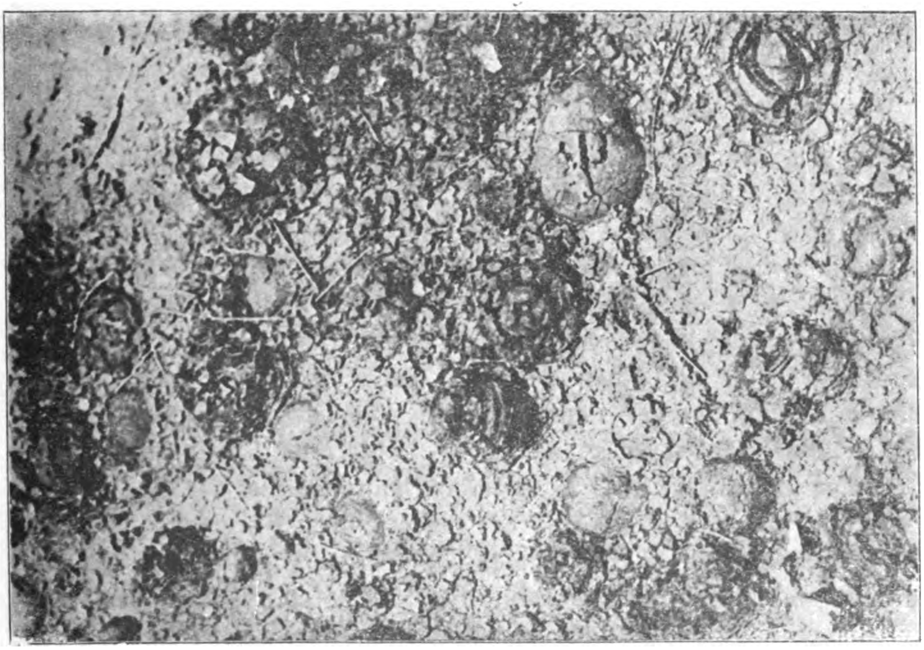


Fig. 2. Manganese nodules in red deep-sea clay, Noil Tobee Central-Timor.

Foto H. G. JONKER.

In the upper portion of the section the brown clay alternates

with bands of yellowish-red clay, which brings out the stratification to greater advantage.

The manganese nodules are numerous and scattered over the entire section, as can be seen in Fig. 2. In the red clay, however, they are more numerous than in the yellow. The relation between these nodules and the clay has been represented in figure 1 only schematically. Compared with the given scale the nodules appear much too large in the drawing. The largest have on an average a longer axis of at most 10 cm., all the others have varying smaller dimensions. Small ones e.g. of a diameter of 1 or 2 cm. are as numerous as larger ones.

Further upstream Jonker discovered another exposure of red deep-sea clay with manganese nodules on the left bank of Noil Tobee, but it is inferior to the one sketched above.

Teeth of *Elasmobranchii* especially of sharks are disseminated in the red clay. Jonker has collected most of them as loose specimens weathered out from the clay. In his collection there are two pieces only in which a shark's tooth constitutes the nucleus of a manganese nodule. Jonker's diary does not give more particulars about the distribution of the shark's teeth in the clay.

a. The red deep-sea clay.

The red deep-sea clay of Noil Tobee has apparently been altered very little by diagenetic processes. It has a greasy feel resembling that of soapstone and can be scratched with the nail; in a dry state it is somewhat plastic, and distinctly so after moistening. Considering it as a rock it could hardly claim the name of clayshale, the term solid clay being more appropriate to its character. In this respect it differs from all other deep-sea shales, hitherto discovered in Timor and in other islands of the East-Indian Archipelago. In Timor mesozoic red clayshale is an important constituent in the structure of the soil, and is found in many places.

In all localities known to me the red deep-sea clay occurs as a non-plastic, fairly hard clayshale, not unfrequently slightly schistose through mountain-pressure, and always altered rather considerably by diagenesis, maybe through silicification, maybe by calcification.

On microscopic examination I found in many cases that such a clay-shale had first been silicified, whereas later a portion of the silica has been dissolved again and leached from the rock whilst a cement of lime had been introduced into the rock.

The locality Noil Tobee, discovered by Jonker, is the only one

known until now where red deep-sea clay of mesozoic age is found unmodified as a true clay. Up to this day fossil red deep-sea clay in an equally unaltered state of preservation, has been found only in one place, viz. the island of Barbados. But here it occurs in much younger deposits, viz. in the so-called "oceanic beds" of miocene age. HARRISON [1]) describes it as "clays having a peculiarly greasy feel, ranging in colour from a dark chocolate-red through various shades of red and pink to yellow and greyish white".

There is some difference in properties between the varieties of the clay of Noil Tobee. The pale red variety displays most distinctly the properties of a true clay and is also fractured least by joints showing slickensides; it no doubt represents the purest and least modified form in which the deep-sea clay here occurs. This explains why only this pale red variety has been used by me as a material for an analysis as well as for microscopic examination. The brown varieties have undergone a more marked modification; they are harder and very much fractured by minor polished fault-planes.

An analysis of a sample of pure, pale red deep-sea clay of Noil Tobee, carried out by Prof. H. TER MEULEN at Delft, shows its chemical composition to be as follows:

SiO_2	57.6	
TiO_2	0.6	
Al_2O_3	19.2	
Fe_2O_3	7.1	
MnO_2	trace	
CaO	1.2	
MgO	1.4	
K_2O	trace	
Na_2O	2.3	
H_2O below 110°	6.2	10.2
H_2O above 110°	4.0	
	99.6	

Trace of Sulphate.

In order to compare this composition with that of recent deep-sea clay and that of the miocene deep-sea clay of Barbados, the results of the analyses have to be brought first into intercomparable form

In the analyses of the numerous samples of recent red deep-sea clay, collected by the Challenger-expedition, BRAZIER has not taken into account the salts which had been dissolved in the seawater

[1]) A. J. JUKES BROWNE and I. B. HARRISON. Lit. 3 p. 189.

adhering to the clay. Such adhering connate salts of course can no more be expected to occur in the samples of fossil deep-sea clay. The quantity of this adhering salt is not inconsiderable and amounts to 3.61 %, as shown by HARRISON and WILLIAMS[1]) in material of the Challenger. The constituents of this salt are NaCl, $MgCl_2$, $CaSO_4$, and a trace of phosphate. Moreover BRAZIER has not determined the alkalis in the samples of the Challenger, so that the figures assigned by him to SiO_2 and other substances are too high. To meet this deficiency in our knowledge of the chemical composition of the recent red deep-sea clay, HARRISON and WILLIAMS[2]) have made a new analysis of the typical red deep-sea clay collected by the Challenger determining the percentage of the alkalis and mentioning separately the quantity and the composition of the adhering sea salt. Leaving out the adhering salt the analysis of the sample of deep-sea clay, examined by HARRISON and WILLIAMS would come to this:

SiO_2	56.12	
Al_2O_3	16.30	
Fe_2O_3	10.94	
MnO_2	1.62	
CaO	1.65	
MgO	1.43	
K_2O	1.95	
Na_2O	3.34	
H_2O	6.92	whilst heating after drying at 100°.
	100.31	

Now we are enabled to compare the above analysis of recent red clay with those of the fossil deep-sea clay, as soon as in both the amount of water, escaping below and above 100°, has been taken into account in the same way. Doing so it is desirable not to take into account the water which escapes on heating to 100°, because this was done neither in the analyses of the miocene deep-sea clay of Barbados, nor in the most recent analyses of recent red deep-sea clay made by G. STEIGER and discussed by CLARKE[3]).

The analysis referred to, of the cretaceous deep-sea clay of Noil Tobee, recalculated in this way runs:

---

[1]) I. B. HARRISON and A. J. JUKES BROWNE. Lit. **6** p. 315.
[2]) Ref. **6** p. 315 and 321.
[3]) F. M. CLARKE. Ref. **4**, p. 785.

SiO_2	61.3	
TiO_2	0.7	
Al_2O_3	20.4	
Fe_2O	7.6	
MnO	trace	
CaO	1.3	
MgO	1.5	
K_2O	trace	
Na_2O	2.5	
H_2O	4.3	Escapes on heating above 110° C.
	99.6	

In this way we have obtained the following analyses in a comparable form:

	1	2	3	4
SiO_2	56.12	54.48	60.99	61.3
TiO_2	2.01 [1]	0.98	3.63 [1]	0.7
Al_2O_3	16.30	15.94	21.03	20.4
Fe_2O_3	10.94	8.66	6.95	7.6
FeO		0.84		
MnO_2	1.62	1.21	1.24	trace
CaO	1.65	1.96	0.—	1.3
MgO	1.43	3.31	2.62	1.5
K_2O	1.95	2.85	0.50	trace
Na_2O	3.34	2.05	1.95	2.5
H_2O after drying at 100°	6.92	7.04	4.72	4.3 [2]
	100.27	99 32	100.—	98.6

[1]) In the samples 1 and 3 TiO_2 is determined from a separate quantity making the figure for Al_2O_3 too high, because TiO_2 is comprised in it.

[2]) Dried at 110°.

1. Recent red deep-sea clay. Pacific Ocean. Challenger station 256. 30° 22′ N. Lat and 154° 56′ W. Long. Depth 5310 m. Anal. HARRISON and WILLIAMS.
2. Average composition of 51 samples of recent deep-sea clay. Anal. G. STEIGER.
3. Miocene red deep-sea clay. Mt. Hillaby, Barbados. Anal. J. B. HARRISON.
4. Cretaceous red deep-sea clay. Noil Tobee. Central Timor. Anal. H. TER MEULEN.

On comparing the analysis 3 of the red clay of Barbados with the analysis 4 of the red clay of Noil Tobee, it strikes us that

the first-named contains 1.24 % MnO_2, whereas the second contains only a trace. This may be accounted for by the fact that no manganese nodules occur in the red clay of Barbados and thus the manganese ore there is not concentrated, as is the case in the red clay of Noil Tobee. For the rest the analyses 3 and 4 resemble each other so much, that we may speak of an almost complete identity as to chemical composition between the cretaceous deep-sea clay of Noil Tobee of Central Timor and the miocene deep-sea clay of Mt Hillaby in the island of Barbados. The differences between the fossil and the recent deep-sea clay are slightly greater. This is easy to understand, as the deposit, directly it had been raised by diastrophism above the sea-level, must in some measure have been modified through diagenetic processes, in spite of its being almost impervious to water. By those processes a portion of the iron has been leached out and removed from the rock, and silica has been introduced into it. Taking this into consideration the chemical composition of the red clay, as well of Barbados as of Central Timor, appears to resemble fairly well that of the recent deep-sea clay brought up at different stations by the Challenger, as is evidenced by the above analyses. The accordance in composition with the samples of red deep-sea clay collected by the Gazelle and analyzed by VON GÜMBEL [1]) is also great.

Microscopic composition of the red clay.

The microscopic examination of four thin slides of red clay of Noil Tobee, carried out by Prof. H. A. BROUWER, yielded the following results: "The major part consists of an extremely fine clay-mass, which cannot be determined more precisely. It contains some larger fragments of minerals and rocks which were recognized as

a. a polysynthetically twinned crystal of plagioclase;

b. a small fragment of a volcanic rock with felsparlaths in the groundmass;

c. a strongly altered fragment of the groundmass probably of a volcanic rock rich in glass;

d. a fragment of a volcanic rock rich in glass, with felsparlaths, featherlets of ore and much glass;

e. some strongly altered (serpentinized) fragments, possibly of olivine originating from a volcanic rock;

f. an amorphous piece of quartz.

Ill-defined remains of radiolaria, the tests of which have mostly disappeared, occur in a small quantity in all the slides".

[1]) W. VON GÜMBEL. Ref. 5, p. 85 and 87.

In one of the sections a nodule of manganese was found to be cut through. The concentration of the manganese ore appears to be perfect in this clay since not a trace of scattered grains of manganese has been found in any of the sections of the red clay in the slide.

Besides the fragment of quartz mentioned sub *f* visible under the microscope, I also observed with the naked eye some white points which proved to be composed of diminutive pieces of quartz. These fragments of quartz I consider to be erratic in the red clay, i. e. to be constituents of terrigenous origin, which for some accidental reason or other have been deposited in the deep sea; they may have been transported by floating treetrunks outside the littoral zone. For Timor and the East-Indian Archipelago in general such an interpretation is admissible, because also in the Mesozoicum this region cannot at any time have been far remote from land.

b. The manganese nodules.

Jonker collected a large number of manganese nodules from the deep-sea clay of Noil Tobee. The largest among them have the size of lemons, the smallest are about equal in size to nuts; in the fragments of red clay a good many occur no larger than peas. The largest specimen measures $10 \times 8\frac{1}{2} \times 6$ cm. Two types are found in the collection, the first type being represented by 90 specimens, the second by 2 specimens only.

Type 1. Nearly all nodules are spherical or ellipsoidal. A few are cylindrical in shape and evidently originated by the coalescence of two individuals.

The surface is tubercular and finely granulated, reminding one of shagreen (Pl. II, fig. 4). The colour is black to brownish black; the stripe is dark brown. The nodules are mostly dull, but display a faint metallic lustre on the projecting parts of the relief, i.e. on the granules and on the tubercles. Their hardness is less than 2. The specific weight is $\pm$ 1.7. This low value is due to the great porosity of the nodules. As to physical properties the composing material is analogous to Waad.

The manganese nodules possess a distinctly concentric structure. A radial arrangement could hardly be perceived in some, in others not at all. Several nodules in the collection were broken in two, and show very well the concentric structure. (Plate II fig. 1 and 2). Some of them are broken on purpose, but Jonker reports that just below the surface he often found the nodules broken in two

pieces. The nodules have often a white or gray nucleus free from manganese in their centre, around which on all sides the manganese has been precipitated in concentric, porous layers [1]). The white nucleus is sharply contrasted with the dark envelope (Fig. 3 in the text and Pl. II fig. 3).

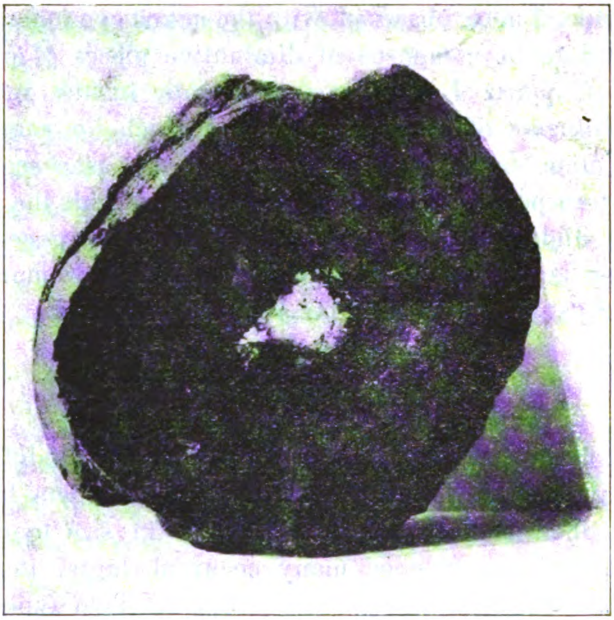


Fig. 3. Manganese nodules with a white nucleus of chert containing radiolaria for the greater part altered into amorphous silica.

In some of the specimens part of the nuclear mass is of a greenish colour and dimly transparent. These parts can easily be recognized as chert with a strong pocket lens.

The nuclei are always brittle and more or less friable. In one case I succeeded in having a thin section made through an entire nodule, nucleus and envelope, without interfering with the structure. This slide is reproduced in Pl. I fig. I.

On microscopic examination this nucleus appeared to consist of radiolarite, being converted for the greater part into white, amorphous silica. In it the radiolaria are packed close together, their casts being filled up with a crystalline mosaic of quartz. The concentric arrangement and the porous character of the manganese envelope round this nucleus are easily recognizable in this figure. In some other slides

[1]) Very rarely a white substance free from manganese, quite similar to that of the nuclei, was found outside the centre of a nodule, between two layers of the manganese envelope.

made from these nodules I could state the presence of some radiolaria also in the manganese substance itself.

The chemical composition is shown in the following analysis of one of these nodules made by Prof. H. TER MEULEN:

SiO_2			24.4	
Fe_2O_3			25.5	
Al_2O_3			9.8	
MnO			16.9	
CaO			1.5	
BaO			0.32	
MgO			0.34	
K_2O			0.15	
Na_2O			1.46	
NiO			0.28	
CoO			0.16	
CuO			0.12	
Cl	(oxyg. aeq.)		0.60	
H_2O	escaping below	110°	7.9	18.1
	,, above	110°	10.2	
			99.63	

Traces of lead, sulphate and phosphate.

This substance might be called a Waad rich in iron and silica. For the sake of comparison I give this analysis of a nodule of Noil Tobee in the following table, after recalculation as if the material had been dried at 110°, next to an analysis of manganese nodules from the recent deep-sea clay brought up by the Challenger at four different stations.

From this table it appears that the composition varies very much in the different nodules [1]). On account of the high percentage of iron they all might be called iron-manganese nodules. The composition of the nodule of Noil Tobee lies, except for its contents of alumina, within the extreme values, found on analyzing the nodules of manganese of the present oceans.

Type 2. Among the manganese nodules of Noil Tobee there are two of a different type, which I have named the second type. One of them is broad and flat, measuring $10 \times 9^1/_2 \times 3^1/_2$ cm.; the second is more spherical and smaller. They have a specific gravity of 4.2 and their hardness is 6. Thus they are much heavier

[1]) An analysis made by A. SCHWAGER of a manganese nodule found in red deep-sea clay by the Gazelle in the Pacific Ocean, is given by W. VON GÜMBEL Ref. 5 p. 102.

and harder than those of the first type. They have a smooth surface and bear a close resemblance to the manganese nodules occurring near Sua Lain in the island of Rotti [1]) in jurassic marls, which enclose numerous radiolaria. They are also very much like those which were collected in Timor near Mt Somoholle in deep-sea clay of presumably triassic age.

Comparative table of analyses of manganese nodules in red clay.

Place	Challenger Stat. 160	Challenger Stat. 252	Challenger Stat. 276	Challenger Stat. 286.	Noil Tobee	Rotti 2d type
Depth in fathoms	2600	2740	2350	2335		
SiO_2	21.80	27.62	13.66	20.01	26.5	2.9
Al_2O_3	22.30	6.60	3.10	2.81	10.6	2.3
Fe_2O_3		17.82	46.40	17.88	27.7	
MnO_2	39.32	25.48	14.82	38.15	18.3	57.7
MnO						10.5
CaO	2.21	2.91	3.53	3.58	1.63	5.6
BaO					0.35	11.7
MgO	0.89	1.27	0.74	0.33	0.37	
K_2O					0.16	
Na_2O					1.59	1.1
NiO			trace	trace	0.30	
CoO			trace	trace	0.17	0.3
CuO	trace	trace	trace	trace	0.13	
Cl (oxyg. aeq)					0.65	
CO_2						small quantity
H_2O escaping above 110°	11.—	15.20	14.40	11.35	11.1	± 15.3

When broken into halves these nodules look quite compact and homogeneous and show no trace of a concentric structure. In the last column of the above table the chemical composition of a nodule of the second type from Sua Lain [2]) in the island of Rotti is given. It differs much from that of the nodules of the first type. These nodules of the second type of Noil Tobee have not been examined any further, because JONKER's notes do not tell us whether

[1]) Ref. **12** p. 326, 393; **2** p. 61 and **9** p. 1064.

[2]) The large percentage of BaO accounts for the high specific gravity,

they originate from the red clay in situ or whether they have been transported by the brooklet from a higher level.

c. The fossils in the red clay and in the manganese nodules.

These fossils have been examined by Dr. L. F. DE BEAUFORT, who has summarized his results, thus far obtained, as follows:

"The fossils derived from the deep-sea deposits of Noil Tobee consist for the greater part of tooth-fragments of *Elasmobranchii*. With a few exceptions only the crown of the teeth has been preserved and of these also the dentine has been dissolved, so that only an enamel sheath remains.

This state of preservation, quite in keeping with what could be anticipated in a deep-sea deposit, renders the determination of the objects very difficult. In many cases it is even impossible to class the fragments as a definite species or even as a definite genus.

By far the greater number of the teeth belong to sharks of the *Lamnidae*. Thus far no older specimens of this family are known than those belonging to the chalk, unless the genus *Orthacodus* of the Upper-Jura be classified among the *Lamnidae*.

This genus, however, is not represented in the collection under consideration. We recognize in it tooth-fragments of *Carcharodon* (known from Chalk and Tertiary deposits), *Lamna* (Cretaceous to Recent), and *Scapanorhynchus* (know only from the Upper Chalk).

Furthermore I include a single fragment among the genus *Hemipristis* (Upper Chalk, Tertiary, and a single recent species) of the family of *Carchariidea*.

Considered merely palaeontologically, the fossils mentioned above might be believed to belong to the Upper Chalk. This view is substantiated in large measure by the presence in the collection of some well-preserved teeth of the easily recognizable genus *Ptychodus*, teeth of this genus, which is looked upon as a precursor of the *Myliobatidae*, being found up to the present only in the Upper Chalk of Europe and North-America.

In the Timor-collection we find teeth of 3, perhaps of 4 species of this genus. They may be assimilated to, or anyhow they are closely related to the following species: *P. decurrens* Ag., *P. dixoni* Dudley and *P. rugosus* Dixon. These three species, which according to SMITH WOODWARD (Quart. Journal Geol. Soc. London, Vol. 67., 1911, p. 276) form an ascending progression, occur according to DIDLEY (l.c. p. 263 seqq) in different layers of the Upper Chalk of England.

Over and above the teeth discussed, the collection also contains some undetermined fish-teeth and a fragment of a tooth of a reptile,

furthermore some fin-spines presumably of *Selachii* and lastly some fragments of bones, indeterminable thus far.

It may be that a closer investigation of these hitherto undetermined pieces from the remarkable collection will reveal that still older types are among them. As yet I can establish only with absolute certainty that *"in the deep-sea deposits of Timor there occur types, known up to now either exclusively from the Upper Chalk or from no older strata than the Upper Chalk."*

The above examination goes to show that the fossils found in the red deep-sea clay and also in some of the manganese nodules are of upper-cretaceous age. From this we can logically infer that the deep-sea clay in which the teeth of *Elasmobranchii* are formed, is also of upper-cretaceous age. This result is divergent from what might be concluded from the stratigraphy of the complex of layers, to which the deep-sea clay belongs, as observed by JONKER. It is evident both from the description and the section (fig. 1) that the red clay directly and conformably overlies a well stratified bedded limestone, in which are found not very well preserved, but clearly recognizable, remains of *Aviculidae* (Halobia). These are only known to occur in deposits of triassic age, and JONKER, therefore, did not hesitate to consider the red deep-sea clay with manganese nodules as triassic.

Although I believe the palaeontological evidence to be conclusive, it appears necessary to look for an explanation of this controversy. Two ways in which the section (fig. 1) may be read deserve consideration in order to account for the apparent contrariety.

First of all the cretaceous deep-sea clay, overlying directly conformably the triassic bedded limestone, may not have been deposited there originally, but may have been brought there afterwards by orogenetic movements. The plane *cc'* (fig. 1) in this case would not be a partition-plane between two superposed formations, but would represent the tectonic contact of two formations of very different age. A large break and a marked stratigraphic gap would then separate the two conformable, successive complexes of layers. Similar stratigraphic hiatus between conformably superposed formations, are of frequent occurrence in Timor with its chaotic tectonic, and are peculiar to regions, which have been considerably disturbed by orogenetic movements with considerable horizontal displacements, as BERTRAND has set forth as early as the year 1890. Most often, however, the difference in age between the conformably superposed formations is not so great as must be assumed in the case of Noil Tobee. Frequently I encountered

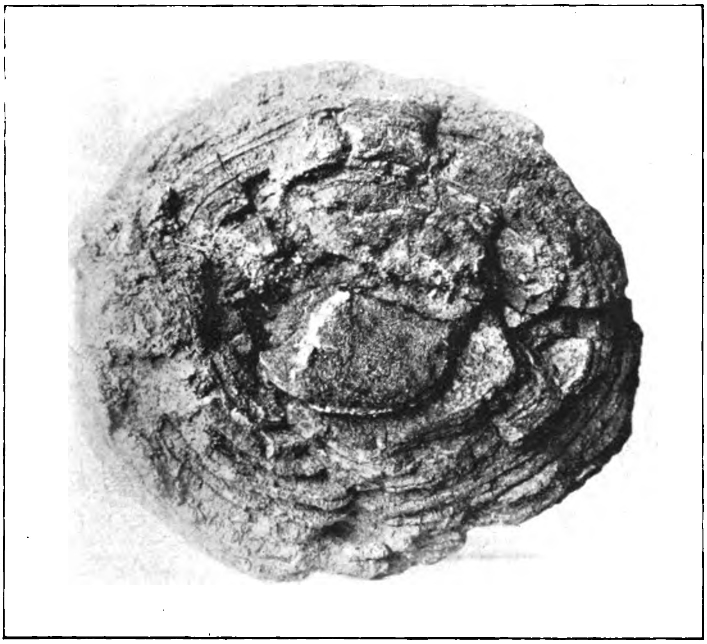


Fig. 1.

Fig. 2.

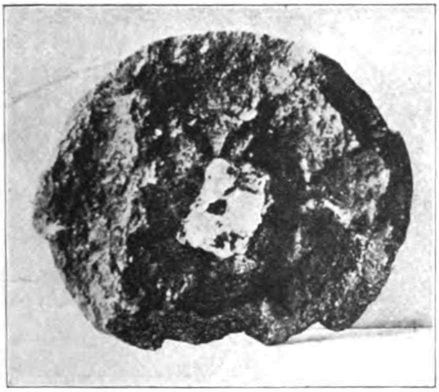


Fig. 3.

Fig. 4.

upper-triassic deposits overlying permian deposits conformably; in the case of Noil Tobee it would appear that an upper-cretaceous deposit is superposed conformably on an upper-triassic sediment. In such a case certain characteristics of the plane of contact between the two formations often reveal it to be a tectonic plane. In fact in such cases the plane is as a rule more or less polished, slickensided or plastered over with a thin layer of gouge. No mention is made of it by JONKER in his notes about the geology of the place. Probably he conceived the plane *cc'* between the red clay and the bedded limestone to be a normal partition between two deposits in normal succession.

Secondly it is possible, that the section after all represents a true undisturbed succession; if so, the red clay with nodules of manganese would embody the sum total of all that has been deposited here, in the deep-sea, from upper-triassic to uppercretaceous time. In such a case one might expect the fish-teeth, described above, not to occur in the lowermost part of the section. The notes on hand do not settle the question. An a priori rejection of this solution would not be warrantable either. True, the thickness of the red clay (in the section rather more than 3 m.) is small if compared with the enormous time its deposition must have taken, but then also the process of sedimentation must have been extremely slow in the deepest parts of the oceans far removed from the land, i.e. in the areas of the red clay.

EXPLANATION OF THE PLATES.

PLATE I.

Fig. 1. Section of a manganese nodule from the cretaceous deep-sea clay of Noil Tobee.

K. Nucleus, consisting of modified radiolaria-chert.

CC Concentric shales of manganese ore.

B.O. Outer surface of the nodule.

Fig. 2. Fragment of red deep-sea clay of Noil Tobee, containing a small manganese nodule with a large, white nucleus. Natural size.

PLATE II.

Fig. 1 and Fig. 2. Broken nodules manganese of Noil Tobee, clearly showing concentric arrangement.

Fig. 3. Nodule of manganese showing concentric layers and a white nucleus.

Fig. 4. Nodule of manganese seen from the outside. The surface is mammillated and is like shagreen in appearance by numerous little rugosities.

REFERENCES.

1. H. A. BROUWER. Geologisch overzicht van het oostelijk gedeelte van den Oost-Indischen archipel. Jaarboek van het Mijnwezen, 46e Jaargang, 1917, especially pp. 52, 61 and 63. Batavia 1919.

2. H. A. BROUWER. Voorloopig overzicht der geologie van het eiland Rotti. T. K. Ned. Aardr. Gen. 2. XXXI p. 611, 1914.

3. A. J. JUKES BROWNE and J. B. HARRISON. On the geology of Barbados Part II. The oceanic deposits. Quart Journ. Geol. Soc. Vol. XLVIII p. 170, 1892.

4. F. W. CLARKE. The composition of the red clay. Journal of Geology XV. p. 783, 1907.

5. W. VON GÜMBEL. Die min. geol. Beschaffenheit der auf der Forschungsreise S. M. S. Gazelle gesammelten Meeresgrund-Ablagerungen. Forschungsreise S. M. S. Gazelle II. p. 69. Berlin 1888.

6. J. B. HARRISON and A. J. JUKES BROWNE. Notes on the chemical composition of some oceanic deposits. Quart Journ. Geol. Soc. Vol. LI p. 313, 1895.

7. G. A. F. MOLENGRAAFF. Geological exploration in Central Borneo pp. 414—418 and Appendix I by E. J. HINDE, Leiden 1902.

8. G. A. F. MOLENGRAAFF. On oceanic deep-sea deposits of Central-Borneo. Proc. Royal Acad. Sc. Amsterdam XII p. 141—147, 1909.

9. G. A. F. MOLENGRAAFF. On the occurrence of nodules of manganese in mesozoic deep-sea deposits from Borneo, Timor and Rotti, their significance and mode of formation. Proc. Royal Acad. Sc. Amsterdam Vol. XVIII, pp. 415—430, 1915.

10. G. A. F. MOLENGRAAFF. L'expédition néerlandaise à Timor en 1910—1912 Archives Néerl. des Sciences exactes et naturelles III B, Tome II, p. 395, 1915.

11. G. A. F. MOLENGRAAFF. Folded mountain chains, overthrust sheets and block-faulted mountains in the East Indian Archipelago. Compte Rendu XIIe Congrès géologique international Toronto 1913, p. 690, Ottawa 1915.

12. R. D. M. VERBEEK. Molukken-verslag. Jaarboek van het Mijnwezen, 37e Jaargang, pp. 694 and seq. Batavia 1908.

Palaeontology. — "*The Proto-Australian Fossil Man of Wadjak, Java*". By Prof. EUG. DUBOIS.

(Communicated at the meeting of May 29 and September 25, 1920).

Tjampur Darat or Wadjak, the capital of the district of Wadjak, is a village (dessa), south-west of the town of Tulung Agung, and about in the meridian of the Wilis-summit. There the plain of Kediri has penetrated, past Mount Këlut, into the Gunung Kidul — the Southern mountain range —, and has obtained a steep Eastern boundary. The origin of this abrupt breaking off of the Tertiary lime-stone mountains has been attributed, no doubt rightly, by VERBEEK and FENNEMA to a fault running along that escarpment, through Tjampur Darat or Wadjak and Gamping [1]). In this southern continuation of the plain of Kediri, separated from the Indian Ocean by a mountain tract only 3 kilometers broad, lies the Rawa Bening (Clear Lake), now for the greater part a marsh, the water of which flows off through the Kali Tjampur, which, after uniting with the Kali Bendo, coming from the West, to form the Kali-Ngrowo, falls into the Brantas on the North of Tulung Agung. Repeated eruptions of Këlut and other volcanoes must gradually have raised the bottom of the lake with volcanic ashes. And while in the similar deposits which were formed downstream, the river easily kept its bed deep, the lake, which was probably very large at first and extended as far as the foot of the lime-stone rocks, had to diminish in extent and depth in course of time. Possibly the upheaval of Southern Java may also have contributed to this effect.

On the slope of that part of the mountain that extends, almost rectilinearly, over a distance of 800 meters in W. S. W. direction, immediately on the south of Tjerme and at 2 kilometers distance S. S. W. of Tjampur Darat, fossil human bones were found in 1889 and 1890. The plain lies there at the foot of the mountain 90 meters above the level of the sea, the plateau more than 140 meters higher, i.e. more than 230 meters above the level of the sea. Near the top the rock rises up almost vertically, for the rest the gradient is on an average 30°, through the accumulation of fallen lime-stone blocks

[1]) Fault N°. XXXI on the map "CVII and DII" of the Geological Atlas of Java and Madura.

and smaller débris, and there are also some small irregular terrace-shaped projections. Where the slope is not very steep and on the plateau, the lime-stone is often covered with a yellowish clay, containing more or less humus, a weathering product, no doubt, of volcanic ashes fallen in former times. In such places where it is somewhat protected against the direct action of the rain, this clay, impregnated with calcite, can unite with fragments of lime-stone to a breccia. Also many bones were wholly or partly inclosed in the hardened clay of such a breccia. For the rest they lay in the loamy clay, only superficially covered with a calcareous concretion.

The first find dates from 1889. In the beginning of this year, when I was carrying out excavations in caves in the surroundings of Pajakombo in the Padang Highlands in Sumatra, Dr. C. Ph. Sluiter, then at Batavia and member of the board of the "Natuurkundige Vereeniging in Nederlandsch-Indië", had the kindness to send me some fossil bones. These fossils had been found by Mr. B. D. van Rietschoten when exploring the described lime-stone rocks for the establishment of marble quarries [1]), and had been sent to the said Society. Mr. van Rietschoten thought these bones to be remains of "the skull of a man or a manlike animal". After having prepared and joined the very fragmentary remains, I recognized in them the not entirely complete skull with right angular part of the lower jaw [2]) and a few other fragments of the skeleton of a fossil man greatly deviating from the Malay type. The resemblance with the Papuan type seemed closest to me [3]).

This important find of Mr. van Rietschoten induced me to carry out excavations near Wadjak the following year. The finding-place of the Wadjak skull I appeared to lie near the middle of the described part of the mountain slope, and at about 50 meters above the plain, in a terrace-shaped projection, formed by blocks and smaller stones with breccia and clay [4]). Here parts were found of a second fossil skull, Wadjak II, with unmistakably similar characters as the first, which, like the first skull, after further preparation, presented an even closer resemblance with the Australian of the present time than

---

[1]) The marble exploitation company, formerly called "Wadjak", is now continued under the name of "Marmoyo".

[2]) Natuurkundig Tijdschrift van Nederlandsch-Indië. Batavia. Deel 49. (1889), p. 209—211.

[3]) The rest of the lower jaw and most of the crowns of the teeth of the upper jaw must have got lost in the digging.

[4]) I had at first erroneously taken an interstice between blocks for a crevice in the rock.

with the Papuan [1]). Besides a large part of the upper jaw and a large part of the lower jaw (Fig. 4 to Fig. 7. The existing fragment of the right ramus mandibulae is not represented), six loose teeth (which are lost in the lower jaw), and several large and small fragments of the calvaria, in which the most important morphological characters can still be recognized, there were found some pieces of other bones of the skeleton and a few fragments of bones of mammals, as far as can be ascertained not different from species now living in Java. All the bones met with were in the same condition of fossilisation; all of them were found scattered in a detached, fragmentary state, quite encrusted, for so far as they were not enclosed in a breccia, with an irregularly thick, yellowish-grey calcareous concretion, forming a rough surface and containing some clay. This so firmly adhered to the white bony substance lying under it, that it mechanically constituted one whole with it; only the difference in colour could serve at its removal. The incrustation was so thin, in most places, that the general morphological characters of the bones were hardly masked by it. That the specific weight of the bones of these fossil australoid men is high, and the fossilisation very complete, is at once perceived when they are taken in the hand; they are really heavy and cold to the touch as stone. From the available remains, the weight of the whole mandible of the Wadjak man II can be calculated at 230 grams, i. e. about a hundred grams more than the maximum of Australian aborigines. Partly this greater weight is, indeed, to be attributed to the very great size and robustness of the fossil mandible, but the specific weight is about 40 per cent. higher than that of fresh bone. For the specific weight of powdered cortical substance of a femur I find 2.78 at 15° C. The specific weight of the cortical substance of recent long bones is 1.98, that of pure calcite 2.72, of apatite on an average 3.19, which is also about the maximum of phosphorite. The fossil bones of Wadjak now contain only a very small quantity of organic substance.

The specific weight of the bones of the fossil man of La Chapelle-aux-Saints, as deduced from a comparison of weight with recent bones of the same dimensions [2]), has increased only in the ratio of about 1 : 1.20, instead of 1 : 1.40, which is about the ratio for the fossil men of Wadjak. This may be partly owing to the more favourable conditions of fossilisation of these latter bones, however it certainly points to great age.

In the absence of direct data for the determination of the geolo-

---

[1]) Verslag van het Mijnwezen, over het Derde Kwartaal 1890. Batavia 1890.

[2]) M. Boule, L'Homme fossile de La Chapelle-aux-Saints, p. 16. Paris 1913.

gical age — also artifacts were not found — another find near Wadjak is of special importance. At the eastern corner of the described rectilinear part of the mountain, at a height of about 120 meters above the plain, in the same kind of breccia and clay and again on a small terrace-shaped projection (behind which was found the entrance of a cave forty meters long, running in the shape of a U, and almost entirely filled up with the same kind of clay, in which nothing of any importance was found), I dug up some parts of a human skeleton in the same year, which are in a very different state of fossilisation, and have a quite different anthropological character. It is also certain that these remains were worked as skeleton by a human hand, for the outer surface of the cranial bones (not the inner surface), the teeth, and also other bones were painted red with a firmly adhering ochre-layer. After this the bones must have been broken, for the fragments were encrusted and partly enclosed in breccia, in a similar way as those of the two Australoids. They are however much less petrified and specifically lighter than these. Besides, the skull was distinctly brachycephalic, in contrast with those dolichocephalic australoid skulls. As this fossil man is certainly prehistoric, the bones of two others, fossilized under similar circumstances, but to a very much higher degree, must probably date from Plistocene time.

The presence of human remains from very different periods may be attributed to the circumstance that this mountain slope belonged to the shore of a lake abounding in fish [1]), the fact that the bones are broken in so many places may be accounted for in this way, that in times lying widely apart, first the two proto-Australians living there, and much later the skeleton placed before the cave which was probably inhabited, were buried and crushed under falling stones and rubble, possibly in earth-quakes. In the lime-stone mountains of Sumatra I a few times witnessed close by the imposing phenomenon of the spontaneous fall of lime-stone rock and rubble, and also once in the Gunung Kidul (Southern Range) in Java. The large quantity of rubble, at the foot and against the slope of these mountains, bears witness to the frequency of the stone-falls. The fragmentary character of the parts of the skeleton cannot be attributed to cannibalism; the fractures are too numerous. The lower jaw of Wadjak II, a very strong bone, was, for instance, broken into at least five large pieces. The fact that in both cases the remains were found on a flat

[1]) Calcareous waters abound in fish as a rule. The Râwâ Bening does so still, and the number of water-fowl is enormous; it is also paradisical through its uxuriant vegetation.

part of the slope under a précipice, and the circumstance that Wadjak I, to all appearance a woman, accompanied Wadjak II, who was certainly a man, that also the skeleton was crushed on the other flat part, in the front of the cave, are facts that quite fit in with the other interpretation of the fragmentary state of the bones. For the same reasons Carnivora (Tiger, Adjag) cannot have broken the bones either. It is further easy to understand that in the progress of the natural change of the mountain slope, many parts of the crushed skeleton were lost.

The skull of Wadjak I is partially filled up with breccia mass, and defective in some places; a few bones have also been slightly dislocated. Consequently some measures can only be taken indirectly, others not at all. After some correction, the former can generally, i.e. when the amount of the dislocation is measurable, still be determined with sufficient certainty.

The general form and the principal dimensions at once show that we have to do with a type deviating altogether from the Malay race. This is already evident on comparison of the norma lateralis with that of a typically Javanese skull placed at the same auricular-bregma line (fig. 1). For further comparison with our fossil skull, as far as its morphological characters are concerned, only the Papuan (in general the Melanesian), the Australian, and the Tasmanian are evidently to be taken into consideration, a group, which morphologically has a great number of characters in common. That the Wadjak man is no more closely related to Homo neandertalensis than those recent human types needs hardly further demonstration nowadays. [1])

The fossil skull of Wadjak I is large, exceptionally large for a woman, to whom it probably belonged (from the comparison with Wadjak II). The greatest length of the calvaria is 200 mm. This is probably never attained by female representatives of the said recent races of man, hardly ever by male Australian skulls (TURNER) [2]), and exceeded by very few by a few millimeters (DUCKWORTH) [3]).

---

1) Cf. M. BOULE, L'Homme fossile de La Chapelle-aux-Saints. Paris 1913. Extrait des Annales de Paléontologie. (1911—1913), p. 231 et seq., and also the treatise by BERRY and ROBERTSON, the last-mentioned paper of note (4), p. 171 et sec., and A. KEITH, The Antiquity of Man, Chapter VIII. London 1920.

2) W. TURNER, Report on the Human Crania and other Bones of the Skeleton. Challenger Reports, Vol. X. (1884); Vol. XVI. (1886).

3) W. L. H. DUCKWORTH, A Critical Study of the Collection of Crania of Aboriginal Australians in the Cambridge University Museum, Journal of the Anthropological Institute of Great Britain and Ireland, Vol. XXIII. (1894), p. 284, and Notes on Crania of Australian Aborigines. Ibid., Vol. XXVI. (1897), p. 204.

The greatest breadth is 145 mm. (measured directly, without the necessary correction 150 mm.), the basi bregmatic height is 140 mm. These measures, too, are near the maxima of the comparable group. For the length-breadth index 72.5 is thus found, for the length-height index 70, for the breadth-height index 96.7. Accordingly the skull is dolichocephalic and tapeinocephalic [Fig. 2. Norma frontalis, and Fig. 3. Norma verticalis].

According to the records of BERRY, ROBERTSON, STUART CROSS, and BÜCHNER [1]) these cranial measures, minima, means, and maxima for 100 Australians, 86 Tasmanians, and 191 Papuans (unsexed), in millimeters, and the mean indices, with which I compare Wadjak I, were as follows:

	Australians			Tasmanians			Papuans			Wadjak I
Maximum Cranial Length	164	181.8	199	163	180.3	198	157	177	197	200
Maxim. Cranial Breadth	120	130.7	143	125	135.1	145	112	128.4	146	145
Basi-Bregmatic Height	115	129.7	144	117	130.3	140	118	131.7	143	140
Length-Breadth Index		71.75			74.94			72.54		72.5
Length-Height Index		71.38			72.19			74.41		70
Breadth-Height Index		99.65			96.33			102.56		96.7

From this appears the close resemblance with this group of modern human types. The approach is closest to the Australians and the Tasmanians, least so to the Papuans. This applies also to other morphological characters of the cranium. The cranial vault has the characteristic rooflike appearance of Australian skulls, and the side-walls are almost vertical (Fig. 2 Norma frontalis), but the height of the cranium is nevertheless comparatively small; the glabella and superciliary ridges are very pronounced; the forehead

[1]) A. W. D. ROBERTSON, Craniological Observations on the Lengths, Breadths and Heights of a Hundred Australian Aboriginal Crania. Proceedings of the Royal Society of Edinburgh, Vol. XXXI. (1912), p. 1. — RICHARD J. A. BERRY and K. STUART CROSS, A Biometrical Study of the Relative Purity of Race of the Tasmanian, Australian and Papuan. Ibid., p. 17. — RICHARD J. A. BERRY and A. W. D. ROBERTSON, The Place in Nature of the Tasmanian Aboriginal as Deduced from a Study of his Calvarium. Part I. His Relations to the Anthropoid Apes, Pithecanthropus, Homo primigenius, Homo fossilis and Homo sapiens. Ibid. p. 41. — L. W. G. BÜCHNER, A Study of the Curvatures of the Tasmanian Aboriginal Cranium. Proceedings of the Royal Society of Edinburgh, Vol XXXIV. (1914), p. 128. — RICHARD J. A. BERRY and A. W. D. ROBERTSON, The Place in Nature of the Tasmanian Aboriginal as Deduced from a Study of his Calvaria. Part II. His Relation to the Australian Aboriginal. Ibid., p. 144.

is more receding; the orbits are low in comparison with their breadth (in all these respects Wadjak II still exceeds the first found skull); the nasal bones are little prominent; the upper jaw is more prognathous, and the floor of the nasal cavity passes gradually into the incisive region; there is even an almost perfect sulcus praenasalis ("Affenrinne") at both crania; the lower jaw is exceedingly strong and the chin more pronounced. In all these characters the fossil cranium is still somewhat nearer the Australian.

Berry, Robertson, and Stuart Cross have decisively shown, apparently, that the present Papuan type is the least pure of the three types mentioned, and, in their opinion, also the Australian is a heterogeneous type, a view which was already accepted by many anthropologists, contra Schoetensack, Klaatsch [1]) and some others.

Berry supposes that a primitive Papuan race may be the common stock type of the Tasmanian, who has remained purer, but varied during the long time of his isolation, and also of the Australian aboriginal, who is the result of the cross between Homo tasmanianus and some unknown other race [2]).

G. Sergi [3]) assumes as the common stock type a primitive *Homo tasmanianus*, characterized by roof-like elevation of the sutura sagittalis and lateral flattening of the cranial walls (lophocephaly), who not improbably would have come from the American continent, across the Pacific Ocean, in early Plistocene, or even late Pliocene times. In Tasmania he then changed to the recent Tasmanian, whom Sergi proposes to call *Hesperanthropus tasmanianus*. In Australia, also according to Sergi, crossing of the *Homo tasmanianus* took place with another, as he supposes, Polynesian element, from which arose the Australian aboriginal of to-day.

It seems to me that the fossil *Homo wadjakensis* of Java, who in some respects possesses more "primitive" characters of the cranium and the lower jaw than these present races, may be considered to be such a stock type. He must then have wandered eastward from

---

[1]) O. Schoetensack, Die Bedeutung Australiens für die Heranbildung des Menschen aus einer niederen Form. Zeitschrift für Ethnologie. Jahrgang 23. (Berlin 1901), p. 127.

H. Klaatsch in "Weltall und Menschheit". Band II. Berlin 1902. — H. Klaatsch. The Skull of the Australian Aboriginal. Reports from the Pathological Laboratory of the Lunacy Department. New South Wales Government. Vol. I, Part 3. Sydney 1908.

[2]) Richard J. A. Berry, A Living Descendant of an Extinct (Tasmanian) Race, Proceedings of the Royal Society of Victoria. Vol. XX. (New Series). Part. I. 1897. Cf. also Proceedings of the Royal Society of Edinburgh. Vol. XXXIV. (1914), p. 186.

[3]) G. Sergi, Tasmanier und Australier. *Hesperanthropus tasmanianus* spec. Archiv. für Anthropologie. Neue Folge, Band XI. (1912), p. 201.

Asia. Though the resemblance to the recent Tasmanian is certainly no less than to the present Australian, I have introduced him here as proto-Australian, because the autochthone of the smallest continent is mostly considered as the principal type of the group. This resemblance and this "primitive" state may further appear from the more detailed comparison and description.

As regards the form of the calvaria in the first place, BERRY and others, with the aid of determinations of minima, means, and maxima for crania of Australians and Tasmanians, find what follows:

	Australians			Tasmanians			Wadjak I
1. **Maximum Cranial Length**	164	181.8	199	163	180.3	198	200
2. **Glabella-Inion Length**	162	179.5	196	157	173.1	188	192
3. Calvarial Height	79.5	95.1	108	87	97	108	100
4. *Calvarial Height Index*	44.9	53	61.5	48.3	56.1	62.7	52
5. Distance of Foot-Point of Calvarial Height from Glabella	88	101.1	123	85	101.9	115.5	123
6. Distance Bregma Foot-Point from Glabella	51.5	61.2	74	43	58.7	71.5	69
7. *Calvarial Height Foot-Point Positional Index*	44.9	56.4	65.3	53.1	59	64.8	64
8. *Bregma Foot-Point Positional Index*	29.2	34.1	38.8	26	34	40.6	36
9. *Breadth-Calvarial Height Index*	60.3	72.7	85.4	65.9	72.2	79.2	69
10. Nasion-Bregma Arc	116	126.8	143	113	126	143	136
11. Nasion-Bregma Chord	100	110.9	124.5	97	109.5	120	119
12. Glabella-Lambda Chord	161	178.7	194	162	173.2	189	190
13. Glabella-Bregma Arc (Frontal Arc)	99	110.2	128	90	111.9	125	122
14. Glabella-Bregma Chord	95	108	121	87	105.2	118	115
15. Greatest Distance Frontal Arc to Chord	13	19.6	28	10	18.9	25	16
16. *Index of Frontal Curvature*	12.5	18.1	24.5	10.3	17.9	23.3	13.9
17. Bregma-Lambda Arc (Parietal Arc)	109	125.9	147	112	125.8	145	130
18. Bregma-Lambda Chord	98	114.6	137	99	113	127	113
19. Greatest Distance Parietal Arc to Chord	17	23.2	30.5	19	23.3	28	23
20. *Index of Parietal Curvature*	15.3	20.2	25.2	17.3	20.6	24.7	20.6
21. *Glabella-Bregma Angle* (BGI).	49°	54.8°	60°	51.5°	56°	64°	54°
22. *Frontal Curvature* (Glabella-Bregma Frontal Angle)	123.5°	139.6°	153°	131.5°	139.5°	149°	148°
23. *Parietal Curvature* (Bregma-Lambda Parietal Angle)	125°	135.7°	145°	125.5°	134.3°	141.5°	138°

Taking the dimensions of the fossil cranium into consideration, the deviations from the Australians and Tasmanians are mostly slight. Then the glabella-inion length, just as the glabella-lambda chord, presents in proportion to the maximum cranial length, the closest agreement with the Tasmanians, the calvarial height index with the Australians. Both the glabella-lambda line and the glabella-inion line are, with regard to the maxium cranial length, shorter in Tasmanian and in Wadjak I than in the Australian. This is in connection with the bulging out of the occiput. The latter is still more strongly pronounced in Wadjak II, so that the lobus occipitalis of the cerebrum ended more or less pointed.

An important difference consists in this that the top of the calvarial height (N°. 7 of the Table) lies relatively much more dorsally in the fossil man of Wadjak I than, on an average, in those recent races, particularly the Australian race. This means that the frontal part of the cranium was comparatively low vaulted, which also appears from the smallness of the index of frontal curvature (N°. 16), and the considerable value of the angle of frontal curvature (N°. 22). It is noteworthy that in all these respects the fossil cranium comes as near, or nearer, to that of the Tasmanian as to that of the Australian.

The comparatively lesser development of the frontal part of the cranium may also be inferred from the measure of the minimum frontal breadth; this is only 99 mm. for the Wadjak cranium, with a maximum length of 200 mm. (in the second cranium, which was certainly still longer, 101 mm.), while in Australian crania the maximum is 104, and the mean 98, according to DUCKWORTH's measurements, and TURNER even met with a maximum of 108. This latter went together with the greatest capacity found by TURNER in Australian crania, 1514 cm.^{3} [1]).

This relatively lesser development of the frontal part must have an unfavourable influence on the capacity of the cranium, as actually appears in the capacities of Australian crania, found by direct measurement.

Nor may the unfavourable influence on the capacity of the roof-like elevation, in comparison with equally high crania with rounded vault, observed in Australian crania, be neglected, when the capacity of Wadjak I has to be estimated, though it cannot be very great here.

[1]) Challenger Reports. Vol. X. (1884). Also in 1897 (Some Distinctive Characters of Human Structure. Toronto Meeting of the British Association for the Advancement of Science) TURNER had not found a greater capacity among 63 Australian skulls than 1514 cm^3 of that cranium of Port Curtis in Queensland.

The length of the basis of the cranium, the basi-nasal line, measures 107 mm. DUCKWORTH found as a mean of 26 male Australian skulls 101 mm., of 5 female skulls 95 mm., and as maximum 109 mm.; FLOWER [1]) found 102.5 mm. for 22 male Australian skulls, 100 mm. for 9 male Tasmanian skulls, and 95.5 mm. for 14 female Australian and also for 4 female Tasmanian skulls. The proportions of this dimension to the principal other dimensions of the fossil skull do not deviate from the recent ones; this cannot be a cause of deviation of the capacity.

Taking all this into consideration, and paying attention to the thickness of the cranial walls, which is 10 mm. near the bregma in Wadjak I, the capacity of the fossil cranium can, in approximation, be calculated from its length, breadth, and height.

Applying the methods of MANOUVRIER [2]), of LEE [3]), and of FRORIEP [4]) I find, taking the above mentioned points into consideration, that the capacity of the Wadjak I skull probably amounts to about 1550 $cm.^3$.

This is a high capacity in comparison with that of the Australians and Tasmanians. TURNER (1897) determined the mean of male Australian crania at 1280, and the maximum at 1514 $cm.^3$, of female crania the mean at 1116 $cm.^3$ and the maximum at 1240 $cm.^3$. The Tasmanian race had a capacity perhaps 50 $cm.^3$ higher. Probably the Wadjak men were taller, at least heavier, than their thin Australian descendants, so that they did not exceed these modern races in the relative development of the neurocranium to the splanchnocranium.

---

1) W. H. FLOWER, On the Size of the Teeth as a Character of Race. Journal Anthrop. Institute of Great Britain and Ireland. Vol. XIV. (London 1885), p. 183.

2) L. MANOUVRIER, Sur l'indice cubique du crâne. Association française pour l'avancement des Sciences, 1880, p. 869. — The mean coefficient 1.2 for male Polynesians, Australians etc. was used, and the capacity calculated in Broca-measure was reduced to real capacity.

3) ALICE LEE, A First Study of the Correlation of the Human Skull. Philosophical Transactions of the Royal Society of London. Series A, Vol. 196. (1901), p. 225—264. Formulae and Tables, p. 243—247. The auricular height, which is a necessary factor in these formulae, is 122 mm. in Wadjak I. The formula (p. 243) for the male Naqada-Egyptian crania was used, with which the similarity in form is relatively greatest (cf. chief dimensions p. 246).

4) A. FRORIEP, Ueber die Bestimmung der Schädelkapazität durch Messung und Berechnung. Zeitschrift für Morphologie und Anthropologie. Band 13, p. 347. (1910). An equal result is also obtained by the method of H. WELCKER (Die Kapazität und die drei Hauptdurchmesser der Schädelkapsel bei den verschiedenen Nationen. Archiv für Anthropologie. Braunschweig 1886. Band 16, p. 1), after some modifications of the chief measures required by the particular shape of the fossil skull.

To estimate this relative development ARTHUR KEITH [1]) has introduced the comparison of the capacity with the "palatal area", this area being the space bounded by the outer margins of the crowns of the teeth in the upper jaw and a line joining the posterior margin of the upper third molar teeth. He found this area of the upper dental arcade for a female chimpanzee skull, of a capacity of 320 cm.^{3}, equal to 36.5 cm.^{2}; hence to 1 cm.^{2} of palatal area came 8.7 cm.^{3} of brain capacity. The upper palatal area of a Tasmanian skull was 36.8 cm.^{2}, the capacity of this skull was 1350 cm.^{3}, which gives a ratio of 1 : 36.7. For the *Homo neandertalensis* of Gibraltar [2]) KEITH found for these values 31.6 cm.^{2} and 1200 cm.^{3}, and the ratio 1 : 38, but for the Aurignac-man of Combe-Capelle the ratio is 1 : 53, about that of modern Englishmen, viz. 1 : 56.3, with 26.6 cm.^{2} palatal area and 1500 cm.^{3} capacity.

With pretty great accuracy — as only the crowns of the incisors and the crown of the right m_1 fail — the palatal area of the Wadjak-man II may be determined at 41.4 cm.^{2}. That of Wadjak I, in which only few tooth-crowns have been left, measures about 35 cm.^{2}. Through its relatively small size this palate presents a striking difference from that of Wadjak II, which is one of the characters that lead me to assume that the first found fossil remains belonged to a woman, the second to a man. Other female characters of Wadjak I are: the more reduced form of the teeth (the upper m_2 and m_3 are almost perfectly three-cusped), the smaller dimensions of the comparable parts of the skull, though not in the same degree smaller as the palate, the less pronounced superciliary ridges and the forehead that does not recede so much, the orbits which are higher with respect to their breadth, the somewhat slighter development of the muscle attachments, the more rounded form of the occiput, the somewhat slighter lophocephaly and dolichocephaly, in so far as the latter can be judged from the fragments of the second skull.

If for the fossil woman of Wadjak 44.3 cm.^{3} brain capacity comes to 1 cm.^{2} of palatal area, it may be assumed that for the man, who had a much larger palate, but probably also a larger neurocranium, this ratio was smaller. Putting his cranial capacity 100 cm.^{3} higher

---

[1]) ARTHUR KEITH, The Antiquity of Man. (London 1920), p. 97, 151, 328.

[2]) For the *Homo neandertalensis* of La Chapelle-aux-Saints I calculate an upper palatal area of 38 cm^2 after the reconstructive drawing of BOULE, which with 1626 cm^3 Broca- or 1530 real cranial capacity yields the ratio 1 : 40.3. But the normal palatal area may have been somewhat larger than that of this man, who had early lost his teeth for the greater part.

than that of the woman, which is a plausible estimation, I find for him the ratio 1 : 40. Thus much appears, at any rate, with certainty that as regards the comparative size of the two chief parts of the skull, the neurocranium and the splanchnocranium, Homo wajakensis resembles those most primitive recent human types, and also the Plistocene Homo neandertalensis.

In many more respects there is unmistakable resemblance between Homo wadjakensis and the recent Australian group of races. But he also presents deviations from this group, which are certainly partly due to a more "primitive" condition.

Both these points may further appear from the description of some other characters.

The strongly marked glabella and superciliary ridges, also of the woman of Wadjak, though not in the same degree as of the man, are certainly australoid characters, but the supraorbital borders and also the lateral orbital borders are somewhat less massive and rounded than in the Australian crania. The height of the orbit is 33 mm., the breadth 42 mm. in Wadjak I, so that the orbital index is 78.6. For the male cranium these dimensions and index are 30, 40, and 75; it is remarkable that the orbit is smaller in the man, but the lower index for the woman presents an important sexual difference in the Australians, according to TURNER. He found for the mean orbital index of Australian crania 84, for that of twenty men 81.4, and of nine women 90; FLOWER in fifty-one Australian crania a mean index of 80.9, Quatrefages and Hamy in thirty-one crania a mean index of 78.8. The inter-orbital breadth of Wadjak II is at least 29 mm. TURNER found as mean of male Australian crania 24.5 mm., and as maximum 28 mm.

The root of the nose is deeply sunk (most in Wadjak II), and the bridge of the nose is very flat, rounded from side to side. The apertura piriformis of Wadjak I measures across 30, (Wadjak II 32), the height is 27 mm., to which corresponds the index 111. In Australian crania this index ranges between 82 and 130 according to KLAATSCH; in European crania it is on an average 70. The spina nasalis is short and blunt. The nasal height is 50 in Wadjak I, the nasal breadth 30 mm. (in Wadjak II 32), the nasal index 60. In Australian crania the following values were found as means for this nasal index: 57.9 by QUATREFAGES and HAMY ($N = 31$), 56.9 by FLOWER ($N = 31$), 53.4 by TURNER ($N = 29$), 55.6 with the maximum 65.1 by DUCKWORTH ($N = 38$).

The sides of the nasal aperture are not sharp-edged, but, as generally in Australian skulls, blunt and rounded off, especially near

the floor of the nose. The more or less direct continuity of this floor into the incisive region, which is frequently found in Australian skulls, is a perfect one in Homo wadjakensis; from the outer edge of the nasal aperture a linear elevation continues on to the latter region, curved downward and inward, which is lost at 6 mm. below the nasal floor. This is a transition form between the infantile type of the lower edge of the nasal aperture and the sulcus praenasalis of the Anthropoids, which may be designated as "Affenrinne", and which is undoubtedly in connection with the strong alveolar prognathism of Homo wadjakensis [1]). The other prognathism, which is indicated by the relative lengths of the basi-alveolar and basi-nasal lines, FLOWER's "gnathic index", which he found to be 103,6 on an average in Australian skulls, while TURNER met with a (female) minimum of 92, and a (male) maximum of 108, cannot be accurately determined in Wadjak I; it can, however, be indicated by the index 91 approximatively. The smallness of this index strengthens me again in the conviction that the first found fossil skull must be considered as female. The alveolar prognathism (Fig. 1. Norma lateralis of Wadjak I and Fig. 4. Upper and lower jaw of Wadjak II) is not slight.

A character which peculiarly distinguishes Homo wadjakensis is the extraordinary great breadth of the dental arcade in the upper jaw, compared with its length. (Fig. 6). The maximum width between the outer edges of the crowns of the 2nd upper molar teeth is 81 mm. for Wadjak II, 71 mm. for Wadjak I. The length of the row of five molars, FLOWER's "dental length", is only 50 mm. at the male skull, and 47 mm. at the female skull. In Australian skulls TURNER [2]) found as maximum of width on the second upper molars 73 mm., as maximum of dental length 51 mm. In the fossil Wadjak men the breadths are to the lengths as 1.62 and 1.51 : 1. The palato-maxillary breadth agrees about with the greatest breadth over the molars. It is 82 mm. in Wadjak II, and 70 mm. in Wadjak I. These breadths are to the dental lengths as 1.64 : 1 and 1.49 . 1. DUCKWORTH determined the average of the palato-maxillary breadths of eleven male Australian skulls at 64.9 mm., and the mean dental length at 46.4 mm.; these dimensions are to each other as 1.40 : 1.

---

[1]) Cf. for these forms of the lower edge of the apertura piriformis: RUDOLF MARTIN, Lehrbuch der Anthropologie. Jena 1914, p. 845 et seq.

[2]) Sir WILLIAM TURNER, The Relation of the Dental Arcades in the Crania of Australian Aborigines. Journal of Anatomy and Physiology. Vol. 25. (1891), p. 461—472. P. ADLOFF (Das Gebiss des Menschen und der Anthropomorphen, p. 28. Berlin 1908) found for this dimension on the maxilla of a Melanesian 75.5 mm.

In a female skull these dimensions and ratio were 63 mm., 46 mm. and 1.37 : 1. Greatest, viz. 1.52 : 1, was the ratio of a male skull with 70 mm. palato-maxillary breadth, and 46 mm. dental length. The mean dental length, determined by FLOWER[1]) from twenty-two male Australian skulls, was 45.9 mm., and from fourteen female skulls 44 mm., from nine male Tasmanian skulls 47.5 mm., and from four female skulls 44 mm.

FLOWER's dental index (dental length $\times$ 100 : basal length) was 44.8 for Australian, and 47.5 for Tasmanian male skulls, 46.1 for Australian and 48.7 for Tasmanian female skulls. At the skull of Wadjak I this index is 44, hence the dental length is relatively small, probably still smaller in Wadjak II.

The breadth between the outer margins of the 2nd upper molars at the fossil skull of Gibraltar is 71 mm. according to KEITH, the dental length, from his drawings (mean of left and right row of teeth) 45 mm.[2]). The breadth is to the length as 1.57 : 1. Almost perfectly the same ratio, 1.56 : 1 for 75 mm. breadth and 48 mm. dental length, is presented by the upper dental arcade of the fossil man of La Chapelle-aux-Saints in the reconstruction of BOULE[3]). Accordingly Homo wadjakensis resembles Homo neandertalensis in this large relative breadth of the upper dental arcade.

This, however, holds only for the greatest breadth (measured at the molars) of the upper dental arcade; the form of this arcade is very different. Whereas the curvature of the arcade of the Neandertal Man continues regularly forward, the arcade curve of the upper teeth of Homo wadjakensis, especially of the male individual, changes its form anteriorly to the molars. The three molars lie in a parabolic line of greater parameter, the foremost half of the teeth row (the praemolars, canini, and incisivi) in a similar line of smaller parameter, so that for the row of the molars the dental arcade narrows, but not gradually. The parabolic line of the foremost half of the dental arcade departs only little from the almost uniform line, in which the whole dental arcade of the lower jaw lies. In fact, the front half of the dental arcade of the upper jaw projects little at the praemolars (of course not at all at the incisivi) beyond that of the lower jaw; the upper molars, however, project greatly outside the lower molars. The width between the outer margins of

[1]) FLOWER, l.c., p. 186.

[2]) KEITH, l.c., p. 149–151.

[3]) MARCELLIN BOULE, L'Homme fossile de la Chapelle-aux-Saints. Extraît des Annales de Paléontologie. (1911—1913). Paris 1913, p. 100, fig. 60. The dental index (FLOWER) was only 38.

the lower 2nd molars is only 69 mm., the arcade in the upper jaw being 12 mm. wider there than in the lower jaw, (in Australians — which race surpasses others in this respect — TURNER found as maximum 8 mm.). The consequence of this remarkable relation between the two dental arcades is that the lingual cusps of the crowns of the molars in the upper jaw have been worn off obliquely from inside and above to outside and below on the buccal cusps of those in the lower jaw, while on the other hand in the upper jaw the buccal cusps, in the lower jaw the lingual cusps of the crowns of the 2nd and 3rd molars are worn off very little, if at all, and the crowns of the 1st molars at least unequally on the buccal and lingual half, the lower ones very obliquely.

Dental arcades resembling the described type, though perhaps not so pronounced, are not seldom met with in Australian and also in Malay skulls; but the type of the Neandertal Man is an entirely different one. Also the molar half of the upper dental arcade projects but little outside that of the lower jaw; the two arcades have the same shape, and cover each other much more, and the wear of the crowns takes place over the whole grinding surface more equally, horizontally. It may be assumed that the food of Homo neandertalensis was of a different nature from that of Homo wadjakensis and of the Australians. This race lives chiefly on animal food; very probably the mode of living of Homo neandertalensis was more vegetarian. In connection with this it is of importance that in an examination with X-rays, made with the collaboration of my brother, Dr. V. DUBOIS, it was found that the teeth of Homo wadjakensis possess roots and pulp-cavities that agree in form and size with the Australian type, and depart entirely from the taurodont type of the Neandertal men.

The following remarks about the most important characters of the teeth and the mandible may now precede a further discussion.

On the whole the teeth are large, though they are still surpassed by those of many Australians. The 2nd and 3rd upper molars present reduction phenomena, especially in Wadjak I.

The mandible (Fig. 7 and Fig. 8) is a very strong bone, clearly built according to a type resembling a common Australian one. The corpus mandibulae is, pretty uniformly, high (40 mm. at the symphysis of Wadjak II. Average of 7 Australians 33 mm., maximum 42 mm. according to FRIZZI [1])) and thick. The ramus is very broad (at the

[1]) E. FRIZZI, Untersuchungen am menschlichen Unterkiefer mit spezieller Berücksichtigung der Regio mentalis. Archiv. für Anthropologie. N. F. Band IX. (1910), p. 252—286.

narrowest place 46.5 mm. in Wadjak II. For 7 Australians, according to FRIZZI, the breadth is on an average 37 mm., maximum 40 mm.). This applies particularly to the mandible of Wadjak II, which I consider as male, but to a certain extent also to the other mandible, of which only little is preserved.

The symphysial or mental angle (between the infradental-pogonium line and the base line) measures 96°. Though there is a strongly developed protuberantia mentalis, yet the perpendicular dropped from the infradental point or incision, falls 3 mm. before its most projecting point, the pogonium. When this projection, which makes the impression of being a separate formation, is thought eliminated, the angle of the chin would be 102°. The other symphysial or mental angle, that with regard to the alveolar line, measures 80°. It would attain 86°, when the protuberantial swelling did not exist. For the Neandertal-mandibulae, which possess no or very small protuberantia, the angle is still considerably greater. La Naulette 94°, Spy 106°, Mauer 105°. From seven Australian mandibles FRIZZI (like WELCKER from fifteen) found a mean of 83° for this angle, the maximum was 94°. But this greater angle of the Australians is also partly owing to the mostly slight development of the protuberantia mentalis. The true angle of inclination of the corpus mandibulae at the symphysis (without that projection), can yet be called peculiarly great in Wadjak II. Hence FRIZZI's "Korrekturvertikale" i.e. the perpendicular drawn to the alveolar border line, close along the deepest point of the chin concavity, only just intersects the protuberantia of Wadjak II. Noteworthy of this fossil mandible is further the relatively thin inferior border or base, and the situation of the small fossae digastricae, behind this border, 23 mm. apart from each other, reminding of the condition of Hylobates syndactylus.

In comparison with the dentition of the Wadjak Man, another find may be mentioned of a fossil man related to the present Australian race, the skull of Talgai in Queensland, Australia, which was discovered in 1884, mentioned by T. W. E. DAVID and J. T. WILSON [1]) in 1914, and elaborately described by STEWART ARTHUR SMITH [2]) in 1918. This skull of a "male youth" (for m_3 was still unerupted), though cracked *in situ* into numerous fragments, which are more or less considerably dislocated, but held in position by thin layers of calcareous earthy matrix cementing them together,

[1]) Reports of the British Association for the Advancement of Science. Sydney Meeting. (1914), p. 531. — Cf. also „Nature". London 1915, p. 52.

[2]) STEWART ARTHUR SMITH, The Fossil Human Skull Found at Talgai, Queensland. Philosophical Transactions of the Royal Society of London. Series B, Vol. 208, pp. 351—387. 7 Plates. Londen 1918.

the condition resembling a coarse mosaic, can yet be clearly recognized as not deviating, in its general features, from the present aboriginal Australian skull. The cranium as a whole, and the palatum, however, hardly admit of any reliable measurements. They could still be made at the tooth-crowns, each in itself, but most of them have more or less receded from each other; the apparent palatal area thus considerably exceeds the real, which, in my opinion, was no larger than that of the Australian native of present times. SMITH supposes that the (upper) canine tooth, in an analogous way as in the dentition of Apes, though without a true diastema in the maxilla, penetrated, almost ape-like, with its apex between the lower canine and the lower first premolar. In my opinion there is reason to doubt this, on the ground of a comparison with the teeth of Wadjak II. The facets on the upper canine, which have been described by SMITH (loc.cit. p. 374 et seq. and figures 6, 21 and 22) and considered by him to have been caused by the projecting between the said teeth in the mandible, are identical in their position with facets on the upper canine in the Wadjak maxilla. One of them, on the distal (posterior) surface, can be clearly recognised as interstitial contact facet (ZSIGMONDY) with the first premolar tooth (in the maxilla). The other placed on the lingual slope of the narrow margin of the mesial surface, by the side of the interstitial contact facet on the mesial surface caused by the contact with the lateral incisor tooth, is to be recognised, by comparison with Wadjak II, as belonging to the general wear of the masticatory surface. In his reconstruction (Fig. 4) SMITH lowers the upper canine tooth to nearly 7 mm. below the level of the mesial margin of the upper premolar, till the upper border of its crown gets very nearly on a level with the upper border of the crown of the premolar. Erroneously, for the crown-border of such a large upper canine tooth as the Talgai canine, is always considerably above the level of the crown-border of the upper premolar; in the maxilla of Wadjak II the distance is 3 mm. The upper canine, therefore, cannot have projected so far downward as is required according to SMITH's interpretation of the distal (posterior) facet. The canine tooth of Wadjak II, which strikingly resembles that of Talgai, is also equally broad as the latter, and if its wear were as little advanced as that of the canine of the boy of Talgai, it would no doubt be as pointed and little shorter than the latter.

If for those reasons I cannot agree with SMITH in ascribing to the fossil skull of Queensland, which indeed he too considers as typically Australian, "characters more ape-like than have been observed in

any living or extinct race, except that of *Eoanthropus*", this skull is nevertheless of great importance, because several circumstances, mentioned by SMITH along with his valuable description, go to show that the aboriginal Australian with Canis dingo already lived, in the smallest continent, by the side of now extinct Marsupialia, which are generally considered as Plistocene. As the said species of the true Canis genus, the only large Placental Mammal of Australia besides Man, has most probably come with the latter from East-Asia, the find of Talgai throws also some more light on the geological age of the fossil Man of Wadjak.

But the "Talgai Man" does not at all indicate a nearer approach to the common ancestor of modern mankind than do the Australian aborigines of the present time.

If, however, the Australians may justly be considered as the most "primitive", the "lowest" type, i.e. that of living races of Man resembling most closely the common stock-type, it might have been reasonably expected that the real predecessor of humanity would be found in their fossil ancestors; unless the Australian type was evolved already long ago and has since remained unchanged.

On account of the unmistakable morphological resemblance, also of the geographical relation and the antiquity, the fossil man of Wadjak may certainly be considered as an ancestor of the present Australian racial group, a proto-Australian. The geographical relation is obvious, and though there are no direct data for the determination of the geological age, this must certainly be considerable; several indirect data which I have mentioned, render it probable that a rather early place in the Plistocene period may be assigned to our fossil Man.

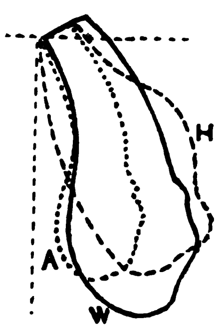
The expectation, however, to find in him a distinctly lower type than the Australian of the present time, has not been realised, for this ancestor had reached the same stage in the evolutional scale as the living race, at least almost.

Striking are the many points of resemblance on the skull and the lower jaw of the Wadjak Man with the Australian group, especially the aboriginal of the largest insular country. The differences may nearly all be attributed to more vigorous development and greater perfection of the type, in surroundings more favourable than those in which the Australian native finds, and has found for a long time, a scanty subsistence. Homo wadjakensis was an optimate form. In the present race the type is evidently in a state of decadence, as also Homo neandertalensis is the less vigorous and less perfect descendant of Homo heidelbergensis. Judging from the lower jaw, also of the latter, the type was purer and in this sense more primitive in the older of the two forms.

At the neurocranium of the Wadjak Man only the somewhat smaller relative size of the frontal part and the jutting backward of the occipital lobe of the cerebrum ("pointed" in the Wadjak II skull, because the occiput is not only flattened in vertical direction, but is also relatively narrow) can be considered as primitive in the true sense, i.e. *phylogenetically*. But even this is somewhat doubtful, in my opinion, where the development of the total brain volume was certainly no less than in the present Australians.

The powerful jaws give a truly bestial appearance to the splanchnocranium. But an absolutely large and strong masticatory apparatus is no evidence as such of a phylogenetically primitive condition. Thus the powerful masticatory apparatus of the Eskimos is only a requirement of the way of living of the hyperboreans, namely their feeding chiefly on raw meat and bacon. Besides, in proportion to the brain capacity, the palatal area of Homo Wadjakensis is certainly no larger than that of the Australians, as was demonstrated above. Taking into consideration that here a *surface* is compared with a *volume*, it is found, that certainly in Wadjak I, and probably also in Wadjak II the mean longitudinal dimension of the masticatory apparatus, *in proportion to that of the brain, is smaller* than in the compared Tasmanian.

The dimensions of the jaws and the teeth of the Wadjak Man, *taken each in itself*, even remain all within the limits of other pachygnathous and megadont fossil and living human types; the deviations are never so considerable as to assume systematic significance. The Wadjak Man is certainly megadont, as the Australian racial group and also Homo neandertalensis, and even the Combe-Capelle Man [1]). In the absolute strength of the masticatory apparatus, *taken as a whole*, the Wadjak Man is, however, only equalled, not surpassed by Homo heidelbergensis. Just as in the whole build of the skeleton, the Australian is a type diametrically opposite to the Neandertalian, as Boule has demonstrated [2]), in the same way Homo wadjakensis is so of Homo heidelbergensis, at least certainly in the lower jaw (compare especially the cross-sections of the symphyses in the adjoined diagram). But these have both the most powerful masticatory appara-



[1]) The size of the teeth may appear from the subjoined comparative tables. (See Tables following page). The maxima of the living races of man are taken from de Terra, Black, Mühlreiter, Adloff.

[2]) L'Homme fossile de La Chapelle-aux-Saints, p. 231—234. Paris 1913.

tus known of their type, the former also of the Homo sapiens-type.

In contrast, however, with Homo heidelbergensis reduction pheno-

Maximum dimensions of crowns of teeth in the maxilla (*mm.*)

	Wadjak II	(I)	Talgai	Combe-Capelle	Krapina	All living races
C sup. Mesio-distal	9.7	—	9.6	8	10.5	9.3
„ „ Labio-lingual	10.2	—	10.9	9	11.3	10.8
P_1 sup. Mesio-distal	8.3	—	8.6	6	8.2	9.5
„ „ Bucco-lingual	11.0	—	12.3	9	11.4	12.5
P_2 sup. Mesio-distal	8.0	(7.6)	8.1	6.5	—	8.2
„ „ Bucco-lingual	10.8	(11.0)	11.0	9	—	11.7
M_1 sup. Mesio-distal	12.0	(11.2)	12.6	10.5	13.3	12.8
„ „ Bucco-lingual	13.0	(13.8)	13.1	12	13.3	14.5
M_2 sup. Mesio-distal	11.0	(10.6)	11.3	10.8	12.0	11.8
„ „ Bucco-lingual	13.5	(14.2)	13.5	12	14.0	14.7
M_3 sup. Mesio-distal	11.0	(8.2)	—	8.2	—	11.7
„ „ Bucco-lingual	13.0	(13.0)	—	11.5	—	14.8

Maximum dimensions of crowns of teeth in the mandible (*mm.*)

	Wadjak II	(I)	Combe-Capelle	Mauer	Krapina	Spy II	All living races
I_1 inf. Mesio-distal	6.2	—	5	5.5	6.2	6.0	6.5
„ „ Labio-lingual	7.2	—	6	7.1	8.1	7.5	7.7
I_2 inf. Mesio-distal	6.8	—	6	6.0	7.5	6.0	7.2
„ „ Labio-lingual	7.6	—	6.5	7.8	8.2	8.0	7.6
C_2 inf. Mesio-distal	8.4	—	8	7.7	8.4	7.5	9.0
„ „ Labio-lingual	9.5	—	9	9.0	10.0	9.0	10.0
P_1 inf. Mesio-distal	8.5	—	6	8.1	8.3	7.5	8.7
„ „ Bucco-lingual	9.0	—	9	9.0	10.0	9.0	9.8
P_2 inf. Mesio-distal	8.3	—	7	7.5	8.5	7.5	9.0
„ „ Bucco-lingual	8.5	—	9.5	9.2	9.9	9.0	10.5
M_1 inf. Mesio-distal	13.7	—	12	11.6	13.8	11.5	12.8
„ „ Bucco-lingual	12.5	—	12	11.2	12.4	11.5	12.2
M_2 inf. Mesio-distal	11.7	(12.0)	12	12.7	12.5	11.0	12.5
„ „ Bucco-lingual	11.0	(11.0)	11.5	12.0	11.4	11.0	12.0
M_3 inf. Mesio-distal	12.0	(13.0)	11	12.2	13.6	12.0	15.0
„ „ Bucco-lingual	11.3	(11.0)	11	10.9	11.0	12.0	13.0

mena occur at the crowns of the molars of Homo wadjakensis in a no less degree than in recent men. The hindmost lingual cusps of the second and the third molar of the upper jaw of Wadjak II, and particularly of Wadjak I are little developed. In connection with this the mesio-distal dimension is relatively small, especially in m_3 of Wadjak I. In Wadjak II the crown of m_1 shows on the backside, in the middle a small accessory cusp, which reminds of what has been described by EMIL SELENKA about the Orang-utan, and which was also found in Man in rare cases.

The total length of the canine of the upper jaw of Wadjak II measured on the skiagram, is 29 mm. MÜHLREITER found 37 mm. as maximum of all living races. Thus measured, the length of the roots of the upper m_1 is 14 mm., of m_2 16 mm., and the distance of the root-ends resp. 9 and 7,5 mm.; this distance is about 10 mm. in m_3. In these respects, and in the vertical depth of the pulp-cavities, which is 15 mm. in m_1, 25 mm. in m_2, the fossil form of Java again resembles the Australians, and differs from Homo neandertalensis (and heidelbergensis).

The folding of the enamel (crenation) is more composite than in Europeans, but not to a higher degree than is also found in Australians.

The two premolars in the lower jaw of Wadjak II are not larger and not more primitive of form than in the fossil mandibles already known and those of the living races of Man. The crowns of the (loose) incisors and canine, like those of the first premolar, are larger, of the second premolar and of the molars on the other hand smaller than in Homo heidelbergensis. The joint length of the two back molars in the latter is 25 mm., in Wadjak II 24 mm., the distance from the incision to the back margin of m_1 being 42 mm. in Homo heidelbergensis and 47 mm. in Wadjak II. The length of the dental arch was 58 mm. in the Heidelberg Man, and probably 60 mm. in Wadjak II. In the Wadjak Man the front part, in Homo heidelbergensis the back part of the dental series was larger. In the latter the molars are all three five-cusped, and the crown of the middle molar is the largest. In Wadjak II, on the contrary, only the front molar is five-cusped; this is also the largest, the second and third molars are four-cusped, and comparatively considerably smaller; the middle one is the smallest. In the fragment that is extant of the lower jaw of Wadjak I (the right back half of the corpus mandibulae and the lower part of the ramus with the angulus, in which the complete m_2 and m_3, besides the greater (back) part of m_1, with much less worn crowns than in Wadjak II), m_3 has three buccal and two lingual cusps; this tooth-crown is also

longer in mesio-distal direction, though the whole of the jaw must have been less large than that of Wadjak II. All these differences between Homo wadjakensis and Homo heidelbergensis are certainly significant of a difference in function of the molars.

A five-cusped m_3 have on an average three of the four Australians and one of the four representatives of the Malay race; in both races m_2 is five-cusped in two out of three individuals on an average. In this respect Papuans agree more closely with Malays than with Australians. In Homo neandertalensis nine of the twelve m_1 were found to be five-cusped, and m_2 nearly always four-cusped (in ten out of eleven of these teeth); probably the crown of m_3 is no less reduced. The two individuals of Homo wadjakensis, therefore, in this respect, closely resemble Homo neandertalensis, and are certainly less on the primitive side than the average Australian native. As has been said, the lower molars of Homo heidelbergensis on the other hand, are all three five-cusped, hence they present the more primitive condition. From what is observed in the living races it seems, however, that both the number of cusps and the size of the dental crowns are in connection with the function.

Some of the most important characters of the maxilla and the mandible of Homo wadjakensis I have already briefly described. The following remarks may now be added.

The protuberantia mentalis, a low trigonal pyramid with rounded edges and vertex, with its base put, as it were, on the uniformly bent outer side of the corpus mandibulae, and rising 3 mm. above this surface (ideal of the "éminence triangulaire, plus au moins bombée à son centre, qui se superpose à la face antérieure de la mandibule" of the European lower jaw, in the description of Topinard), may be clearly recognised as a formation that has arisen independently of the growth of the basal part of the corpus mandibulae. According to KLAATSCH [1]) such a "trigonal prominence" is also what is found, as a rule, in Australian mandibles. The basal part is by no means bent outward as in many modern lower jaws, but the external surface of the corpus mandibulae is straight to the inferior border. The internal surface at the chin, apart from the spina mentalis placed on it, is only slightly convex, and inclines almost uniformly from above downward. The spina is of a type frequently occurring in Home sapiens, which I will designate as scissor-shaped of outline, as it really presents a close resemblance with the outline of a

[1]) H. KLAATSCH, The Skull of the Australian Aboriginal. Reports from the Pathological Laboratory of the Lunacy Department. New South Wales Government. Vol. I. Part III, p. 155. Sydney 1908.

shorter or longer pair of tailor's scissors; it consists of a median crista geniohyoidea, 9 mm. long, and two round tubercula genioglossa lying 6 mm. apart above it. The foramen mentale is placed under the interval between p_2 and m_1, and directed backward.

An incisura submentalis, so considerable in the mandible of Mauer (Homo heidelbergensis) (10 mm. deep), is scarcely perceptible (1 mm. deep) in that of Wadjak II. The incisura praeangularis s. praemasseterica (BONNET) is, on the other hand, uncommonly deep. There also exists a very considerable tuber massetericum (BONNET). These two prove that the musculus masseter was exceedingly powerful.

In strength the lower jaw of Wadjak II is not inferior to that of Homo heidelbergensis. This may already be inferred from the vertical sections in the symphysis-line in comparison also with the most frequently occurring Australian type and with the common European lower jaw. (Fig. 8 of Plate II). For in the symphysis the lower jaw has to resist the greatest violence.

The strength of this bone in Wadjak II may further appear from the following measures. The height of the corpus mandibulae at the symphysis, 40 mm. in Wadjak II, is about 33.5 mm. in Homo heidelbergensis, 36 mm. at the mandibula of Spy, 30 mm. at that of La Naulette. The median thickness at the symphysis, 16 mm. in Wadjak II (above the spina mentalis) is on the other hand 17.5 mm. in Homo heidelbergensis, but only 15 mm. at the mandibula of Spy, and also at that of La Naulette. The height, measured between p_2 and m_1 is 37 mm. in Wadjak II, and the thickness there 17 mm.; in Homo heidelbergensis these measures are resp. 33 and 19.4 mm.

The greatest thickness of the body of the lower jaw, under m_2, is 21 mm. in Wadjak II, which is equal to the "enorme Breite" found by OETTEKING, and also by GORVANOVIĆ-KRAMBERGER, each once, in lower jaws of Eskimos, in which race this bone is peculiarly strong as a rule [1]). The lower jaw of Mauer is 23.5 mm. thick at the same place, that of La Naulette only 15, and of Spy 16 mm., Australians not seldom attaining 19 mm. The height at m_2 is about 35

[1]) B. OETTEKING, Ein Beitrag zur Kraniologie der Eskimo. Abhandlungen und Berichte des Königl. Zoologischen und Anthropologisch-Ethnographischen Museums zu Dresden. Band XII (1908). N⁰. 3, p. 38.

K. GORJANOVIĆ—KRAMBERGER, Der Unterkiefer der Eskimos (Grönländer) als Träger primitiver Merkmale. Sitzungsberichte der Königl. Preuss. Akademie der Wissenschaften. Jahrgang 1909, p. 1282—1294. Taf. XV und XVI, p. 1283.

mm. at the lower jaw of Wadjak II, at that of the Heidelberg man 30 mm.

The condylar height of the ramus mandibulae of Wadjak II is about 70 mm., the breadth, at the narrowest place, 46.5 mm. At the lower jaw of the Heidelberg Man these dimensions are resp. 69 and 52 mm., but the angulus (s. arcus, BONNET) is, as it were, cut off obliquely, just as in the Neandertal Man of La Chapelle-aux-Saints, whereas it forms a round projection at the lower jaw of Wadjak II. The external surface of the ramus cannot be measured accurately, because this part of the lower jaw has broken off with the loss of some parts that cannot be accurately determined, but it is very large, and may be estimated at only 2 cm.^{2} less than that of the Heidelberg Man. In the latter this surface is enlarged, it is true, by the very considerable breadth of the processus coronoides, but on the other hand the angular part is much larger in the Wadjak Man. The outer surface of the ramus at an average European lower jaw is 16 cm.^{2} smaller, at an Australian lowerjaw (according to KEITH) 12 cm.^{2} smaller than at that of the Heidelberg Man. This means that the area of attachment of the muscles of mastication of the Wadjak Man is almost as large as that of the latter — in the Heidelberg Man the musculus temporalis preponderated, in our Javanese Australian the masseter — and much larger than that of the present European, and even of the Australian aborigines.

The condylus is in transversal direction as large as that of the Mauer-mandibula, on the other hand in sagittal direction much smaller and rounder. Also the glenoid fossa is of the present type. The articulation was evidently, as in general in Homo sapiens,more hinge joint, for movement up and down of the lower jaw, than gliding joint, hence less adapted to grinding motion of the lower jaw than that of Homo heidelbergensis. The important differences of the same nature, which exist between the temporo-mandibular joint of Homo neandertalensis and modern Man, have been set forth by BOULE in his masterly description of the fossil Man of La Chapelle-aux-Saints. The very wide and shallow articular cavities of the latter were certainly adapted to grinding movement, almost as in the Anthropoids.

The processus coronoides is narrower, but higher and consequently the incisura mandibulae is deeper than in Homo neandertalensis and heidelbergensis.

The external surface of the ramus as a whole inclines somewhat towards the outside from above downward, so that the two rami diverge. This is still more pronounced for the regio angularis,

because this is, besides, in itself strongly bent outward, which is especially apparent when the posterior and inferior border are considered. This part of the ramus is thick and strong. This thickness and the bending of the angular part of the ramus outward mean strong development of the musculus masseter, absolute and in comparison with the musculus pterygoidens internus [1]). In the morphology of the ramus mandibulae described, as in that of the chin-region, the lower jaw of Wadjak II represents very perfectly the type of modern Man (Homo sapiens).

Entirely opposite to this is the type of the lower jaw of Homo neandertalensis, which is exhibited in its greatest purity by the Mauer mandible (Homo heidelbergensis). Here no protuberantia mentalis (in the older form, the Heidelberg Man) or only traces of it (in some representatives of the later form, the Neandertal Man proper), nor outward bending of the inferior border of the corpus mandibulae. On the other hand, on the inner side of the regio mentalis, particularly in this most original and powerful jaw of the type, a considerable strengthening of the arch of the mandible by means of a *torus mentalis internus*, closely corresponding to that of the Anthropoids and of most of the lower Monkeys, and in connection with this no spina mentalis, or one that is only little developed. The two rami converge from above downward, and the thin pars angularis is bent inward (at least not outward). In the Homo neandertalensis of La Chapelle-aux-Saints Boule has also described and drawn this important obliqueness of the rami mandibulae with regard to the sagittal plane of the skull, and the greatly narrowed pars angularis, which makes the said obliqueness more apparent, in that it "se déjette en dedans, au lieu de se déjetter en dehors, comme dans la plupart des mandibules humaines actuelles" [2]); he has also pointed out its occurrence in many cynomorphous Monkeys and in the Orang-utan among the Anthropoid Apes, also seeing in this an indication for the comparatively great strength of the musculi pterygoidei (which may also be inferred from the extensive surfaces of their origin and insertion).

It is clear that thus in the Man of Wadjak, just as in a more or less degree in general in Homo sapiens, the directions of the right and the left musculus masseter, which muscles moreover had their

[1]) According to Theile the musculus pterygoideus internus, in a strong European, has not even half the weight of the masseter, the musculus temporalis on the other hand one and a half times the weight.

[2]) Marcellin Boule, L'Homme fossile de La Chapella-aux-Saints. Paris 1913, p. 93—94 and p. 65, fig. 45.

origin from much less projecting zygomatic arches, than in Homo neandertalensis, strongly diverged from each other downward, which must have gone together with peculiarly strong divergence in that direction of the musculi pterygoidei interni. Jointly with the musculi temporales and pterygoidei interni, the masseters drew, in their principal action, not only the lower jaw upward at the angles, but at the same time the two angles towards each other, through which its arch was greatly strained, most at the symphysis, where the curvature of the arch is greatest, and caused there on the outside of the lower jaw very considerable stretching strains. In general in Homo sapiens the resulting contraction direction of all the muscles of mastication is converging upward, and stretching strains arise of this nature.

In the mandibular type of Homo neandertalensis, on the contrary, strong strains must have arisen in the mandibular arch on the inside of the symphysis, in consequence of the convergence of the two musculi masseteres, which was still increased by the peculiar projection of the zygomatic arches — the considerable phaenozygy. The same we find in the Apes, for also those Anthropoids in which the ramus mandibulae is not directed obliquely to the sagittal plane from above outside to below inside, yet present convergence of the two musculi masseteres in consequence of the projecting far beyond the sides of the skull of the zygomatic arches, from which these muscles take their origin; the phaenozygy is here still more considerable than in Homo neandertalensis.

It will remain WALKHOFF's great merit that he was the first to draw attention to muscular action as an explanation of the chin of Homo sapiens. In his conception Homo neandertalensis and Homo heidelbergensis must have been almost or entirely speechless, which, taking the great brain-capacity of the Neandertal Man into consideration, is very doubtful. But VAN DEN BROEK is justly of opinion that other muscles than those which are directly active in speech, namely the facial (mimic) muscles and the muscles of mastication, may have given rise to the particular form of the chin of modern Man. He chiefly thinks of the facial (mimic) muscles [1]. Here stress may be laid on the action of the muscles of mastication, by the side of whose strength, which acts not less continually than the facial muscles and which is to be measured by more than a hundred kilograms even in Europeans, the power of the lingual and hyoid

[1] A. J. P. VAN DEN BROEK, Ueber Muskelinsertionen und Ursprünge am Unterkiefer; ein Beitrag zur Kinnfrage. Zeitschrift für Morphologie und Anthropologie. Band 21, p. 227. Stuttgart 1920.

muscles taken into consideration by WALKHOFF and of the facial muscles, becomes negligible [1]).

In order to be able to resist the stretching strains described above, which are caused by the action of the masticatory muscles, the lower jaw had to be reinforced at the symphysis. This has happened: first, by a general uniform heightening or thickening, both in the mandibular type with stretching strains on the outside and in that with stretching strains on the inside; secondly, through locally restricted strengthening, and then: in the type of Homo sapiens (and Homo wadjakensis) with stretching strains on the outside, through: (*a*) a protuberantia mentalis, (*b*) the lower border bent outward (torus marginalis), which is not found in the Wadjak-Man; in the type of Homo neandertalensis (and Homo heidelbergensis) and most Apes, with stretching strains on the inside, through: (*a*) a torus mentalis internus, (*b*) a lamina submentalis (KEITH's "simian plate, shelf or ledge"), which latter is only met with in Monkeys, not in the Neandertal-Heidelberg Man [2]).

The existence of these strengthenings of the mandible need not only be accepted as mechanically efficient and necessary, such a growth of bony substance may also be considered as a definite consequence of muscular action, which — as AICHEL [3]) has demonstrated — causes directly or indirectly stretching strain ("*Zug*"), and with it physiological stimulation of the periost.

What then explains further the difference in direction of the muscles of mastication, which is the cause of the two mandibular types? Why is the direction of the musculus masseter slanting from above and outside towards below and inside in the type of Homo neandertalensis-heidelbergensis and the Monkeys, and on the contrary, at least the resulting direction of contraction of the muscles of mastication in the type Homo sapiens-wadjakensis from above and inside downward and outside?

The explanation is to be found in the special function of the masseter and the other muscles of mastication. A different direction

---

1) In the large Anthropoids (Orang-utan) the strength of the muscles of mastication, estimated by their weight, is three times as great as in Europeans.

2) This torus mentalis internus is another than the torus mandibularis met with by C. M. FÜRST (Verhandlungen der Anatomischen Gesellschaft 22. Versammlung, p. 295. Jena 1908) in about 80% of the lower jaws of Eskimos examined by him, on the inside of the premolars.

3) O. AICHEL, Vorläufige Mitteilung über Entstehung und Bedeutung der Augenbrauenwülste, zugleich ein Beitrag zur Abänderung der Knochenform durch physiologische Reizung des Periostes. Anatomischer Anzeiger, Band 49 (1916), p. 497.

and unequal strength of them must go hand in hand with different and unequally strong function. In the last-mentioned type, that of modern Man, the musculus masseter is comparatively stronger, the musculi pterygoidei are weaker than these muscles were in the type of the Neandertal Man. The direction of the musculus pterygoideus internus in modern Man is such that it strengthens the action of the masseter to a considerable degree, thus helping to elevate the lower jaw, whereas in the Neandertal type the more transverse direction of this muscle (which is besides stronger), with regard to the ramus, caused it and the musculus pterygoideus externus, with the musculus temporalis, to be especially active in the grinding movement. The latter muscle was more developed broadwise (in sagittal direction), less as to its height (vertical direction) in Homo neandertalensis, which could be inferred not only from the form of its attachment area on the skull [1]), but also from its broad insertion, appearing in the shortness, but considerable breadth of the processus coronoides and the shallowness of the incisura mandibulae. The backmost part of the muscle, active in the masticatory movement, was evidently, compared with the type of Homo sapiens-wadjakensis, relatively stronger than the front part, which assists the masseter.

Thus the masticatory apparatus of the type Homo neandertalensis-heidelbergensis was undoubtedly more adapted for grinding movement; that of Homo sapiens-wadjakensis, on the other hand, particularly suitable for biting, cutting, and crushing of the food. The latter type was most perfect in the Wadjak Man. The lower dental arch is here at the molars narrower by the width of a crown than the upper dental arch, so that, as I have already mentioned, the buccal cusps of the lower molars are worn off very obliquely against the lingual cusps of the upper molars, whereas the lingual cusps of the lower, and the buccal cusps of the upper molars have remained intact (m_2 and m_3), or are worn off a good deal less (m_1). Grinding mastication, with horizontal movement of the lower jaw, as in the other type, must not have been possible with this obliqueness of the masticatory surfaces and great inequality of the two dental

[1]) Described by M. BOULE, loc. cit. p. 43, of the skull of La Chapelle-aux-Saints. Compare also: R. VIRCHOW (Zeitschrift für Ethnologie. Berliner Gesellschaft für Anthropologie, Ethnographie und Urgeschichte. 1872, p. 8) on the Neandertal-skull and J. FRAIPONT (JULIEN FRAIPONT et MAX LOHEST, Recherches ethnographiques sur des ossements humains découverts dans les dépots quaternaires d'une grotte à Spy. Archives de Biologie. Vol. VII. (1886), p. 720. Gand 1887) on the Spy-skulls.

arches. This type of dental arch and teeth of the Wadjak Man, to some extent analogous to that of the Carnivora among the Mammals, was certainly particularly suitable for animal food. In the Australians, which live chiefly carnivorously, the difference in breadth of the two dental arches is greater than in any other living race, perhaps with the exception of the Eskimos, but even in Europeans the upper dental arch is, as a rule, wider at the molars than the lower arch; this is a general character of Homo sapiens [1]).

KLAATSCH considers this wide lateral prominence of the upper dental arch of the Australians as a character of the primitive state; the dentition of his *Homo aurignacensis* of Combe-Capelle had lost this "Primitivität" of the Australians [2]). This can only refer to an original type of Man, not to a prehuman stage; for in the Anthropoids and most other Monkeys the upper molars certainly do not extend further beyond the lower ones than in modern Man. Such conditions, with narrow lower dental arch and oblique wearing off of the teeth, as are met with in the Wadjak Man, have even been described of jaws of the Eskimos, who belong to the Mongoloids, but feed chiefly on raw meat and bacon [3]).

Entirely different was the type of the relation of the dental arcades in the Neandertal- (and probably also the Heidelberg-) Man. The two dental arches must have covered each other perfectly or the upper molars must have extended but little outside the lower ones, as in most Monkeys; for the crown of these teeth were ground off horizontally, at least uniformly over their entire breadth. The prematurity of the wearing off in comparatively still young individuals, has struck many investigators; it is universally attributed to coarseness and impurity of the vegetable food, which was often mixed with small quantities of earth. This renders it probable that Homo neandertalensis found his food mostly on (or in) the ground; this can also be deduced from particularities of his skull and skeleton, which will be discussed further.

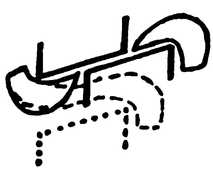
As meat and fish, in general animal food, contain the nourishing substances in a relatively pure state, and are mostly not hard, they need not be ground particularly fine to be digested. Biting off by the front teeth, tearing, and crushing by the molars is sufficient; thus the food can rapidly pass, almost *linea recta*, through the mouth-

[1]) SIR WILLIAM TURNER, The Relations of the Dental Arcades in the Crania of Australian Aborigines. Journal of Anatomy and Physiology. Vol. 25 (1891), p. 461—472.

[2]) In Praehistorische Zeitschrift. Band I (1910), p. 313. Berlin 1910.

[3]) K. GORJANOVIĆ-KRAMBERGER, l. c.

cavity to the gullet. The direction in which the masticatory muscles draw the lower jaw against the upper jaw, was in the Wadjak-Man from below and outside towards above and inside, in which direction the masticatory surfaces of the molars have been ground off against each other. For such a jaw is also frequently active alternately left and right, in which the half of the mandible which was first somewhat abduced when the mouth was opened — in such a way that the molar-crowns are directly above each other — is moved obliquely from below and outside towards above and inside. Thus between the buccal crown-halves of the upper and lingual crown-halves of the lower molars, which have remained unworn for the greater part, and form two rows of cusps, pieces of meat and fish are stretched in such a way that when the jaws are firmly closed, particles are comparatively easily pinched off by the other crown-halves, which have been ground off against each other.



The two sides can also act simultaneously, but then more crushing. In any case these jaws are almost as unsuited for grinding movement as those of the Carnivora among the Mammals. The converging direction of contraction of the masticatory muscles and in connection with this the formation of the chin in the type of Homo sapiens-wadjakensis is, therefore, to be explained by the more carnivorous masticatory apparatus of this type of Man.

Vegetable food, however, on which the men of the type Homo neandertalensis-heidelbergensis chiefly lived, like the Monkeys, is generally much poorer in nourishing substances, contains them at least in less concentrated condition, or else it is very hard. If sufficient quantities of nourishing substances were to be absorbed and digested, the masticatory apparatus had to be very active and the food had first to be ground very fine. This took place through grinding mastication, in which the food was continually pushed automatically by the tongue and the not masticating side of the lower jaw — the grinding movement is chiefly alternately one-sided — under the masticating teeth of the upper jaw of the other side (which takes place on opening the mouth in the other type of jaws and teeth). The direction of the movement of the lower jaw, determined by the direction of contraction of the muscles of mastication, had therefore to be from below and inside to above and outside. And the directions of muscular contraction of the two sides thus diverging in this diluvial type of Man, as in the Monkeys, led to the formation of a torus mentalis internus, in the latter besides

to a lamina submentalis, all this in connection with a more vegetarian way of living.

Thus the comparison of the masticatory apparatus teaches us that Homo wadjakensis and Homo neandertalensis were types of an entirely opposite way of living. The former will have chiefly subsisted by hunting and fishing, the other must have found his vegetable food on, in or near the ground, for there can be no doubt of his biped locomotion. The same contrast in mode of living can also be deduced from a comparison of the neurocranium and the other parts of the skeleton.

The most striking and important characters of the neurocranium of Homo neandertalensis are the platycephaly (with flattening of the forehead and of the occiput, the latter leading to the formation of a torus occipitalis transversus), and the torus supraorbitalis. These two, in the first place, have been considered as simian morphological characters of the Neandertal Man, as attributes of low and quantitatively small development of the brain. Chiefly in virtue of these characters, G. SCHWALBE[1]) has tried to justify, with great conviction, the epithet *primigenius,* assigned by WILSER to this Plistocene human type, by comparative measurements and morphological investigations. Homo neandertalensis would be the direct stock form of Homo sapiens, modern Man, from whom he would be distinguished by essential peculiarities. The latter is diametrically opposed to what HUXLEY stated in his famous treatise "Evidence as to Man's Place in Nature" in 1863, and what, as far as platycephaly is concerned, was again advocated by SERA, ten years ago, though with an entirely different purpose in view, in an elaborate study[2]). In HUXLEY's opinion, and in that of others a torus supraorbitalis, though in a less degree, would even be found in some cases among the present Australians, the lowest and most primitive of living races. In both conceptions a type might have been expected in the probably Plistocene

---

[1]) Especially in his "Studien zur Vorgeschichte des Menschen", Zeitschrift für Morphologie und Anthropologie. Sonderheft (228 pp.) Stuttgart 1906.

[2]) G. L. SERA, Sul significate della platicefalia con speciale considerazione della razza di Neanderthal.Archivio per l'Antropologia e la Etnologia. Vol. 40, p. 381—432 (1910); Vol. 41, p. 40—82 (1911). SERA, indeed, considers the platycephaly of the Neandertal Man as a typical property, but not as simian. It is sporadically met with in living races, it would, however, have occurred constantly in this diluvial Man, pathologically or semi-pathologically, then as a passive adaptation to the glacial climate. The characters of the masticatory apparatus discussed here, which are in connection with the form of the neurocranium are incompatible with this conception; so is the fact that the typical masticatory apparatus of this fossil Man in early diluvial time was more perfect (*Homo heidelbergensis*).

australoid Man of Wadjak, which approaches somewhat nearer to the type of the Neandertal Man. The contrary is found. Between the Javanese proto-Australian and the Neandertal Man the contrast, as regards the splanchnocranium, is still sharper than between the latter and the Australian of the present time; attention may be drawn here to the characters of the external nares. Nor is there found a trace of a torus supra-orbitalis at the neurocranium of the proto-Australian, or platycephaly; many an Australian of modern times is in this respect even somewhat less far from the Neandertal Man. Evidently the two types were distinct from the very beginning. Indeed since it is known that the capacity of the brain of Homo neandertalensis was not smaller than that of present Man, nay even exceeded this, it will not do to consider his platycephaly and the torus supraorbitalis attending it as characters of a still low and simian brain-development. Homo neandertalensis was perfectly human, and this resemblance in characters to the Apes can only be explained as functional analogy.

The torus supraorbitalis of Homo neandertalensis cannot be accounted for by his powerful masticatory apparatus, for in this respect he is inferior to Homo wadjakensis, who nevertheless does not possess a torus supraorbitalis. No more can such an explanation apply to the Monkeys, among which this torus is almost universally found.

It is an important fact that there is no torus supraorbitalis at the skull of the Orang-utan, whereas this is strongly developed in the other Anthropoid Apes. The neurocranium is also comparatively short and round and less flattened in the Malay Ape. The primary deviation is evidently the absence of the torus supraorbitalis. This can again not be attributed to a difference in the comparative size of the jaws, for this is certainly no less than in the Chimpanzee, and equals in large individuals of Sumatra that of the Gorilla.

Now there is one organ in the Orang-utan very peculiarly developed, entirely different from what is found in the other Anthropoid Apes, and this is in connection with the mechanism of the movements of the skull, and indirectly with its shape. In all orang-utans, female as well as male ones, the throat poach or laryngeal sac, properly two sacs, homologue to the small ventricles of the mucous membrane of the larynx, which are known by the name of ventriculi Morgagni in the anatomy of Man, is or are not enclosed between the lower jaw and the trachea, as in the Siamang, or (apart from axillary and other deepseated recesses and c. q, of transverse sacs under the lower jaw) restricted to the median front side of the neck only, as in the

Chimpanzee or the Gorilla, but are developed to a large air-cushion which, embracing the neck, extends far over the breast and the shoulders, and on which the head rests in front and on the sides [1]). DENIKER and BOULART, and also HECK are inclined to consider the large laryngeal sacs of the Orang-utan as support for lower jaw and parts of the head, the muscles of the neck being much less strong than in the other large Anthropoids [2]). Probably not in contrast with the much smaller laryngeal sacs of the other Anthropoids mentioned, their functional meaning is certainly not connected with the voice, for the Orang-utan is almost dumb. Nor is their large size in the Orang-utan in connection with an extra-ordinary weight of the head; that of many Chimpanzees is no less heavy, and the head of the Gorilla is certainly generally heavier. The laryngeal sacs of the Orang-utan grow with the general growth of the animal, and are larger in males than in females, largest in gigantic old males. As the head gets heavier, the laryngeal sacs increase in volume. They support the head also on the side, and it seems that they can be assisted in this by the cheek lobes, for where these occur, the laryngeal sacs are comparatively less large [3]).

But in Anthropoids and other Apes, in contrast with what happens in Homo sapiens, the head is not carried poised on the vertebral column, but in most it is carried strongly hanging over; the centre of gravity then lies far before the supporting line of the condyles, and very powerful muscles of the neck carry this overhanging weight. The muscles of the neck in the Orang-utan are directed much less steeply with regard to the "horizontal planes" of the skull, consequently the planum nuchale is steeper than in the other Anthropoids, for instance in the Chimpanzee. The angle of the basio-nasal line with the basio-inion line is about 30° smaller in the Orang-utan, and the angle of the plane through the middle of the condyles and the nasion with the condylo-inion-plane 22° smaller than in the Chimpanzee. This means that only with an elevation of the head of the Orang-utan of 22°, when in front it certainly rests

[1]) R. FICK (Vergleichend anatomische Studien an einem erwachsenen Orang-Utang. Archiv für Anatomie und Entwickelungsgeschichte. (W. HIS), Jahrgang 1895, p. 75) found at the dead body that when the laryngeal sacs are swollen, the head was greatly lifted up backward, without his seeing in this an indication of the vainly sought functional meaning of that air sac.

[2]) DENIKER and BOULART in Nouvelles Archives du Muséum d'histoire naturelle. Paris 1895. sér. 3, t. VII, p. 47—48. BREHMS Tierleben. Vierte Auflage. Säugetiere, Bd. IV, Primates. Bearbeitet von L. HECK, p. 630. Leipzig. 1916.

[3]) Cf.: C. KERBERT, Reuzen-Orangoetans, in „Natuur- en Wetenschap". Eerste Jaargang, p. 7. Brugge 1914.

no longer on the air-cushion, the muscles of the neck pull at the occiput in as favourable a direction, thus raising the front of the head as in the Chimpanzee. But in the Orang-utan these muscles then pull in the direction of the orbital arch of the frontal bone, while through its elevation the front part of the head hangs much less heavily at the occipital part of the cranium, which is besides shorter; they draw in the Chimpanzee in the direction of the crown of the cranium, the frontal part of the head still hanging with its full weight at the occipital part of the cranium, which is besides longer. When, as occurs frequently during locomotion, the body is moved up with great velocity, or is checked in its speed obtained by gravity, the heavy head will fall forward with great force through inertia, unless it is stopped. This takes place in front in the Orang-utan, by means of the elastic laryngeal air sac, in the other Anthropoids and most lower Apes only by means of the muscles of the neck, which acting behind the transverse axis of rotation, pull the skull backward. The stretching strain thus arising between the front part and the back part of the calvaria, is comparatively small in the Orang-utan, great in the other Anthropoids, whose cranial vault would certainly run a risk of breaking, if there were no mechanism to strengthen it, through transference and dispersion of the excited strain. In Man of the present type the head turning about the condylar axis, never hangs over forward so heavily, because in the ordinary erect attitude it balances on the vertebral column, the planum nuchale lies very flat, and the muscles of the neck, which thus act almost straight downward, pull the head backward, which causes the strain excited between the occiput and the front to be much less great in all positions. However, also in Man and in the Orang-utan, the cranial vault might possibly not always be able to resist it, without the mechanism in question, now to be described, which is however less strong here [1]).

Apparently the strain is borne certainly not entirely, probably only for a very small part by the brittle bony substance, but for the greater part by the very elastic apparatus of the *musculus epicranius* or *occipito-frontalis*, the two-bellied flat muscle, whose uniting tendon, the strong epicranial aponeurosis or galea aponeurotica, which chiefly consists of longitudinal fibers, and is loosely

---

[1]) The principal functional meaning of the air pouches, found in so many Monkeys, most probably consists, in general, in this that they help the muscles of the neck to prevent sudden stretching of the encephalon and the medulla spinalis of which there might be a danger from the generally heavy front part of the head, and the situation of the foramen occipitale under the back part of the cranium.

attached to the calvarial bone upon which it glides, but firmly bound to the hairy skin of the head, extends over the calvaria between the fascia temporalis of the two sides, in which it is lost. The backmost belly, formed on either side by the musculus occipitalis, starts, in modern Man, in different extension and coherency from the occipital bone, above the superior curved line, and laterally to the basis of the mastoid process of the temporal bone, hence above the muscles that pull the head backward. The front muscle belly, formed by the two musculi frontales, rises from the epicranial aponeurosis, and its fibers terminate, in Man of the modern type, besides in the skin of the root of the nose and of the brows over their entire length, at the median part of the frontal bone and at the outside of the orbital arch, but here in very various extension and coherency; most uniform is still the lateral part of this attachment, namely near and at the processus zygomaticus frontalis. More coherent is this bony origin (directly or indirectly by fascia) in Apes that possess a torus supraorbitalis [1]).

This apparatus must have a more important function and especially (in the Neandertal Man) have had a more important function than elevating the eye-brows and wrinkling the forehead. Its principal action apparently is, as was stated above, the distribution of the strain, which is exited by the muscles of the neck, by transference to the frontal orbital arches, and as stress, to the malar bones and elastic zygomatic arches back to the occipital bone.[2])

The functional significance of the torus supraorbitalis in most Apes, and its absence in the Orang-utan and Homo sapiens thus becomes clear; besides, its formation can also be explained directly mechanically by the application of AICHEL's demonstration.

The validity of this view may also appear from what is found in American Apes (Chrysothrix, Cebus, Ateles). Here the planum nuchale, to which the muscles that draw the head back, are attached, makes much smaller angles with the transversal glabella-inion plane, hence no very great strain can arise in the cranial vault, and there was not developed a torus supraorbitalis.

---

[1]) In an analogous way as the apparatus of the musculus epicranius protects the calvaria from the violence of the cervical muscles, the strong fascia temporalis, stretched out between the temporal crest and superior temporal line, and the zygomatic arch, and serving for partial attachment of the musculus temporalis, protects it against the violence of the latter muscle, and the zygomatic arch against that of the musculus masseter. This apparatus, though exceedingly strong in the Apes, does probably not contribute to the formation of the lateral part of the torus supraorbitalis, but only of the temporal crest.

[2]) Cf. on those muscles in Apes and Man: G. RUGE, Untersuchungen über die Gesichtsmuskulatur der Primaten, p. 37—51 and 84—93. Leipzig 1887.

Apparently the torus supraorbitalis of Homo neandertalensis must be explained in a similar way, as mechanically efficient, and as having arisen mechanico-physiologically, if the head was not carried erect, resting in equilibrium on the vertebral column, as in Homo sapiens-wadjakensis, but bent forward, supported by the muscles of the neck. And actually a number of characters of the former, of which some had been known already for some time, others were described by MARCELLIN BOULE for the first time, from the fossil man of La Chapelle-aux-Saints, could not be explained differently. The characters of the occiput lead us to assume that in Neandertal Man the muscles of the neck were very strong and supported the head also in a position of rest. This latter appears among others from the steepness of the planum nuchale. For the glabella-inion-opisthion angle or lower inion angle amounts to 51.5° at the Neandertal calvaria, to 54° at the skull of Spy I, to 53° in Spy II, to 44,5° at the skull of La Chapelle-aux-Saints (BOULE), to from 31° to 40° in Homo sapiens; in Wadjak I this angle cannot be determined accurately; it is probably 40°, but the planum nuchale is still less steep as a whole on account of the depression under the inion well-known also of Australian skulls. The foramen occipitale is placed somewhat further backward in the Neandertal Man than generally in modern Man (and the Wadjak Man), and the angle of the plane of this foramen with the plane of the orbital axes (BROCA) is open in front, in the same way, though not so widely, as in the Anthropoids, in contrast with the angle open to the back of the modern type of Man and of the Wadjak Man (not to be measured accurately at the skull of the latter). Accordingly, the plane of the foramen occipitale must turn strongly forward (16.5° in comparison with the Australian skull, 22° with the European skull, according to BOULE), if the orbits are to assume the same direction with regard to the vertical. The spinous processes of the two lowest cervical and first dorsal vertebrae are not directed obliquely downward, as in Homo sapiens, but about horizontally, as in the Anthropoids, and the curvature of the cervical vertebral column is little pronounced. The figure of Neandertal Man was short, especially in the legs, but broad and thickset, the posture less perfectly vertical, with legs slightly bent in the hip and knee joints. The mastoid processes are comparatively small, so that the musculi sternocleidomastoidei, which turn the head, (hardly feasible with bent head) were comparatively weak muscles. The orbits are (quite different from those in the Wadjak Man) very large, deep, and round; the eye-balls must have been large. Like arboreal animals,

and those that move very rapidly (Horse, Ostrich), Homo neandertalensis had large eyes, in order to be able to distinguish details in the field of view sharply, as served his requirements when seeking vegetable food on, in, or near the ground, at any rate in his close neighbourhood, like the arboreal animals. It was different with the hunting and fishing Wadjak Man, to whom the minute details in his field of view were not so important. In accordance with his mode of living, the latter, judging from the preserved parts of the femur and the tibia, was equally slenderly built as the Australian aborigines are as a rule. He was, indeed, taller; therefore the bones are absolutely heavier (thicker). The diaphysis of the femur measures in the middle, sagittally 30 mm. (Neandertal 30, Spy 31), transversally 29 mm. (Neandertal and Spy 30); under the trochanter minor, sagittally 28 mm., (Neandertal 29, Spy 27), transversally 33 mm. (Neandertal 34, Spy 35). The caput femoris has a vertical diameter of 47 mm. (Neandertal 52, Spy 53) and the same transversal diameter (Neandertal 50, Spy 52). The breadth of the proximal epiphysis of the tibia is 75 mm. (Spy 81). Consequently the Wadjak Man was much slenderer than the Neandertal Man (whose legs were much shorter).

In all these points the Neandertal Man was the direct opposite of the Wadjak Man. The other peculiarity of the skull, so characteristic of the former, of Pithecanthropus, and of the Apes, namely the platycephaly, which generally goes together with a torus supraorbitalis, and which, with this latter, is entirely absent in the Wadjak Man, can now be explained as *mechanically efficient*: first to obtain a longer lever for the force of the muscles of the neck carrying the exceedingly heavy head, that hangs forwards, through the "chignon"-like bulging out of the occiput; secondly to get a more favourable direction of the musculus epicranius in the conveyance of the strain from the occipital bone to the frontal bone, in the direction of the longitudinal axis of the calvaria and of the zygomatic arches; thirdly to make the head, which was always to be carried by muscular force, less top-heavy, by transference of brain below the transversal glabella-inion plane (which I propose to demonstrate further in a following communication). Also the physiological pressure of the musculus epicranius, which worked exceedingly energically, may be considered as a *direct* cause of the platycephaly — in an analogous way as in the artificial deformation of the Marken skulls, according to BARGE's investigation [1]).

[1]) J. A. J. BARGE, Beiträge zur Kenntnis der niederländischen Anthropologie. II. Schädel der Insel Marken. Zeitschrift für Morphologie und Anthropologie, Band 16, p. 465—524, with one table and 6 plates. Stuttgart 1914.

It appears thus firmly established that Homo neandertalensis (with Homo heidelbergensis) and Homo wadjakensis belong to two types of Man opposite in every respect, and that it is especially impossible to derive this form of the type Homo sapiens (though it is very old), from the other type. They may, nay they must, indeed, have sprung from a common Hominide branch in a time geologically much more remote than that from which their fossil remains date. It need hardly be said that the latter cannot be identified with the time of their origin, nor without further proof, with the optimum of their existence, nor with the end.

Homo heidelbergensis and Homo wadjakensis were both optimate forms of their type. The best time of existence of the first type the Neandertal Man proper had certainly already long behind him. From the Second or Mindel-Riss Interglacial period, from which the lower jaw of Mauer (Homo heidelbergensis) dates, till the Third or Riss-Würm Interglacial period, from which most fossil remains of the Neandertal Man are, the type has greatly deteriorated, judging from the masticatory apparatus. It then disappears soon, probably in the last or Würm-Glacial period (Spy), making place in Europe for several already very differentiated forms of the type Homo sapiens (Cro-Magnon, Combe-Capelle, Grimaldi). In the vegetable world which got poorer and poorer during the Plistocene epoch, a Man specially equipped for a vegetarian mode of living must have experienced greater and greater difficulty in finding his food, whereas a carnivorous Man could always find an ample supply of food in the animal world. The adaptation to the unfavourableness of the climate by the assumption of a more carnivorous way of living, could only be very limited in such a very specialised type as the Neandertal Man; the very small morphological approach in the masticatory apparatus to the type of Homo sapiens, may be accounted for in this way. In the latter type, however, such an adaptation to a more omnivorous way of living, was indeed possible, which facilitated the feeding; it was still more improved by the use of fire in the preparation of the food, all which contributed to the development of the type Homo sapiens in his present form.

It needs no further argument that the Neandertal-Heidelberg type cannot have arisen in the Plistocene epoch. It is also impossible to assume this for the Homo sapiens type, because these two types must certainly come from a common stock, as is proved by the human shape of their bodies, and especially, because they had both already reached the height of modern Man in the principal human character, the very exceptionally large size of the encephalon; it is

PLATE I.

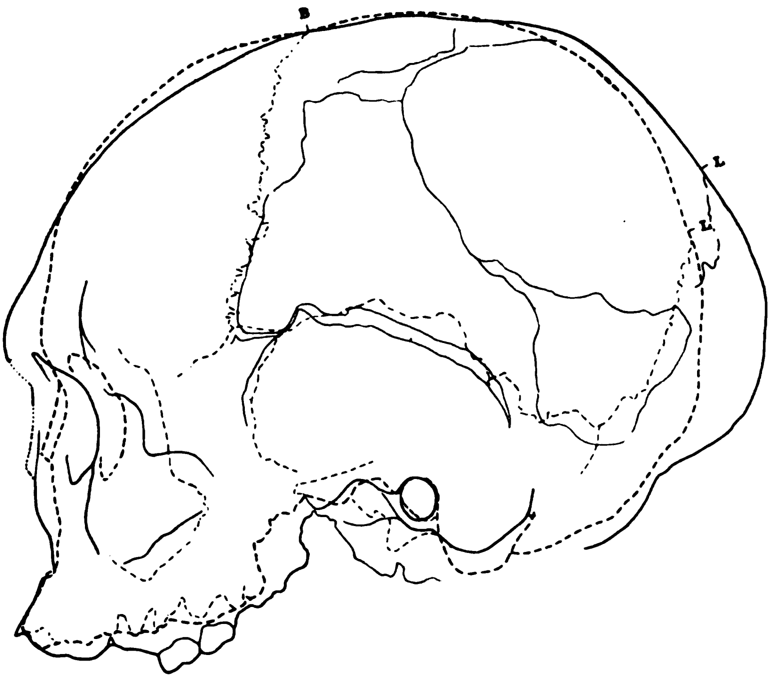


Fig. 1.

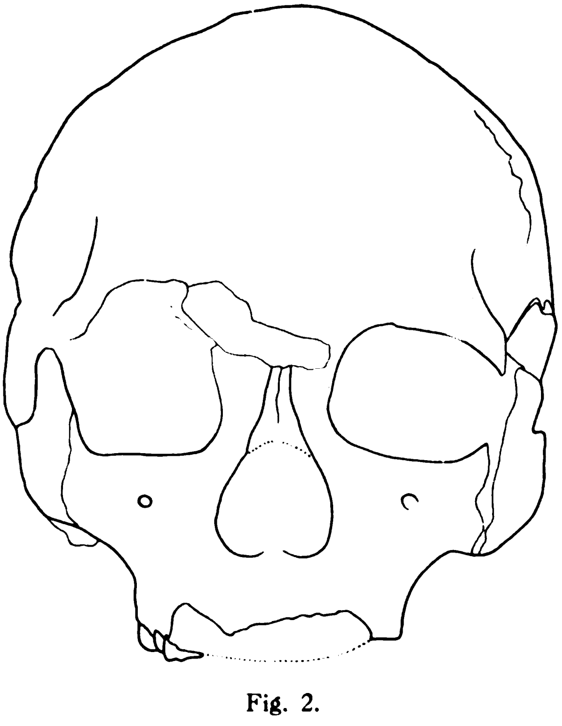


Fig. 2.

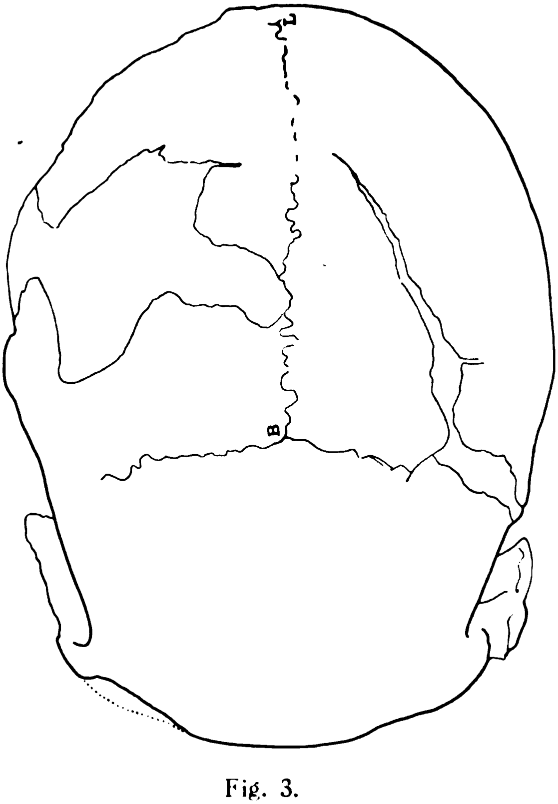


Fig. 3.

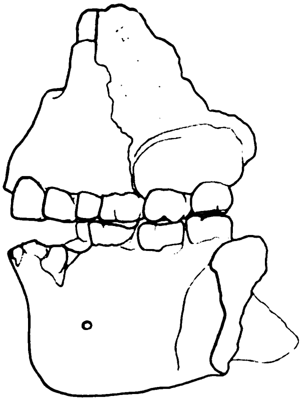


Fig. 4.

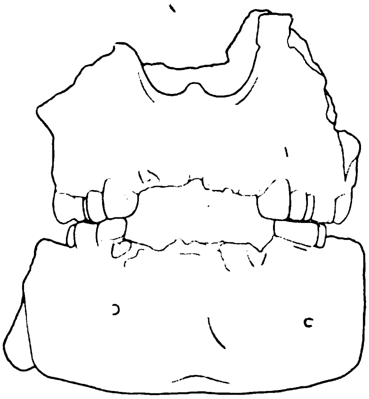


Fig. 5.

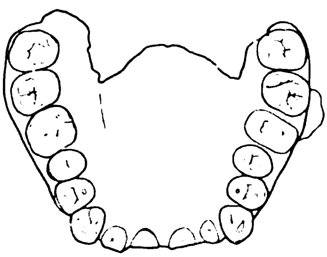


Fig. 6.

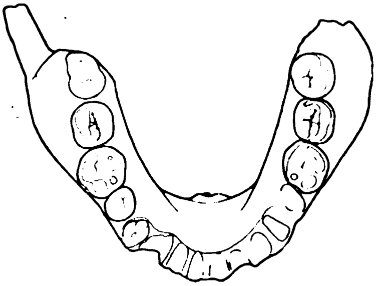


Fig. 7.

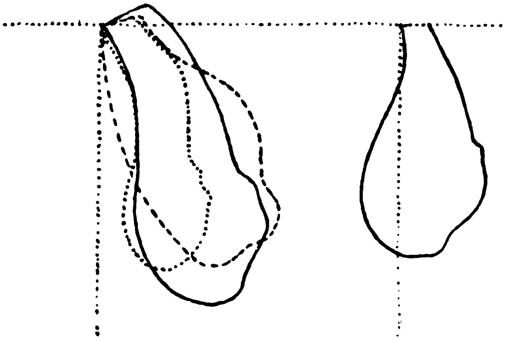


Fig. 8.

moreover impossible to assume this on account of the said early differentiation of sapiens-forms, also manifest in the Wadjak Man, probably the oldest, certainly the most primitive of the forms of this type known up to now.

If therefore the Neandertal type and the Wadjak (sapiens) type existed already before the Plistocene epoch as real Human beings, which had a common human stock, it must have been in still earlier times that their common ancestor sprang from a biped, though only Man-l i k e transitional type, possessing a less large encephalon.

EXPLANATION OF THE PLATES.

PLATE I.

Fig. 1. Wadjak I. Norma lateralis of the skull. As horizontal the Frankfurt plane. Stippled outline of a typical Javanese skull.

Fig. 2. Wadjak I. Norma frontalis of the skull. As horizontal the Frankfurt plane.

Fig. 3. Wadjak I. Norma verticalis of the skull. As horizontal the Frankfurt plane.

Figures 1—3 in $^1/_2$ natural size.

PLATE II.

Fig. 4. Wadjak II. Maxilla and mandibula. Left side. As horizontal the alveolar plane.

Fig. 5. Wadjak II. Maxilla and mandibula. Facial view. As horizontal the alveolar plane.

Fig. 6. Wadjak II. Maxilla from below. Alveolar plane. The crowns of the right p_2 and of the left m_2 and m_3 must lie 1 mm. more to the outside, the crown of the left p_2 0.5 mm. in the figures 4, 5, and 6, on account of deformation in correspondence to twice the amounts in the original.

Fig. 7. Wadjak II. Mandibula from above. Alveolar plane.

Figures 4—7 in $^1/_2$ natural size.

Fig. 8. Vertical cross-sections in the symphysis line of the mandibula of Wadjak II (full line), *Homo heidelbergensis* (broken line), and an Australian (stippled line), by the side of it a Frenchman. The two latter according to BOULE (loc. cit., p. 88, Fig. 56). All these from plaster casts, except the Mauer-jaw, which is from a figure of the original (O. SCHOETENSACK, Der Unterkiefer des Homo Heidelbergensis. Jena 1908, Table 8, Fig. 20 and Table 13, Fig. 48). Natural size.

Physics. — "*On the application of* EINSTEIN'S *theory of gravitation to a stationary field of gravitation.*" By H. A. KRAMERS. (Communicated by Prof. H. A. LORENTZ).

(Communicated at the meeting of September 25, 1920).

§ 1. *Definition and invariant properties of a stationary field of gravitation.*

We will call a field of gravitation stationary when the expression for the line element can be put into such a form $ds^2 = g_{\mu\nu} dx_\mu dx_\nu$ [1]) (x_0 time-coordinate, x_1, x_2, x_3 space-coordinates) that the gravitation potentials $g_{\mu\nu}$ do not depend on the time x_0. A special case of the stationary field of gravitation, defined in this way, forms the so-called "static" field of gravitation, which appears when it is possible by a suitable transformation to make the quantities g_{01}, g_{02} and g_{03} equal to zero. It is simply seen, that when the line element of a stationary field of gravitation is brought in the above mentioned form, the most general transformation of coordinates, for which the $g^{\mu\nu}$'s remain independent of the time, and for which a point at rest remains at rest, is given by the formulae

$$\left.\begin{aligned} x_k &= \varphi_k (x'_1, x'_2, x'_3), \qquad (k = 1, 2, 3) \\ x_0 &= a x'_0 + \psi (x'_1, x'_2, x'_3). \end{aligned}\right\} \quad . \quad (1)$$

Here φ_k and ψ are arbitrary functions of x'_1, x'_2, x'_3, while a is a positive. constant. The quantities $g_{\mu\nu}$ and their derivatives show, with regard to the transformation group expressed by (1), certain invariant and covariant properties, which we will now investigate. The line element may be written in the following form,

$$\left.\begin{aligned} ds^2 = g_{\mu\nu} dx_\mu dx_\nu = -\Sigma G_{kl} dx_k dx_l + \frac{1}{g_{00}} (g_{00} dx_0 + g_{01} dx_1 + g_{02} dx_2 + g_{03} dx_3)^2, \\ G_{kl} = - g_{kl} + \frac{g_{0k}\, g_{0l}}{g_{00}}, \end{aligned}\right\} . \quad (2)$$

[1]) Just as EINSTEIN we have omitted the signs of summation for summations which have to be extended over indices, which occur twice in a product.

where the summation has to be extended over $k,l = 1, 2, 3$. When in the following an index can assume one of the values 1, 2, 3, we will denote this index by a Latin letter. If on the contrary an index can assume one of the values 0, 1, 2, 3, it will always be denoted by a Greek letter. In case of summation over an index, occurring twice in a product, the sign of summation will be omitted in both cases. If now we perform the transformation (1), the expression $G_{kl}\,dx_k\,dx_l$ becomes again a quadratic form of the differentials of the space coordinates, and $\frac{1}{g_{00}}(g_{0\mu}\,dx_\mu)^2$ becomes again the square of a linear differential form. Since the separation in two parts of the expression of the line-element given by (2) is only possible in one way, we may conclude, that the expressions

$$G_{kl}\,dx_k\,dx_l \quad \text{and} \quad \frac{(g_{0\mu}\,dx_\mu)^2}{g_{00}}$$

are invariants with regard to the transformation (1). Consequently the quantities $g_{0\mu}/\sqrt{g_{00}}$ possess the character of a vector, and from this we conclude again, that the bilinear differentialform

$$g_{00}^{s-\frac{1}{2}} \left\{ \frac{\partial}{\partial x_\mu}\left(\frac{g_{0\nu}}{g_{00}^s}\right) - \frac{\partial}{\partial x_\nu}\left(\frac{g_{0\mu}}{g_{00}^s}\right) \right\} dx_\mu\,\delta x_\nu \quad . \quad . \quad . \quad . \quad (3)$$

is also invariant with regard to the transformation (1). The constant s may be chosen arbitrarily, because the quantity g_{00} appears only multiplied by a constant factor after the transformation. Choosing the special value $s = 1$, we see that all terms for which $\mu = 0$ or $\nu = 0$ become equal to zero, so that in this case we may omit the index 0 under the summation, and we obtain the result, that the expression

$$\sqrt{g_{00}} \left\{ \frac{\partial}{\partial x_k}\left(\frac{g_{0l}}{g_{00}}\right) - \frac{\partial}{\partial x_l}\left(\frac{g_{0k}}{g_{00}}\right) \right\} dx_k\,\delta x_l \quad . \quad . \quad . \quad . \quad (4)$$

is invariant. As the coefficients of this differential form are antisymmetrical with regard to the indices k and l, we may consider the expression (4) as a linear form of the differentials $dx_{kl} = dx_k\delta x_l - \delta x_k dx_l$. Now for a threedimensional extension, the expression $\Sigma\sqrt{G}\,Dx_m dx_{kl}$, remains invariant for an arbitrary transformation of coordinates, where G represents the determinant of the coefficients G_{kl} in the expression $d\varrho^2 = G_{kl}\,dx_k dx_l$ for the invariant line-element, and where under the summation the indices k,l,m assume the sets of values 1, 2, 3 and 2, 3, 1 and 3, 1, 2. Consequently the quantities $\sqrt{G}dx_{kl}$ are transformed as the components of a covariant vector (if we, in the usual way, call the transformation of the components dx_k of a

small displacement contravariant). From the invariance of the expression (4) we may thus conclude, that the quantities [1])

$$R^m = \tfrac{1}{2}\sqrt{\frac{g_{00}}{G}}\left\{\frac{\partial}{\partial x_k}\left(\frac{g_{0l}}{g_{00}}\right) - \frac{\partial}{\partial x_l}\left(\frac{g_{0k}}{g_{00}}\right)\right\}, \quad . \quad . \quad . \quad . \quad (5)$$

where k,l,m again may assume the sets of values 1, 2, 3 and 2, 3, 1 and 3, 1, 2, with regard to the transformation (1) are the contravariant components of a vector in the three-dimensional extension with the invariant line-element $d\varrho^2 = G_{kl}\,dx_k dx_l$. The invariant absolute value of this vector is given by $R = \sqrt{G_{kl}\,R^k R^l}$.

If the components R^m are everywhere equal to zero, we have to do with a static field of gravitation. In fact, from (5) follows that in this case the quantities g_{0k}/g_{00} may be deduced from a potential φ in such a way that $g_{0k} = g_{00}\dfrac{\partial\varphi}{\partial x_k}$; but from this follows again, that the line element may be written in the form $ds^2 = -\,G_{kl}\,dx_k dx_l + g_{00} dx_0'^2$, where $x_0' = x_0 + \varphi$.

If the components R^m are not equal to zero, these quantities determine in every point what might be called the "rotatory" properties of the stationary field of gravitation. This may be illustrated by considering the motion of a masspoint, the velocity of which is small compared with the velocity of light. In general the "worldline" of a masspoint is determined by the equations

$$\frac{d^2x_\lambda}{ds^2} + \left\{\begin{matrix}\mu\nu\\ \lambda\end{matrix}\right\}\frac{dx_\mu}{ds}\frac{dx_\nu}{ds} = 0. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

If now we assume, that by a suitable transformation of the kind (1), the line-element has been given the form $ds^2 = dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2$ at a given point P of the worldline, it may be easily verified that the equations (6), looking apart from small terms of the same order of magnitude as the square of the velocity, in the point P assume the simple form

$$\frac{d^2x_k}{dx_0^2} = 2\left(R^l\frac{dx_m}{dx_0} - R^m\frac{dx_l}{dx_0}\right) - \tfrac{1}{2}\frac{\partial g_{00}}{\partial x_k}, \quad . \quad . \quad . \quad . \quad (7)$$

where k,l,m just as before may assume the sets of values 1, 2, 3 and 2, 3, 1 and 3, 1, 2. From these equations we learn that the "force" which the field of gravitation in the point P exerts on a mass point of unit mass may be described as the sum of a Coriolis-force perpendicular to the velocity and proportional to it and of a force, which may be derived from the potential $\tfrac{1}{2}\,g_{00}$.

[1]) If we admit only such transformations, for which the functional determinant is positive, we may by the root-sign in this expression always understand the positive root.

If next we consider the motion of a mass-point in a conservative field of force, such as would take place according to Newtonian mechanics, we obtain equations of motion of the same form as (7), if the Cartesian coordinates describing the position of the mass-point refer to a system of coordinates which rotates uniformly in the space. In fact, if the equations of motion in the non-rotating system of coordinates possess the form

$$\frac{d^2 x_k}{dt^2} = -\frac{\partial \varphi}{\partial x_k}$$

they obtain in a system of coordinates x'_1, x'_2, x'_3, which rotates round an axis through the origin with an angular velocity R which, considered as a vector, possesses the components $-R^1, -R^2$ and $-R^3$, [1] the form:

$$\frac{d^2 x'_k}{dt^2} = 2\left(R^l \frac{dx'_m}{dt} - R_m \frac{dx'_l}{dt}\right) - \frac{\partial(\varphi - \psi)}{\partial x'_k},$$

$$\psi = \tfrac{1}{2}(R)^2 (x'^2_1 + x'^2_2 + x'^2_3) - \tfrac{1}{2}(R^1 x'_1 + R^2 x'_2 + R^3 x'_3)^2,$$

which coincides exactly with the form of the equations (7), if we put $t = x_0$ and $\varphi - \psi = \frac{1}{2} g_{00}$. The essential difference with the equations of motion in the non-rotating system of coordinates lies consequently in the appearance of the Coriolis-forces, and we are justified in denoting in the following the vector R^k as the "*rotation-vector*".

The character of the rotation-vector may also be examined in the following way. We will try by a transformation of coordinates of the form (1) to give the line-element of a stationary field of gravitation such a form, that in a given point P not only the relations $g_{\mu\nu} = \varepsilon_{\mu\nu}$ are valid, where the quantities $\varepsilon_{\mu\nu}$ are defined by

$$\varepsilon_{00} = -\varepsilon_{11} = -\varepsilon_{22} = -\varepsilon_{33} = 1, \qquad \varepsilon_{\mu\nu} = 0\ (\mu \neq \nu), \quad . \quad (8)$$

but that at the same time the quantities $\frac{\partial g_{\mu\nu}}{\partial x_k} = g_{\mu\nu,k}$ as many of them as possible become equal to zero. If it was possible to make all the latter quantities equal to zero, we should obtain in this way a system of coordinates, which is "geodetic" in P. Now it is always possible in many ways by means of a transformation (1) to make the quantities $g_{\mu\nu}$ assume the values $\varepsilon_{\mu\nu}$ in the point P, but in general it will not be possible to make all quantities $g_{\mu\nu,k}$ equal to zero. In

[1] Here and in the following we will assume the usual rule, that to a rotation in a plane corresponds a direction of the normal of this plane in such a way, that, for a rotation in the x_1, x_2-plane from the positive x_1-axis to the positive x_2-axis through an angle smaller than π, the corresponding normal points to the same half-space as the positive x_3-axis.

the first place this is easily seen to hold for the quantities $g_{00,k}$, because g_{00} by the transformation (1) is only multiplied by a constant factor; but neither the quantities $g_{0,kl}$ can all of them at the same time be reduced to zero, because this would mean, that the components of the rotation-vector would be equal to zero, and this is in general not the case. On the other hand it is obviously always possible to perform such a special transformation, that the system of coordinates in the three-dimensional extension with the line-element $d\sigma^2 = G_{kl}\, dx_k\, dx_l$ becomes geodetic in the point P. In this way we get $\frac{\partial G_{kl}}{\partial x_n} = 0$ and consequently also $g_{kl,n} = 0$, since $g_{kl} = - G_{kl} + \frac{g_{k0} g_{l0}}{g_{00}}$ and since the quantities g_{k0} are equal to zero in the point P.

Let us now imagine, that by means of (1) the line-element has been given such a form, that in a given point P the following relations are valid

$$\left.\begin{aligned} a)\quad & g_{\mu\nu} = \varepsilon_{\mu\nu} \\ b)\quad & g_{kl,\nu} = 0 \\ c)\quad & g_{0k,l} + g_{0l,k} = 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

As regards the third of these conditions it will be observed, that it is always possible by a suitable transformation of the time to effect, that the symmetrical quantities $A_{kl} = g_{0k,l} + g_{0l,k}$ become equal to zero in P. In fact, it is easily shown that, if the conditions (a) and (b) are already fulfilled, but not yet (c), the transformation $x'_0 = x_0 + \frac{1}{2} (A_{kl})_P (x_k - (x_k)_P)(x_l - (x_l)_P)$ leads us to the desired purpose. Thereby we have denoted the value of a quantity in the point P by adding the index P on the right below. Let us now perform a transformation of coordinates, which corresponds to a uniform rotation, around an axis through P, of the x_1, x_2, x_3-space (considered as a Euclidean space with the line-element $d\sigma^2 = dx_1^2 + dx_2^2 + dx_3^2$), the angular velocity of which considered as a vector has the components R^1, R^2, and R^3. After the performance of this "*rotation transformation*", which does not belong to the group of transformations (1), the relations (a) and (b) are still valid, but also all the quantities $g_{0k,l}$ have become equal to zero. This may be proved by a direct calculation, and the proof becomes especially simple, if we assume, that in P the quantities R^1 and R^2 are equal to zero, so that we have to do with a uniform rotation round the axis of the coordinate x_3 with an angular velocity $R^3 = \omega$. Since in the point P the quantities R_k reduce to $\frac{1}{2}(g_{0m,l} - g_{0l,m})$, we have in consequence of relation (c), that of all

quantities $g_{0k,l}$ only the two quantities $g_{01,2}$ and $g_{02,1}$ are different from zero in such a way that $\omega = g_{02,1} = -g_{01,2}$. The rotation-transformation can now be written in the form

$$x_1 - (x_1)_P = x'_1 \cos \omega x_0 - x'_2 \sin \omega x_0,$$
$$x_2 - (x_2)_P = x'_1 \sin \omega x_0 + x'_2 \cos \omega x_0.$$

If now by means of this formula the line-element (2) is transformed, and if we make use of the above mentioned relations (a), (b) and (c) it is easy to verify:

That for the transformed quantities $g'_{\mu\nu}$ in the point P the relations (a) and (b) are still valid.

That, although the $g_{\mu\nu}$'s will contain the time x_0, their first derivatives with respect to the time will be equal to zero in P, and that equally all derivatives of g_{01} g_{02} and g_{03} have become equal to zero.

We thus see in the first place that, after the rotation-transformation, the equations for the world-line of a mass point, the velocity of which is small compared with that of light, assume in the point P the simple form

$$\frac{d^2 x_k}{dx_0^2} = -\tfrac{1}{2} \frac{\partial g_{00}}{\partial x_k},$$

so that the term corresponding to a Coriolis-force appears no more, which was naturally to be expected from the above considerations. Let us further consider the special case, that in the point P the mass point can remain in equilibrium; that is, that in this point the quantities $g_{00\,k}$ in the original system of coordinates are equal to zero. In this case we find, that in the new system of coordinates, to which the rotation-transformation has given rise, all quantities $g_{\mu\nu,\rho}$ without exception disappear in the point P, so that this system of coordinates is geodetic in that point.

§ 2. *On the field of gravitation, which is produced by stationarily moving masses.*

Let us consider a space-time-extension, for which the line-element at large distances from the zero-point of the coordinates approaches to $ds^2 = dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2$ [1]) and in which there exist masses, which perform stationary motions; that is, the components $T_{\mu\nu}$ of the energy-tensor of matter do not depend on the time. The field of gravitation, to which these masses give rise will then be stationary

[1]) Here and in the following we shall always assume that the centimeter has been chosen as unit of length. The unit of time is then determined by the condition that the velocity of light is equal to 1.

in the sense described in § 1, and is determined by the equations of EINSTEIN

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = - 8 \pi \varkappa T_{\mu\nu}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

in which $R_{\mu\nu}$ is a tensor of the second order, which depends only on the g's and their derivatives:

$$R_{\mu\nu} = \frac{\partial}{\partial x_\nu} \left\{ \begin{matrix} \mu\varrho \\ \varrho \end{matrix} \right\} - \frac{\partial}{\partial x_\varrho} \left\{ \begin{matrix} \mu\nu \\ \varrho \end{matrix} \right\} + \left\{ \begin{matrix} \mu\varrho \\ \sigma \end{matrix} \right\} \left\{ \begin{matrix} \nu\sigma \\ \varrho \end{matrix} \right\} - \left\{ \begin{matrix} \mu\nu \\ \varrho \end{matrix} \right\} \left\{ \begin{matrix} \varrho\sigma \\ \sigma \end{matrix} \right\},$$

while $R = g^{\mu\nu} R_{\mu\nu}$. $\varkappa$ is the gravitation constant of EINSTEIN, which, if we choose the gram as unit of mass, is equal to $7{,}4 . 10^{-29}$. If we assume that the $g_{\mu\nu}$'s of the line-element which corresponds to the field of gravitation, only differ little from $\varepsilon_{\mu\nu}$, there exists a simple method indicated by EINSTEIN to obtain in first approximation a solution of the equations (10). This solution is obtained by writing

$$g_{\mu\nu} = \varepsilon_{\mu\nu} + \gamma_{\mu\nu}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

where the functions $\gamma_{\mu\nu}$ everywhere possess a very small value, and introducing the quantities $\gamma'_{\mu\nu}$ defined by

$$\gamma'_{\mu\nu} = \gamma_{\mu\nu} - \tfrac{1}{2} \varepsilon_{\mu\nu} (\varepsilon_{\alpha\beta} \gamma_{\alpha\beta}),$$

which give

$$\gamma_{\mu\nu} = \gamma'_{\mu\nu} - \tfrac{1}{2} \varepsilon_{\mu\nu} (\varepsilon_{\alpha\beta} \gamma'_{\alpha\beta}), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

the values of the $\gamma'_{\mu\nu}$ in the point $\overline{x}_1, \overline{x}_2, \overline{x}_3$ and at the moment $\overline{x}_0$ may be calculated as retardated potentials by means of the formulae

$$\gamma'_{\mu\nu} = - 4 \varkappa \int \frac{T_{\mu\nu} [x_0 = \overline{x}_0 - r]}{r} dS \quad . \quad . \quad . \quad . \quad (13)$$

Here dS represents the space-element $dx_1\, dx_2\, dx_3$ and r represents the distance from that space-element to the point

$$\overline{x}_1, \overline{x}_2, \overline{x}_3 \; (r^2 = (x_1 - \overline{x}_1)^2 + (x_2 - \overline{x}_2)^2 + (x_3 - \overline{x}_3)^2).$$

The addition $[x_0 = \overline{x}_0 - r]$ means that everywhere the value of $T_{\mu\nu}$ at the moment $\overline{x}_0 - r$ has to be used. If we apply the formulae (13) to our case where the $T_{\mu\nu}$'s do not depend on the time, we may clearly omit the latter condition, and we obtain the usual formulae for static potentials

$$\gamma'_{\mu\nu} = - 4 \varkappa \int \frac{T_{\mu\nu}}{r} dS \; . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

We will now calculate the components of the rotation-vector in the point $\overline{x}_1, \overline{x}_2, \overline{x}_3$. Neglecting small terms which relative to the main terms are of the same order of magnitude as the $\gamma'_{\mu\nu}$'s, we obtain for the x_1-component of the rotation-vector

$$R^1 = \tfrac{1}{2}\sqrt{\frac{g_{00}}{G}}\left\{\frac{\partial}{\partial x_2}\left(\frac{g_{30}}{g_{00}}\right) - \frac{\partial}{\partial x_3}\left(\frac{g_{20}}{g_{00}}\right)\right\} = \tfrac{1}{2}\left(\frac{\partial \gamma'_{30}}{\partial x_2} - \frac{\partial \gamma'_{20}}{\partial x_3}\right)$$

$$= -2\varkappa\int\left\{T_{03}\frac{\partial}{\partial \overline{x}_2}\left(\frac{1}{r}\right) - T_{02}\frac{\partial}{\partial \overline{x}_3}\left(\frac{1}{r}\right)\right\} dS =$$

$$= 2\varkappa\int\frac{(\overline{x}_2 - x_2)\, T_{03} - (\overline{x}_3 - x_3)\, T_{02}}{r^3}\, dS,$$

and analogous expressions for the components R^2 and R^3. With neglect of small terms of the same order of magnitude as the square of the velocities of the masses we have further the formulae

$$T_{00} = m, \qquad T_{01} = -m v_1, \qquad T_{02} = -m v_2, \qquad T_{03} = -m v_3,$$

where m is the density of mass of matter, and v_1, v_2, v_3 denote the components of the velocity of matter. If we substitute these values in the expression found above, we get

$$R^1 = 2\varkappa\int m\frac{(x_2 - \overline{x}_2)\, v_3 - (x_3 - \overline{x}_3)\, v_2}{r^3}\, dS \quad . \quad . \quad . \quad . \quad (15)$$

and correspondingly R^2 and R^3. This formula teaches, that the contribution of every mass particle to the rotation-vector in point P is equal to the moment of momentum of the mass particle with respect to the point P, divided by the cube of the distance from P, and multiplied by twice the gravitation constant of EINSTEIN.

Formula (15) can be applied to a problem which has been treated by H. THIRRING [1]) in order to illustrate the influence of rotating masses on the field of gravitation. A homogeneous spherical shell with mass M and radius a rotates with constant angular velocity ω in a space, in which no other matter is present, and for which the quantities $g_{\mu\nu}$ approach to $\varepsilon_{\mu\nu}$ at infinite distance from the centre. It is asked to determine the influence of the spherical shell on the motion of a mass point, which is lying just at the centre O. The field of gravitation produced by the shell is stationary. From symmetry we may further conclude that in O a mass point can remain in equilibrium; that means that in this point the quantities $\frac{\partial g_{00}}{\partial x_k}$ disappear. Approximately, that means omitting small terms proportional to ω^2, g_{00} will even be constant in the space within the shell, and the quantities $\frac{\partial^2 g_{00}}{\partial x_k \partial x_l}$, which determine the force exerted on a mass point at rest just outside O, will in general be proportional to ω^2, but cannot be determined if the constitution of the shell has not

[1]) H. THIRRING, Phys. Zeitschr. XIX, p. 33 (1918).

been defined in nearer details. The rotation-vector in O, however, may directly be calculated approximately by means of (15). Its direction is of course parallel to the axis of rotation. We introduce a system of Cartesian coordinates x, y, z, the origin of which coincides with O, and the z-axis of which coincides with the axis of rotation. Let the mass of unit surface of the shell be denoted by m. The contribution to the value of R^z which is due to a ring of the shell, the angular distance of which to the z-axis is equal to ϑ, will then be equal to

$$2\varkappa \times 2\pi a^2 \sin\vartheta \, d\vartheta \times m \, \frac{x \, . \, x\omega + y \, . \, y\omega}{a^3} = \frac{\varkappa M \omega}{a} \sin^3\vartheta \, d\vartheta$$

and for R^z itself we thus get

$$R^z = \frac{\varkappa M \omega}{a} \int\limits_0^\pi \sin^3\vartheta \, d\vartheta = \frac{4\varkappa M\omega}{3a} \quad . \quad . \quad . \quad . \quad . \quad (16)$$

From this we learn that if in O we introduce a system of coordinates, which rotates uniformly in the same sense as the shell with an angular velocity equal to $\frac{4M\varkappa}{3a}$ times that of the shell, the Coriolis-forces will disappear from the equations of motion for a mass point in O. This result is in agreement with the results obtained by Thirring in his above mentioned paper.

Another application of formula (15) may be obtained in connection with the following problem. Let us imagine a uniformly rotating sphere, such as e.g. the earth, and let us suppose that the Foucault's pendulum experiments are performed at the northpole. Then it will be found, that the plane in which the pendulum moves, will not remain at rest with respect to the fixed stars, but will rotate slowly in the same sense as the earth. The angular velocity of this slow rotation is given by the absolute value of the rotation vector at the pole, which by means of (15) may be found by simple integration. We find

$$R = \frac{4\varkappa M \omega}{5a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

where M denotes the mass of the earth, which is supposed to be homogeneous, while a and ω represent the radius and the angular velocity of the earth. The factor $\frac{4\varkappa M}{5a}$ is of course so small (circa 5.10^{-10}), that it will be impossible to detect this rotation of the plane of the pendulum. Also at lower latitudes a similar influence

on the result of FOUCAULT's experiments must be expected, but we will not enter here into this problem.

§ 3. *Influence of a stationary field of gravitation on the motion of a rigid body round its centre of gravitation.*

In the former § we have given an example of the appearance of the rotation-vector; the present § forms a direct continuation of § 1 and gives the necessary preparation for the treatment of the problem, which will be discussed in § 4, and which deals with the influence of the sun's field of gravitation on the precession of the axis of the earth.

If in the following we speak of a rigid body, we mean only a body, which is practically rigid, and which can move in the way well known from classical mechanics, characterised by 6 degrees of freedom, without changes of form or the appearance of enormously high stresses. Thus we will assume that the linear dimensions of the body are so small, that the "geometry" inside the body, which is determined by the quantities $g_{\mu\nu}$ and their derivatives deviates very little from the Euclidean geometry, and also that the relative velocities, which the different parts of the body possess relative to each other, are very small compared with the velocity of light. For such a rigid body it is possible directly to determine the values of the components of the energy-tensor of matter to an approximation, which may be exactly defined. In fact, if we introduce such a system of coordinates that in every point within the body the line element only differs very little from $ds^2 = dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2$ — as a consequence of the above mentioned assumptions this will always be possible — and if we denote by v a small quantity of the same order of magnitude as the velocity $\frac{dx_k}{dx_0}$ of the different parts of the body, we have — neglecting small terms, which relative to the main terms are of the order v^2 and of the same order as the small deviations of the $g_{\mu\nu}$'s from the $\varepsilon_{\mu\nu}$'s (see (8)) —:

$$T_{oo} = m, \qquad T_{ok} = -m \frac{dx_k}{dx_o} \quad . \quad . \quad . \quad . \quad . \quad (18)$$

while the quantities T_{kl} $(k,l = 1, 2, 3)$ which are connected with the stresses existing in the body, and which can only be determined, if the constitution of the body is known more closely, will be small compared with the quantities T_{ok} and may be considered as being of the order of magnitude v^2. The quantity m in the formulae (18) represents the mass per unit of volume. Further it may be remarked,

that apart from the sign, we need not distinguish between covariant, contravariant and mixed components of the energy-tensor[1]). Theoretically there will be a difference, but this difference will be of the same order of magnitude as the small deviations of the $g_{\mu\nu}$'s from the $\varepsilon_{\mu\nu}$'s, and small terms of this order have already been neglected in the establishment of the formulae (18).

In the former we have fixed the properties of a rigid body with an approximation, sufficient for our purpose. Let us now imagine such a body at a certain time to be placed in a stationary field of gravitation in such a way, that its centre of gravitation is at rest and coincides with a point P of the field, where all the derivatives $\frac{\partial g_{00}}{\partial x_k}$ are equal to zero, and we propose to discuss the influence, which the stationary field of gravitation will have on the motions, which will be executed by the body. We will begin by proving, that the centre of gravity will remain at rest in P. For this purpose we will use the equations of energy and impulse of matter:

$$\frac{\partial \mathfrak{T}_{\varkappa}^{\lambda}}{\partial x_{\lambda}} - \tfrac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x_{\varkappa}}\mathfrak{T}^{\mu\nu} = 0, \quad \mathfrak{T}^{\mu\nu} = T^{\mu\nu}\sqrt{-g}, \quad . \quad . \quad . \quad (19)$$

where g represents the determinant of the quantities $g_{\mu\nu}$. We will assume, that by means of a suitable transformation of the form (1) the coordinates in P are made to fulfill the conditions (9). Then the $g_{\mu\nu}$'s may in the neighbourhood of P be represented by

$$\begin{aligned} g_{00} &= \varepsilon_{00} + \tfrac{1}{2}(g_{00,mn})_P\, x_m x_n, \qquad \left(g_{\mu\nu,mn} = \frac{\partial^2 g_{\mu\nu}}{\partial x_m \partial x_n}\right) \\ g_{0k} &= (g_{0k,m})_P\, x_m + \tfrac{1}{2}(g_{0k,mn})_P\, x_m x_n, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20) \\ g_{kl} &= \varepsilon_{kl} + \tfrac{1}{2}(g_{kl,mn})_P\, x_m x_n. \end{aligned}$$

Here we have assumed for the sake of simplicity, that the coordinates x_1, x_2 and x_3 are equal to zero in the point P and have neglected small terms of the order of magnitude x^3, x^4 etc., that is, terms which would contain products of three or more x'_k's.

Let us now consider a closed surface in the x_1-x_2-x_3-space, which encloses the body under consideration, and in the inside of which the relations (20) hold, and let us integrate both sides of (19) over the space inside this surface. Then we get, denoting the space element $dx_1\, dx_2\, dx_3$ by dS:

---

[1]) $T^{0k} = T_0^k = -T_k^0 = -T_{0k} = m\frac{dx_k}{dx_0}$, $\qquad T^{00} = T_0^0 = T_{00} = m.$

$$\frac{d}{dx_0}\left(\int T^0_i \, dS\right) = \tfrac{1}{2}\,(g_{oo,ik})_P \int x_k \, T^{oo} \, dS + (g_{ok,i})_P \int T^{ok} \, dS + \\ + (g_{ok,li})_P \int x_l \, T^{ok} \, dS + \ldots \ldots \quad (21)$$

Here we have omitted terms, which would be of the order v^2 (terms with T^{kl}) and of the order x^2. The left side of (21) represents, apart from the sign, the variation with the time of the total momentum of the body in the direction of the x_i-axis. The integral in the first term on the right side (order of magnitude x) represents the total moment of the body with respect to a plane through P perpendicular to the x_k-axis and is equal to zero, because we have assumed that P coincides with the centre of gravity. The integral in the second term on the right side (order of magnitude v) is equal to the total momentum of the body in the direction of the x_k-axis and is also equal to zero, because we have assumed that the centre of gravity was at rest at the moment under consideration; finally the third term is a small term of the order of magnitude xv and may be neglected, since we already have omitted terms of the order of magnitude x^2 and v^2. From this we see that in first approximation the momentum of the body remains zero in the course of time, and that consequently also the centre of gravity remains at rest. (Here it may be of interest to mention, that it is impossible to fix the centre of gravity of a body in an invariant way; if we try to keep to the classical definition, there always exists a small uncertainty in the position of the centre of gravity, the order of magnitude of which may be easily indicated). If we assume that the equilibrium of the body in P is stable, equation (21) allows us also to calculate the small oscillations, which the centre of gravity can perform in the neighbourhood of P, but we shall not enter further into this point.

We will now proceed to consider the possible motion, characterised by three degrees of freedom, of the rigid body round its centre of gravity. This may be done most easily by calculating the rate of variation with the time of the moments of momentum of a body round the axes of coordinates. For this purpose it will be of advantage to introduce the system of coordinates, which was discussed at the end of the first §, and which appears by the "rotation-transformation" mentioned there (see p. 1056). This new system of coordinates rotates uniformly round the point P with respect to a system of coordinates, which is at rest in the stationary field of gravitation, with an angular velocity, the components of which coincide with the components R_k of the "rotation-vector", and we shall investigate

the moment of momentum of the body with respect to these coordinates. In the point P this system is geodetical at any moment, but the quantities $g_{\mu\nu}$ will, in contrast to what was the case before, depend on the time x_0 and will be periodical with respect to the time in P with a period $T = \frac{2\pi}{R}$, where R represents the absolute value of the rotation-vector. In the neighbourhood of P we have now, instead of (20), the following formulae

$$g_{\mu\nu} = \varepsilon_{\mu\nu} + \tfrac{1}{2}\,(g_{\mu\nu,mn})_P\, x_m\, x_n, \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

where the quantities $(g_{\mu\nu,mn})_P$ are periodical functions of the time. In order now to determine the variation with the time of the moment of momentum round the x_1-axis we make again use of the impulse equations (19), and get from these:

$$x_2 \frac{\partial T_3{}^{\lambda}}{\partial x_\lambda} - x_3 \frac{\partial T_2{}^{\lambda}}{\partial x_\lambda} = + \tfrac{1}{2}\,(x_2\, g_{\mu\nu,3} - x_3\, g_{\mu\nu,2})\; T^{\mu\nu}.$$

Integrating again over a closed surface in the x_1—x_2—x_3-space, which encloses the body, we find with neglect of terms of the order xv^2, xv^2, xv^3, and higher orders:

$$-\frac{d}{dx_0}\left(\int (x_2\, T_3{}^0 - x_3\, T_2{}^0)\, dS\right) = -\tfrac{1}{2}\int (x_2\, (g_{00,3k})_P - x_3\, (g_{00,2k})_P)\, x_k\; T^{00}\, dS. \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

The left side represents the variation with the time of the moment of momentum of the body round the x_1-axis; the right side may directly be interpreted as the x_1-component of the couple, which a field of acceleration with potential $+\tfrac{1}{2}\, g_{00}$ exerts on the body, and is obviously closely connected with the integrals $\int x_k x_l\, T_{00}\, dS$, which determine the ellipsoid of inertia of the body. In case of a homogeneous spherical body they are as is well known equal to zero. By means of (23) and of the two analogous equations, which refer to the moment round the x_2-axis and the x_3-axis, the motion of the body round its centre of gravity in the stationary field of gravitation may thus be determined completely. It may be described as a Poinsot-motion, which is more or less disturbed by the influence of a field of acceleration with potential $+\tfrac{1}{2}\, g_{00}$ (right side of (23)), and on which is superposed a uniform rotation, the components of the angular velocity of which are given by R_1 R_2 and R_3. The latter rotation is quite independent of the properties of the body, in contrast to the influence of the field of acceleration, which is intimately connected with these properties, and which e.g. disappears, if we have to do with a homogeneous spherical body.

Until now we have neglected the influence on the field of gravitation due to the body itself, but in the applications to special cases such a neglect might not be justifiable. When e.g. in the next § we will discuss the praecession of the axis of the earth, we have to do with a body, the "own" field of gravitation of which is much stronger, e.g. at its surface, than the field of gravitation arising fróm the sun (which appears as is well known in the forces, which cause the tides). We might imagine that in such a case other forces might influence the motion round the centre of gravity, which are much stronger than the forces just considered, or which disturb these forces essentially. A closer consideration shows, however, that if the mass of the body is so small, that at large distances it can only cause small changes in the original stationary field of gravitation, the own field of gravitation will only cause a small change in the motion of the body, which may be considered superposed on the influences of the stationary field of gravitation considered above, and which will be proportional to the mass of the body.

In order to show this let us first imagine the body placed in a space, in which no other matter is present, and the line-element of which approaches to $ds^2 = dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2$ at infinite distance from the origin. Then it is easily seen, that in first approximation the own field of gravitation will have no influence at all on the POINSOT-motion of the body, because the "forces" determined by the $g_{\mu\nu}$'s, which the different parts of the rigid body exert on each other in first approximation will fulfill the principle of action and reaction, just as is the case in NEWTON's theory of gravitation. This may easily be proved by applying EINSTEIN's approximative solution of the field equations, described on page 1058, on the impulse energy equations (19), but for the sake of brevity we will not enter into this proof.

Let us again imagine the body placed in a stationary field of gravitation with its centre of gravity at the point P. Let us suppose, that the original values of the $g_{\mu\nu}$'s only undergo small changes $\Delta_{\mu\nu}$ on account of the presence of the body, and let the new values of the $g_{\mu\nu}$'s be denoted by $g'_{\mu\nu}$, so that

$$g'_{\mu\nu} = g_{\mu\nu} + \Delta_{\mu\nu} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Then we obtain, by applying the field-equations (10), for the $\Delta_{\mu\nu}$'s a set of 10 partial linear inhomogeneous differential-equations of the second order, of which we will assume that there exists a regular solution. (If necessary boundary conditions must be given. If the stationary field of gravitation is such that the line-element every-

where differs very little from $ds^2 = dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2$ and becomes equal to this expression at infinity, the $\Delta_{\mu\nu}$'s might simply with a high degree of approximation be calculated by means of the formulae (13) of EINSTEIN). Inside the body and in its neighbourhood this solution will be the same as in the absence of the stationary field of gravitation, and just as before there will be no direct influence of the own field of gravitation on the motion of the body. Further ,it is easily seen, that the values of the $\Delta_{\mu\nu}$'s at large distance from the body to a high degree of approximation will depend only on the total mass of the body, because at such large distances the influence of the body can be considered as that of a singular point (or as a singular line in the space-time-extension) characterised by the integrals of the quantities $T_{\mu\nu}$ over the volume of the body. But, always neglecting small quantities of the order v^2 and higher orders, the integrals involving the T_{kl}'s may be neglected, while those of the T_{ko}'s disappear, because the centre of gravitation is at rest; so that we only have to do with the integral of T_{00} extended over the volume, that is with the total mass M of the body. Thus we find that the body will exert small forces proportional to M on the bodies, which give rise to the stationary field of gravitation. The motion of these bodies will therefore undergo a small perturbation, and as a consequence of this the $g_{\mu\nu}$'s of the stationary field of gravitation itself will again undergo a modification. Instead of (24) we must therefore write

$$g'_{\mu\nu} = g_{\mu\nu} + \Delta_{\mu\nu} + \Delta'_{\mu\nu}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

where the $\Delta'_{\mu\nu}$'s represent the modifications just mentioned. The $\Delta'_{\mu\nu}$'s are terms, which will be small compared with the $\Delta_{\mu\nu}$'s, and which will be proportional to M; in contrast to the terms $\Delta_{\mu\nu}$ they will, however, in general have an influence on the motion of a body, which clearly will be proportional to M.

In order to discuss this influence we will confine ourselves, for simplicity, to the case that the $\Delta'_{\mu\nu}$'s are independent of the time. In this case the quantities $\frac{\partial \Delta'_{00}}{\partial x_k}$ will in general not be equal to zero in the point P, so that there must be found a point P' in the neighbourhood of P, where the centre of gravity of the body may remain at rest. Further in order to determine the components of the rotation-vector we will have to introduce in the formulae (5) instead of the quantities $g_{\mu\nu}$ the values of the quantities $g_{\mu\nu} + \Delta'_{\mu\nu}$ and of their derivatives in the point P'. In this way a small modification proportional to M will be found in the values of the rotation-

vector at the point, where the body is situated, and in consequence of this a corresponding modification in the perturbing influence, which the field of gravitation exerts on the motion of the body. It will also be clear that the influence which according to equation (23) is exerted on the motion of the body, will undergo a modification proportional to M.

The results of this § may be briefly stated as follows. If a rigid body is situated in a stationary field of gravitation with its centre of gravity at rest, the POINSOT-motion, which the body in the absence of the field of gravitation would perform in the way well known from classical mechanics, will be disturbed by a superposed uniform rotation, which is independent of the properties of the body, and at the same time by the influence of a conservative field of acceleration, an influence, which is closely connected with the properties of the body (e.q. with the ellipsoid of inertia). The "own" field of gravitation of the body will — in first approximation — have the effect that all quantities, which characterise the position and the just mentioned perturbations in the motion of the body, will undergo small modifications proportional to the mass M of the body.

In practical applications all these influences may of course be of quite different orders of magnitude, and it may happen that some of them practically may be neglected, while others are so large, that the approximation perhaps has to be carried on further than indicated in this §.

§ 4. *Influence of the field of gravitation of the sun on the rotation of the earth.*

The line element of the field of gravitation, to which the sun, which is supposed to be at rest, gives rise, can be written in the following form, which for the first time was given by SCHWARZSCHILD:

$$ds^2 = \left(1 - \frac{a}{r}\right) dT^2 - \frac{1}{1 - \frac{a}{r}} dr^2 - r^2 (d\vartheta^2 + \sin^2 \vartheta\, d\varphi^2), \quad (26)$$

where T represents the time, the unit of time being chosen such that the velocity of light at large distance from the sun approaches to unity. r, ϑ, φ are polar coordinates, which determine the position of a point in space, and a is a constant, which is connected with the mass $M_\odot$ of the sun by the formulae

$$a = 2\varkappa M_Z, \quad \ldots\ldots\ldots \quad (27)$$

where $\varkappa$ again denotes the constant of gravitation (see p. 1058).

Let us now introduce a new system of coordinates, which rotates round the axis of the original system of coordinates with an angular velocity ω. The line element in the new coordinates may then be calculated by means of the transformation

$$\varphi = \psi + \omega T$$

This gives

$$\left.\begin{aligned} ds^2 = &\left(1 - \frac{a}{r} - r^2 \sin^2 \vartheta\, \omega^2\right) dT^2 \\ &- 2r^2 \sin^2 \vartheta\, \omega\, d\psi\, dT - \\ &- \frac{dr^2}{1 - \frac{a}{r}} - r^2 (d\vartheta^2 + \sin^2 \vartheta\, d\psi^2) \end{aligned}\right\} \quad \ldots\ldots \quad (28)$$

The field of gravitation corresponding to this line element is stationary. We will first try to find a point P, where a mass-point can remain in equilibrium. In such a point the first derivatives of $g_{TT} = 1 - \frac{a}{r} r^2 \sin^2 \vartheta \omega^2$ are equal to zero. This gives the following conditions, which must be fulfilled by the coordinates of P:

$$\frac{\partial g_{TT}}{\partial r} = \frac{a}{r^2} - 2r \sin^2 \vartheta\, \omega^2 = 0, \qquad \frac{\partial g_{TT}}{\partial \vartheta} = -\, 2r^2 \sin \vartheta \cos \vartheta\, \omega^2 = 0.$$

From this we see, that a mass-point can remain at rest at every point P, which lies in the equatorial plane $\vartheta = \frac{\pi}{2}$, and for which the distance A from the sun fulfills the relation

$$2\, A^3\, \omega^2 = a, \quad \ldots\ldots\ldots \quad (29)$$

This relation gives us therefore the connection between the angular velocity and the orbital radius of a planet, which moves in a circle round the sun.

In order to discuss the rotation of such a planet round its axis we shall begin by calculating the rotation-vector in P. In order to find, by means of (5), its contravariant components we want to know the value of the determinant G of the quantities $G_{kl} = -\, g_{kl} + \frac{g_{ko} g_{lo}}{g_{oo}}$.

We find

$$G_{rr} = \frac{1}{1-\frac{\alpha}{r}}, \quad G_{\vartheta\vartheta} = r^2, \quad G_{\psi\psi} = r^2 \sin^2 \vartheta + \frac{(r^2 \sin^2 \vartheta\, \omega)^2}{1-\frac{\alpha}{r} - r^2 \sin^2 \vartheta\, \omega^2} =$$

$$= r^2 \sin^2 \vartheta \frac{1-\frac{\alpha}{r}}{1-\frac{\alpha}{r} - r^2 \sin^2 \vartheta\, \omega^2},$$

$$G_{r\vartheta} = G_{\vartheta\psi} = G_{\psi r} = 0, \quad G = \frac{1}{1-\frac{\alpha}{r}} \cdot r^2 \cdot r^2 \sin^2 \vartheta \frac{1-\alpha/r}{1-\alpha/r - r^2 \sin^2 \vartheta\, \omega^2} =$$

$$= \frac{r^4 \sin^2 \vartheta}{g_{TT}}.$$

Since the derivatives of g_{TT} are equal to zero in the point P, the expressions (5) reduces in that point to

$$(R^m)_P = \tfrac{1}{2} \frac{1}{\sqrt{g_{00} G}} \left\{ \frac{\partial g_{0l}}{\partial x_k} - \frac{\partial g_{0k}}{\partial x_l} \right\}.$$

This gives

$$(R^r)_P = -\tfrac{1}{2} \frac{1}{\sqrt{g_{00} G}} \frac{\partial}{\partial \vartheta} (r^2 \sin^2 \vartheta \omega) = 0,$$

$$(R^\vartheta)_P = \tfrac{1}{2} \frac{1}{\sqrt{g_{00} G}} \frac{\partial}{\partial r} (r^2 \sin^2 \vartheta\, \omega) = \left(\frac{r \sin^2 \vartheta\, \omega}{\sqrt{r^4 \sin^2 \vartheta}} \right)_P = \frac{\omega}{A}, \quad (R^\psi)_P = 0.$$

Consequently the rotation-vector in P is perpendicular to the equatorial plane, and for its absolute value R we find

$$R = \sqrt{G_{kl} R^k R^l} = R^\vartheta \sqrt{G_{\vartheta\vartheta}} = \frac{\omega}{A} \cdot A = \omega, \quad . \quad . \quad . \quad (30)$$

According to § 1 p. 1057 this result means, that for an observer placed at the earth the sun rotates with an angular velocity ω with respect to a system of coordinates, in which no Coriolis-forces are present, i. e. in which the Galileian law of inertia holds. On the other hand, from the point of view of the same observer, the sun rotates in the same sense with respect to the fixed stars with an angular velocity equal to the product of ω and the ratio of the time-unit of an observer on the earth and an observer, which is at rest at infinite distance from the sun, since, according to the formulae in the beginning of this §, ω represents the angular velocity of the earth round the sun, if the last mentioned time unit is used [1]).

[1]) Originally the writer had simply put the angular velocity of the sun with respect to the fixed stars equal to ω, and as a consequence of this obtained he result that EINSTEIN's theory of gravitation did not claim a non-Newtonian

From formulae (28) the ratio of the two time units in question is found to be equal to the value of $\sqrt{1-\frac{a}{r}-r^2\omega^2 \sin^2\vartheta}$ at the point where the earth is situated. Since at this point $r=A$, $\vartheta=\frac{\pi}{2}$, the ratio in question is equal to $1-\frac{3a}{4A}$.

It is therefore seen, that a system of coordinates in which the law of inertia holds, at the point where the earth is situated will rotate with an angular velocity $\frac{3a}{4A}\omega=\frac{3}{2}v^2\omega$ (where v is the velocity of the earth in its circular orbit) with respect to the fixed stars in the same sense as that, in which the earth rotates round the sun.

From this result, and from the results in the former §, we may therefore conclude that according to the gravitation theory of EINSTEIN there will be a contribution to the precession of the axis of the earth in progressive sense, which is independent of the constitution of the body of the earth, and which amounts to an angle equal to one and a half times the ratio of the velocity of the earth to the velocity of light, i.e. to 0,019 arc seconds anually. The existence of a non-Newtonian precession of this kind has for the first time been suggested in a paper by Professor SCHOUTEN. [1]) In this paper attention was drawn to the circumstance, that the field of gravitation of the sun is such, that a small body, which was made to move geodetically along a circle round the sun with radius A, would no longer have the same position as before at its return to the same point, but that it would be turned by a small angle equal to $\frac{\pi a}{A}$, in the same sense as that, in which the body had moved along the circle, and it was pointed out, that this result suggested a possible precession of the axis of the earth with respect to the fixed stars.

Now it remains to investigate the influence, which what we have called the "own" field of gravitation of the earth, may exert on its motion. According to what has been said in the former § (p. 1067), we may expect, that this influence will cause perturbations as well in the orbit of the earth as in its motion round the centre of gra-

---

precession of the kind described. I am indebted to Dr. FOKKER for the remark, that in doing so I had overlooked the difference in the time unit, in which the two angular velocities in question were expressed. Compare A. D. FOKKER, These Proc. Vol. XXIII. N°. 5, p. 729 (1920).

[1]) These Proc. Vol. XXI, p. 533 (1918).

vity, and these perturbations will be small quantities proportional to the mass M_A of the earth. From classical mechanics we know already in first approximation the influence on the orbit: the sun is not at rest, but describes round the centre of gravity O of sun and earth an orbit similar to that of the earth, in such a way that its distance to O is always equal to the distance of the earth to O multiplied by $\frac{M_A}{M_Z}$. Assuming that the orbit of the earth is again a circle, we shall still have, that the product of the square of the angular velocity ω and the cube of the distance earth-sun will have a constant value, but this value will no longer be equal to $\frac{\alpha}{2}$, but to $\frac{\alpha}{2}\left(1+\frac{M_A}{M_Z}\right)=\frac{(M_A+M_Z)\alpha}{2\,M_Z}$. If we again introduce a system of coordinates rotating with angular velocity ω, with respect to which the centre of gravity of the earth is at rest, the field of gravitation will again be stationary in these rotating coordinates, (if we look apart from the motion of the sun and of the earth round their respective centres of gravity [1]) but the distance from the earth to O corresponding to this angular velocity is no longer the same as when the mass of the earth was neglected, but smaller in the proportion $\left(1-\frac{2M_A}{3M_Z}\right)$. Thus we have to do with a displacement of the point, where the centre of gravity of the earth may remain at rest, which is a consequence of the own field of gravitation of the earth, and which is proportional to the mass of the earth. According to the considerations on p. 1066 such a displacement was to be expected.

We must further consider the possibility that the absolute value of the rotation-vector at the centre of the earth is no longer exactly equal to ω, but may, e. g. be written in the form $\omega\left(1+k\frac{M_A}{M_Z}\right)$, where k is a numerical factor of the same order of magnitude as unity. Here it must be remembered, that when speaking of the rotation-vector at the centre of the earth, we mean the quantity, which according to the scheme indicated in the former § (p. 1066)

[1] From some interesting considerations by EINSTEIN, Berliner Berichte 1916 p. 695, it follows, that the field of gravitation in question may only be regarded as stationary to a certain degree of approximation, because we must expect that, analogous to what according to the classical theory of electrons would take place in a system of moving electrified particles, a system as that considered here will radiate energy in space in the form of socalled gravitational waves.

may be calculated by means of formula (5), if we so to say neglect the field of gravitation, which is directly due to the earth, or, more exactly, if we replace the $g_{\mu\nu}$'s of the original stationary field of gravitation by the quantities $g_{\mu\nu} + \Delta'_{\mu\nu}$, where the $\Delta'_{\mu\nu}$'s represent the small terms proportional to M_A, which are discussed on p. 1066. We shall now prove, that the just mentioned factor k is equal to zero, at any rate with neglect of small quantities of the order $\frac{\alpha}{A}$. This may most easily be proved by using the results of classical mechanics, which — with neglect of small quantities of the relative order of magnitude $\frac{\alpha}{A}$ — will coincide with the results of EINSTEIN's theory of gravitation. In fact, it is well known, that according to classical mechanics, the precession of the axis of rotation of the earth will entirely be due to the inhomogeinity of the Newtonian field of gravitation due to the sun at the place of the earth. In the mathematical expression for this precession there will therefore, whether we take the mass of the earth into account or not, not occur terms independent of the constitution of the earth and of the kind discussed on p. 1070, terms, which only appear, when the modifications in the phenomena of gravitation required by EINSTEIN's theory are taken into account. Hence we have the result, mentioned above that, with neglect of small quantities of the order $\frac{\alpha}{A}$, that is of the same order as the square of the velocity of the earth, the mentioned factor k will be zero.

Consequently we find, that the influence of the earth's own field of gravitation on the non-Newtonian contribution to the precession of the rotation-axis of the earth, considered in the present §, will at most be a small quantity of the order $\omega \times \frac{\alpha}{A} \times \frac{M_A}{M_Z}$. It is furthermore clear that, if the mass of the earth is not neglected, also the contribution to the value of the precession, which is due to the inhomogeinity of the sun's field of gravitation at the place of the earth will be altered by small quantities, which relative to the main term are of the order $\frac{M_A}{M_Z}$, and that this alteration will be the same as that calculated by means of Newtonian mechanics.

In connection with this result it might be of interest to draw attention to the fact that we have made use of the circumstance, that for the system under consideration, which consisted of bodies moving under mutual gravitational influence, the results obtained

on Newton's theory will differ from those obtained on EINSTEIN's theory only by small quantities of the same order as the square of the velocities of the bodies. This circumstance is remarkable on account of the fact, that in EINSTEIN's theory all gravitational influence is propagated in space with the velocity of light, as it is e.g. indicated by the formulae (13), which have the form of retarded potentials. From this we might at first hand infer, that there might be discrepancies between the results of EINSTEIN's and of NEWTON's theories of the same order of magnitude as the first power of the velocity of the earth. A closer consideration, into which for the sake of brevity we will not enter here, shows however, that for the system under consideration small terms of this order will just compensate each other, a circumstance which is completely analogous to similar well-known phenomena with which we meet in the theory of electrons, whose interaction may be calculated by means of retarded potentials.

Conclusions of this paragraph.

It has been found that, in confirmation of an idea for the first time put forward by SCHOUTEN, the gravitation theory of EINSTEIN leads to the result, that theoretically there will exist a contribution to the value of the precession of the rotation axis of the earth, which did not appear on NEWTON's theory, and which is independent of the constitution of the body of the earth, and which amounts to a progressive precession of 0,019 arc seconds annually. In the calculations, the influence of the mass of the earth was neglected; if it is taken into account, there may arise a modification in the value of the precession, which relative to the main term in the expression for this precession is of the same order of magnitude as the ratio of the mass of the earth to that of the sun. In the considerations no regard has been paid to the contribution to the precession arising from the influence of the moon.

Chemistry. — "*Catalysis* — Part XII. *Some induced reactions and their mechanism*". By NIL RATAN DHAR. (Communicated by Prof. ERNST COHEN).

(Communicated at the meeting of January 29, 1921).

When an aqueous solution of mercuric chloride is boiled with oxalic acid, there is no reduction of the mercuric chloride to the mercurous state, but as is well known this mixture of mercuric chloride and oxalic acid decomposes at the ordinary temperature in sunlight according to the equation,

$$2HgCl_2 + H_2C_2O_4 = 2HgCl + 2HCl + 2CO_2.$$

The same change, however, takes place in the dark as the author has observed if a few drops of a deci-normal potassium permanganate are added to the mixture. As soon as the color of the permanganate is discharged, mercurous chloride begins to separate out.

This phenomenon appears to be of general occurrence. Thus the reduction of mercuric chloride and bromide by oxalic acid, tartaric acid, citric acid, malonic acid, malic acid, glycollic acid, cane sugar, glycerine, lactic acid, hydroxylamine hydrochloride, hydrazinehydrochloride etc., the reduction of gold chloride by several reducing agents, the reduction of silver nitrate, cupric chloride and selenious acid (to selenium) by various organic acids, cane sugar etc. are promoted by the addition of such oxidising agents as potassium permanganate, potassium persulphate, manganese dioxide, potassium nitrite, hydrogen peroxide, ceric salts etc.

It is a remarkable fact that this effect is particularly noticeable in those reactions which are sensitive to light.

In all these instances, chemical changes are taking place in a homogeneous system. I have also investigated several cases of induced reactions taking place in heterogeneous systems, and I have made a special study of oxidations effected by oxygen of the air. The following are some of the experimental results obtained:

Primary change	Induced change
Sodium sulphite + oxygen	Sodium arsenite + oxygen
Sodium sulphite + oxygen	Sodium nitrite + oxygen
Sodium sulphite + oxygen	Sodium oxalate + oxygen
Sodium sulphite + oxygen	Sodium formate + oxygen
Sodium sulphite + oxygen	Ferrous ammonium sulphate + O_2
Sulphurous acid + oxygen	Ferrous ammonium sulphate + O_2
Stannous chloride + oxygen	Ferrous ammonium sulphate + O_2
Manganous hydroxide + oxygen	Sodium arsenite + oxygen
Cobaltous hydroxide + oxygen	Sodium arsenite + oxygen
Acetaldehyde + oxygen	Sodium arsenite + oxygen
Formaldehyde + oxygen	Sodium arsenite + oxygen
Benzaldehyde + oxygen	Sodium arsenite + oxygen
Ferrous ammonium sulphate + oxygen	Sodium oxalate + oxygen
Ferrous ammonium sulphate + oxygen	Sodium tartarate + oxygen
Ferrous ammonium sulphate + oxygen	Sodium citrate + oxygen
Ferrous hydroxyde + oxygen	Sodium arsenite + oxygen
Ammoniacal cuprous hydroxide + oxygen	Sodium arsenite + oxygen.

In all these cases at first the primary change, that is, the oxidation of the easily oxidisable substance takes place and this primary change induces or promotes the secondary or the induced change that is, the oxidation of the difficultly oxidisable substance. In other words, the potential chemical change between oxygen and sodium arsenite is activated by the previous oxidation of sodium sulphite. The oxygen divides itself, as it were, between the two reducing agents and the proportion in which it divides itself between the two reducing agents is the next point of interest.

It is well known that a solution of sodium arsenite is not oxidised by atmospheric oxygen under ordinary conditions. On the other hand a solution of sodium sulphite is readily oxidised to sodium sulphate. Now if we mix the two together both the oxidations take place simultaneously. At the same time a curious phenomenon takes place. The velocity of the oxidation of sodium sulphite becomes very small in presence of sodium arsenite, that is, sodium arsenite which is undergoing a slow oxidation acts as a powerful negative catalyst in the oxidation of sodium sulphite. Similarly a solution of an oxalate which also undergoes slow oxidation in presence of sodium sulphite which itself is being oxidised, decreases to a marked extent the oxidation of sodium sulphite by atmospheric oxygen. It appears, probable, therefore that the phenomenon of negative catalysis is

possible only when the catalyst is liable to be oxidised. These cases are of great importance in connection with the controversial question of negative catalysis.

In a previous paper (Jour. Chem. Soc. 1917, **111**, 707) I have shown that manganous salts act as a powerful negative catalyst in the oxidations of formic and phosphorous acids by chromic acid and manganous salts can easily pass into the manganic state. Moreover it has been shown by myself as well as other investigators that various organic substances notably hydroquinone, brucine etc. act as negative catalysts in the oxidation of sodium sulphite by oxygen and all these organic substances are themselves readily oxidised. It is well known that the oxidation of phosphorus by oxygen of the air is retarded by the vapours of various organic substances e.g. ether, alcohol, turpentine etc. and the oxidation of chloroform is retarded by the presence of a small quantity of alcohol. Now all these negative catalysts are good reducing agents and are themselves readily oxidised. Hence in oxidation reactions the phenomenon of negative catalysis takes place when the catalyst itself is liable to be readily oxidised.

A study of the slower oxidations that take place at ordinary temperatures has not only shown that the process of oxidation is complicated by the presence of water, but the question has been raised that just so much oxygen takes part in the induced reaction as combines with the substance undergoing oxidation.

SCHÖNBEIN (Jour. prakt. Chem. 1858, **75**, 99; 1864, **93**, 25; **105**, 226, 1868) first noticed that when certain substances are undergoing oxidation spontaneously by atmospheric oxygen, one part of the oxygen combines directly with the substance undergoing oxidation whilst another part of it is converted into ozone, hydrogen peroxide or simultaneously oxidises some other substance. SCHÖNBEIN (*loc. cit.*) still further demonstrated that just so much oxygen is rendered active as is consumed by the substance which is being oxidised or in all slow oxidations the same amount of oxygen is required as is consumed in the formation of hydrogen peroxide from water or is consumed in the induced oxidation.

Later investigators like JORISSEN (Zeit. phys. Chem. 1897 **23**, 667) ENGLER and WILD (Ber. **33**, 1109, 1000) have verified the law of SCHÖNBEIN in several cases. If we expose a mixture of sodium sulphite and sodium arsenite to atmospheric oxygen according to SCHÖNBEIN (loc. cit.) one atom of oxygen should go to oxidise sodium sulphite, while the other atom would oxidise a molecule of sodium arsenite in the same time. The oxidation of sodium arsenite is a very slow

chemical change and in order that SCHÖNBEIN's law be applicable, it follows immediately that the oxidation of sodium sulphite, which is fairly rapid, becomes a slow change and the velocity of this oxidation becomes equal to that of the oxidation of sodium arsenite, because the same amount of oxygen will be taken up by the reducing agents in the same time. As a matter of fact from my experiments I have observed that in presence of sodium arsenite or potassium oxalate the velocity of the oxidation of sodium sulphite becomes very small. We assume that a molecule of oxygen splits up in this reaction into two atoms and each atom oxidises one of the reducing agents. Now as a solution of sodium sulphite is much more readily oxidised than a solution of sodium arsenite it becomes difficult to understand why the other oxygen atom instead of attacking the readily oxidisable unacted sodium sulphite attacks the much more difficultly oxidisable sodium arsenite. Or if we assume that at first a peroxide of the type of BODLÄNDER's benzoyl peroxide (Ahrens' Sam. **3**, 470 1899) is formed as a combination of the sodium sulphite with a molecule of oxygen, we are still encountered with the same difficulty. In this case we shall have to assume that this peroxide instead of attacking the readily oxidisable and unattacked sodium sulphite, will attack the less readily oxidisable sodium arsenite by preference. It seems to me therefore that the only course left to us is to find out the explanation in the view of the formation of a complex of sulphite and arsenite or of sulphite and oxalate and that this complex is oxidised as a whole. It is well known that complex oxalates and sulphites do exist. OSTWALD thinks that in order to explain positive catalysis by the hypothesis of intermediate compound formation it is necessary to show that the intermediate reactions actually take place more readily than the direct reaction under the given conditions, because if a reaction goes more slowly via the intermediate product than the direct path, it will take the latter and the possibility of intermediate products can have no influence on the process. "Hence", adds OSTWALD, "I see no possibility of explaining retarding catalytic influences by the intermediate products" (Nature, 1902, **65** 522).

I have observed in a previous paper (Proc. Akad. Wet. Amsterdam 1920) that in the oxidation of sulphites and sulphurous acid the sulphite ion is the active agent. If we can decrease the sulphite ion we can decrease the chemical change, and a solution of sulphurous acid which is a weak acid containing few sulphite ions is oxidised less readily than a solution of sodium sulphite of the same concentration. On the addition of an arsenite to a sulphite a complex which

itself is oxidised as a whole is formed. At the same time the velocity of the oxidation of the sulphite becomes less due to the decrease in the concentration of the sulphite ions, arising out of the formation of a complex of sulphite and arsenite or of sulphite and oxalate. Here it seems to me that the only plausible explanation of this negative catalysis stands on the hypothesis of the formation of an intermediate complex compound.

On investigating these various cases of induced reactions I was naturally led to the more general conclusion that one chemical change should induce another chemical change of the *same type* and I tried to verify this conclusion. I found that the reduction of mercuric chloride by such different reducing agents as formic acid, sulphurous acid, phosphorous acid etc. induce in all cases the reduction of the same substance (e.g. mercuric chloride) by sodium arsenite. I also investigated other changes, as for instance, the decomposition of unstable substances. It is well known that ammonium dichromate decomposes readily into nitrogen, water and chromium oxide. Also the decomposition temperature of potassium persulphate is lower than that of potassium, chlorate, and I have found that in presence of decomposing ammonium dichromate or persulphate the decomposition temperature of potassium chlorate is appreciably lowered. In this connection it will be of interest to investigate whether the presence of an easily decomposable explosive will lower down the decomposition temperature of a difficultly decomposable explosive, and this investigation will throw light on the stability of mixed explosives. As far as my experiments go I am inclined to the view that *one chemical change will either promote or induce another chemical change of the same nature.*

A solution of ferrous ammonium sulphate is very slowly oxidised by atmospheric oxygen. If an oxalate is added to this solution the rate of oxidation is greatly increased, and the ferrous iron readily passes into the ferric state in presence of atmospheric oxygen, and the oxalate is also slowly oxidised. The same sort of behaviour is noticeable if we add a tartrate or a citrate instead of an oxalate to a ferrous salt solution. The potential reducing activity of iron is increased by the addition of an oxalate or a tartrate. Hence solutions of ammonium ferro-oxalate, or ferro-citrate or ferro-tartrate are better reducing agents than ferrous ammonium sulphate and are largely used as developers in photography.

Now there is no chemical change between potassium oxalate and mercuric chloride in the dark at the ordinary temperature, though the following chemical change takes place in light

$$K_2C_2O_4 + HgCl_2 = 2HgCl + 2KCl + 2CO_2.$$

If to this mixture of oxalate and mercuric chloride one adds a ferrous salt, one gets a slight precipitate of ferrous oxalate and at the same time mercurous chloride is formed. A solution of ferrous salt cannot reduce mercuric chloride at the ordinary-temperature, but in presence of an oxalate it becomes a better reducing agent, and reduces mercuric chloride to the mercurous state, and at the same time the potential reducing power of an oxalate is activated in bringing forth the reduction of mercuric chloride to the mercurous state. Evidently the reducing power of the ferrous salt as also that of the oxalate is activated by their mutual presence. Tartrates and citrates behave in a similar manner in presence of ferrous salts.

WINTHER (Zeit. wiss. Phot. 1909, **7**, 409) has brought forward argument to show that the light sensitiveness of a mixture of an oxalate and mercuric chloride is due to the presence of iron and the investigator suggests that the purest mixtures of an oxalate and mercuric chloride so far prepared have contained iron. In absence of iron the mixture is not sensitive to visible rays. It seems to me that the real function of iron, if it is present, is not that of a photoferment or photocatalyst, as suggested by WINTHER, but that of an inductor. The ferrous salt in presence of an oxalate reduces the mercuric chloride to the mercurous state and at the same time activates the potential reducing power of the oxalate in inducing the reduction of mercuric chloride to the mercurous state. This induced reaction takes place also in the dark. Evidently the WINTHER hypothesis seems to be doubtful.

Chemical Laboratory Muir Central College
Allahabad India.

KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN
TE AMSTERDAM.

PROCEEDINGS

VOLUME XXIII

N°. 8.

President: Prof. H. A. LORENTZ.
Secretary: Prof. L. BOLK.

(Translated from: "Verslag van de gewone vergaderingen der Wis- en Natuurkundige Afdeeling," Vols. XXVIII and XXIX).

CONTENTS.

Physios. — "*Stationary streaming caused by a body in a fluid with friction*". By Prof. J. M. BURGERS. (Communicated by Prof. P. EHRENFEST).

(Communicated at the meetings of 18 Dec. 1920 and 29 Jan. 1921).

§ 1. In this paper a summary will be given of some types found in the literature of the stationary streaming in a fluid caused by the uniform rectilinear motion of a simple symmetrical body. The fluid will be supposed to be incompressible, illimited in all directions and adhering to the sides of the body. Neither the possibility of lability of the stationary (or laminary) currents will be attended to, nor peculiarities or differences occurring when the body deviates more or less from the spherical form or when we pass from the three-dimensional problem to the two-dimensional one.

The body however is assumed to have in both cases its axis of symmetry parallel with the direction of the current and to possess no sharp edges.

The character of the streaming is wholly governed by the *number of* REYNOLDS:

$$R = \frac{Ud}{\nu} \quad \ldots \ldots \ldots \ldots \quad (1)$$

U is the velocity of the body with respect to the undisturbed fluid; d is a dimension of the body f. i. the greatest dimension perpendicular to the direction of the motion, μ is the cinematic coëfficient of friction of the fluid ($= \mu/\varrho$) [1]). At one limit, friction infinite or motion infinitely slow, $R = 0$, at the other limit, fluid without friction, $R = \infty$.

Remark. For shortness sake the expression: *absolute* current is used here to indicate the current seen by an observer for whom the fluid at an infinite distance is at rest while the body is moving with the velocity U. The streamlines are displaced with the body;

[1]) This number of comparison was introduced by O. REYNOLDS in the investigation of the flow through tubes, Phil. Trans. Lond. **174**, p. 935, 1883. It is of great importance for all model-experiments in hydrodynamics and in aerodynamics. See f.i. L. BAIRSTOW, Applied Aerodynamics, London 1920, Ch. VIII (p. 372) and other new textbooks.

in this case they are not identical with the paths of the particles of the fluid.

By *relative* flow will be meant the image of the flow as seen by an observer that regards the body as being at rest. This is therefore the real stationary state [1]). As will be known both states may be derived from each other by increasing or decreasing by U all velocities parallel to the axis of motion.

In the diagrams of the distribution of the vortices the density of the vorticity is indicated by vertical hatching. In some sketches the "opposite" vorticity (see § 3) is indicated by broken horizontal hatching.

§ 2. *Introduction.*

The state of flow is governed by the propagation or the dispersion of the *vorticity*. This is caused by two phenomena: *diffusion* and *convection*. The equation for the vortex motion which may be deduced from the equations of Euler:

$$\frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{\varrho}\nabla p + \nu \Delta \mathbf{v} - (\mathbf{v} . \nabla)\mathbf{v} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

by differentiation is:

$$\frac{\partial \mathbf{w}}{\partial t} = \nu \Delta \mathbf{w} - (\mathbf{v} . \nabla)\mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{v} . \quad . \quad . \quad . \quad . \quad (3)$$

The first term at the right hand, with the coefficiënt of friction ν, gives the diffusion of the vorticity. As will be known the velocity of diffusion has no definite value: it depends on the concentration gradient of the diffusing matter or state. There is no diffusion front or a propagation of waves.

Besides this term we see at the right hand side the convection terms: $-(\mathbf{v}. \nabla)\mathbf{w} + (\mathbf{w}. \nabla)\mathbf{v}$, which express:

1° that the vorticity is carried along by the current;

2° that the vortex vectors turn with the fluid particles and are deformated together with these.

In one case the last term fails: viz. in the plane or two dimensional motion [2]). In this case the operation $(\mathbf{w}. \nabla)$ (the differentiation in the direction of $\mathbf{w}$) is zero. The first term is of more importance

---

[1]) Some investigators as f.i. F. Ahlborn speak of *lines of force* and of *current* to indicate what has been called here absolute and relative flow.

[2]) In the original paper the three-dimensional motion was mentioned here too; this is not correct, as has been remarked to me by Prof. Prandtl.

and defines the true convection. Therefore we shall take this equation for the vortex motion:

$$\frac{\partial \mathbf{w}}{\partial t} = \nu \Delta \mathbf{w} - (\mathbf{v} \cdot \nabla) \mathbf{w} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

It is evident that for a small velocity U and a high value of ν (R therefore small) the vorticity will come everywhere by the diffusion. When, on the contrary, U is great and ν small (R therefore great) practically no eddies will diffuse against the current; all is drawn back.

§ 3. *Elementary description of the flow for R not too small* (fig. 1—4).

I. When in a fluid originally at rest a body is suddenly set into motion, we may consider a surface σ_u that surrounds the body at a very short distance ε. Outside this surface we have, during the first moments, only to do with pressure forces and as these are continuous an irrotational motion (without vortices) will arise. Let us consider the flow at a moment τ after the beginning of the motion of the body, then ε will be smaller as τ is smaller. The initial flow (because of the condition of continuity) must therefore be determined by the well-known boundary condition for the potential φ:

$$\frac{\partial \varphi}{\partial n} = V_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where V_n is the normal component of the velocity of the body at the point in question of the surface. Thus, the original flow is the irrotational motion of classic hydrodynamics (PRANDTL); this has been proved experimentally [1]).

Between σ_u and the body a thin *vortex layer* is formed, the intensity of which is defined by:

$$\int w \, dn = \frac{\partial \varphi}{\partial s} - V_s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

[1]) L. PRANDTL, Verhandl. des III. internat. Mathematiker-Kongresses, Heidelberg 1904, p. 484.

H. RUBACH, Forschungsarbeiten herausgeg. v. Verein deutscher Ingenieure, **185**. 1916.

Also in the limiting case of very great friction ($R \rightarrow 0$) we find by the calculation method of STOKES that the original motion is the ordinary potential flow. See A. B. BASSET, Hydrodynamics II (Cambridge 1888), p. 289 (Art. 505).

See in connexion with this also the note of § 7.

(s indicates here a tangential direction at a point of the surface) [1]).

Fig. 1 illustrates the "relative" flow for the two-dimensional case, the body being a circular cylinder.

II. The above mentioned vortex layer flows out by the diffusion, becomes thicker; the vorticity comes into the current and is carried along to the back of the body. See fig. 2 (the dotted circle in the vortex region gives the direction of rotation). When the vorticity behind the body has got a definite intensity, a part of the fluid there begins to rotate as a whole or to flow in closed orbits i. o. w. behind the body at both sides of the axis of symmetry there are formed circular currents (see fig. 3) [2]). Behind the body we therefore have a current towards the left; in front of it the current remains as it was to the right. Therefore we must have at both sides a point S, where the current leaves the surface (for solids of revolution this will take place along a parallel circle).

At the back of the body a vortex layer is now formed the rotation of which is opposite to that at the front (in the figure indicated by dotted, horizontal hatching).

III. After some time we might expect a stationary state to be created, in the way as has been sketched in fig. 4, where the diffusion and the convection neutralize each other. In reality this is not the case. After passing a state as is represented by fig. 3. the flow begins to fluctuate; it becomes more or less "turbulent". In stead of the regular vortex distribution a more or less irregular one is formed; the vortices "coagulate" so to say, so that regions with strong vortex motions (vortex cores) are formed, dispersed in a mass with weaker vortex motion.

A more detailed discussion of these phenomena will be omitted here [3]).

---

[1]) The limit of ε for $\tau = 0$ is determined by the sphere of action of the molecular forces at the surface of the body. As long as the fluid is treated as a continuum this may be regarded as infinitely small.

[2]) These considerations have of course the same purpose as those of PRANDTL (l.c.); the above form has been chosen to illustrate the propagation of the vortex motion.

That the accumulation of vortices gives rise to circular currents in the fluid, will probably only be true for high values of R (when the vortex motion is not too diluted). Only in the limiting case of very great R it can be proved by means of the formulae (see below, § 7 and 8).

On the photo's of RUBACH l.c. we can see that the circular currents are formed on a small scale behind the cylinder at both sides of the point where the flow unites. Theoretically this has been investigated by BLASIUS (see l.c. § 8).

[3]) This fluctuating motion may also be described in another way. When R is high, so that the velocity of diffusion of the vortex motion is small, then the vortex sheet

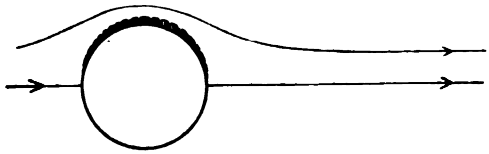


Fig. 1.

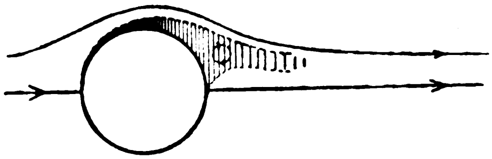


Fig. 2.

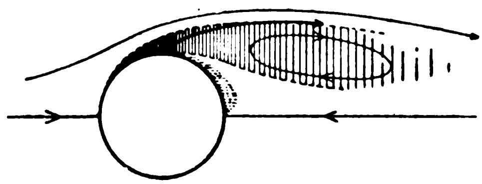


Fig. 3.

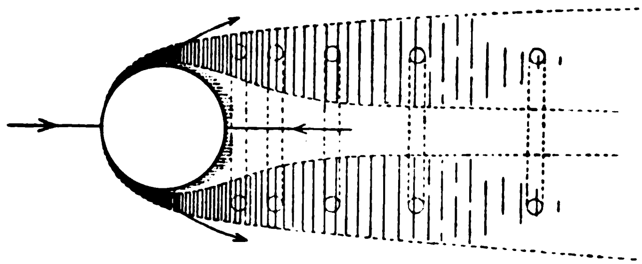


Fig. 4.

formed in front of the body and leaving it at S to enter the fluid, is *very thin*, so that it represents approximately a surface of discontinuity for the distribution of the velocity. This sheet can be seen on photo's of the flow in the immediate neighbourhood of the body. Sometimes the stream lines are bent so sharply that we may speak of a discontinuity.

Thus a vortex layer leaves the body at both sides; as will be known these layers (at least for the two-dimensional flow) have the property to *curl* themselves in the way of *spirals* to a row of vortex threads. The vortex threads formed from both layers are placed more or less regularly (von Kármán, Phys. Zeitschrift **13** p. 49, 1912 and others).

This may be demonstrated by coloured fluid flowing from holes at the front surface of the body. This fluid comes into the boundary or vortex layer, participates its motion and in this way makes its form manifest. Especially fine cinema-

Laminary motion for different values of R.

§ 4. *The flow round a sphere according to* Stokes. The method indicated by Stokes for the solution of the hydrodynamical equations neglects the convection terms. It gives therefore the flow at the end $R \rightarrow 0$ of the series; the velocity U must be very small, so that the influence of the friction dominates. Equation (4) is reduced to the simple equation for the diffusion:

$$\frac{\partial \mathbf{w}}{\partial t} = \nu \Delta \mathbf{w} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The most known solution of this equation is that for the *stationary motion of a sphere* [1]). The vortex motion diffuses from the surface symmetrically forward and backward; its intensity is given by:

$$w = \frac{3}{2} Ua \frac{\sin \theta}{r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

(a is the radius of the sphere; r and θ are polar coordinates with the centre of the sphere as pole and the direction of motion as axis). This distribution is represented in fig. 5.

The stream function for the absolute flow is:

$$\psi_{abs} = \frac{3}{4} Ua \left(r - \frac{a^2}{3r} \right) \sin^2 \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

By means of this the figures 6 and 7 for absolute and relative flow have been drawn. These figures too are symmetrical at both sides.

§ 5. *The motion according to* Oseen.

Oseen has remarked that the considerations leading to the neglection of the convection terms hold in the immediate neighbourhood of the sphere; at great distances from it the velocity of diffusion diminishes to zero, while the convection current always keeps the same velocity U: hence the convection will predominate here.

---

tographic photo's are found in: E. F. Relf, Technical Report Advisory Committee for Aeronautics, London 1912—13, p. 133 (Rep. N°. 76); and in L. Bairstow, Applied Aerodynamics, London 1920, p. 345, fig. 167 (photo by J. L. Nayler, see Techn. Rep. etc. Rep. N°. 332 of May 1917).

The vortex layer behaves as a so-called „filament-line", (see L. Bairstow, l.c. p. 348).

[1]) See f.i. H. Lamb, Hydrodynamics, Cambridge 1916, p. 587. Lamb also gives a fine diagram of the absolute flow.

Fig. 5—7. Flow caused by a sphere for $R \to 0$ according to Stokes.

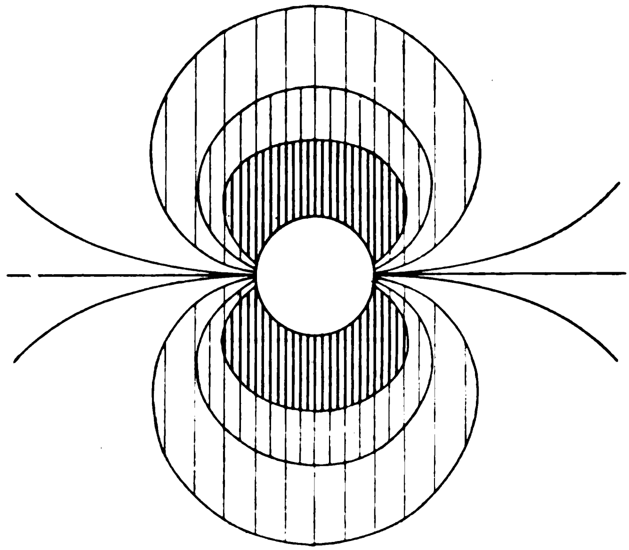


Fig. 5. Distribution of the vorticity.

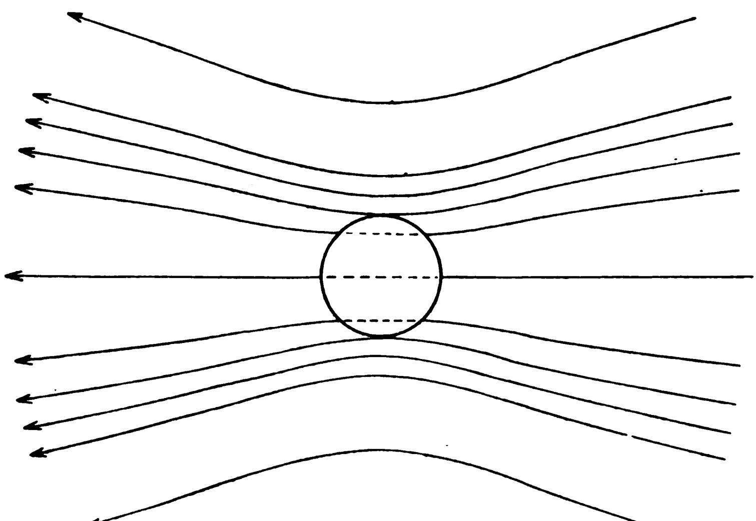


Fig. 6. Absolute flow.

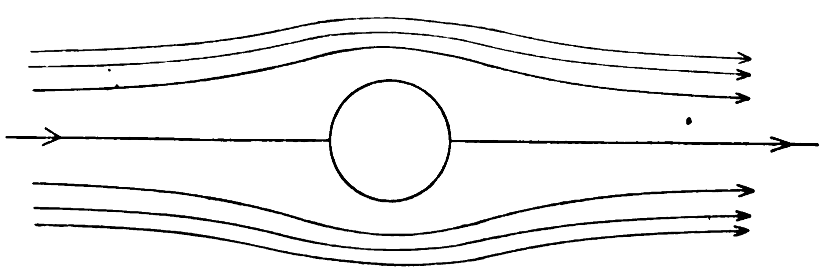


Fig. 7. Relative flow.

To take this into consideration OSEEN has kept the constant translational current U in the equation; equation (4) has not been simplified to (7) but to:

$$\underline{\frac{\partial \mathbf{w}}{\partial t} = \nu \Delta \mathbf{w} - U \frac{\partial \mathbf{w}}{\partial x}}\ ^{1)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

In the nearest neighbourhood of the body this does not give an amelioration, on the contrary rather a little change for the worse, but the inaccuracy remains of the same order of magnitude as in STOKES' solution viz. of the order $\frac{Ua}{\nu}$, or of the order of R.

For the stationary motion of a *sphere* OSEEN finds in the immediate neighbourhood of the sphere the same solution as STOKES and therefore the same value of the resistance. This was also found by LAMB in another way. At a great distance however all has been "drawn backward". The distribution of the vorticity is defined by:

$$w = \frac{3}{2}\, Ua\,.\, \frac{1 + Ur/2\nu}{r^2}\,.\, \sin\theta\,.\, exp \left\{ -\frac{U(r-x)}{2\nu} \right\}, \quad . \quad . \quad (11)$$

and represented in fig. 8 $\left(\text{the figures have been drawn for } R = \frac{2aU}{\nu} = \right.$ $\left. = 0{,}4\right)$. The asymmetry between front and back side is evident. Because of the exponential factor at the end of (11) w is very small outside a parabolical space f.i. bounded by:

$$r - x = \frac{20\nu}{U},$$

(where at the outside the exponential function is smaller than 0,000045). Here the motion becomes therefore nearly irrotational.

The stream function is given by:

$$\psi_{abs} = \frac{3\nu a}{2}(1 + \cos\theta)\left[1 - exp\left\{-\frac{Ur(1-\cos\theta)}{2\nu}\right\}\right] - \frac{Ua^3}{4r}\sin^2\theta \quad (12)$$

See fig. 9 and for the relative flow fig. 10. Fig. 11 gives on a smaller scale (i.e. for higher values of r) the distribution of the vorticity and the absolute flow; it shows that outside the parabolically bounded space the motion approaches a radial current

---

[1]) C. W. OSEEN, Arkiv f. Mat. Astron. och Fysik. Bd. 6, N⁰. 29 (1910).

H. LAMB, Phil. Mag. (6) 21, p. 112, 1911 and Hydrodynamics, p. 595 seqq. LAMB gives a discussion of the character of the motion (from which these remarks have been taken) and also gives a solution for the corresponding two-dimensional problem. (In this last case STOKES' method does not give a solution).

Fig. 8—10. Flow caused by a sphere for $R = 0,4$ according to OSEEN and LAMB.

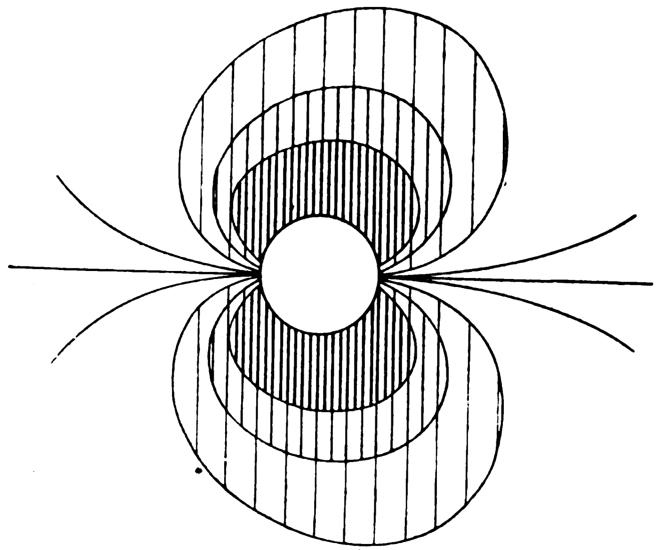


Fig. 8. Distribution of the vorticity.

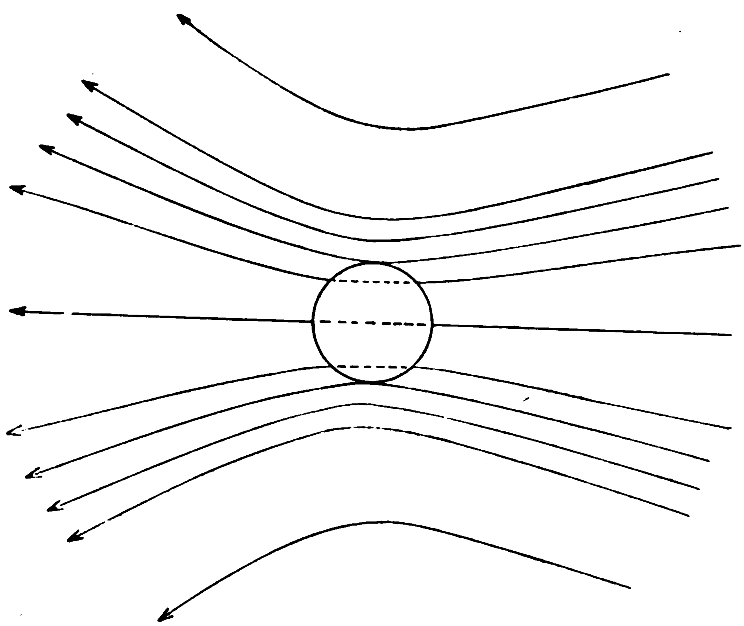


Fig. 9. Absolute flow.

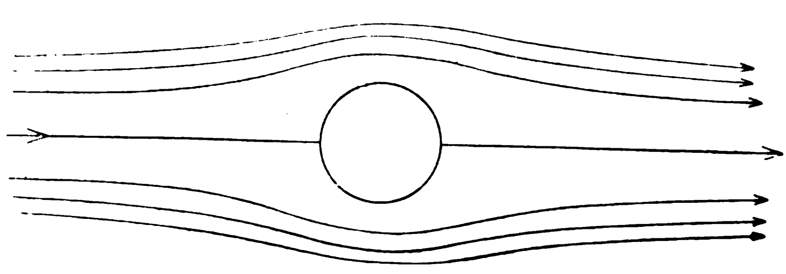


Fig. 10. Relative flow.

(with $v = 3va/2r^2$), while inside this space the fluid is carried along with the sphere [1]).

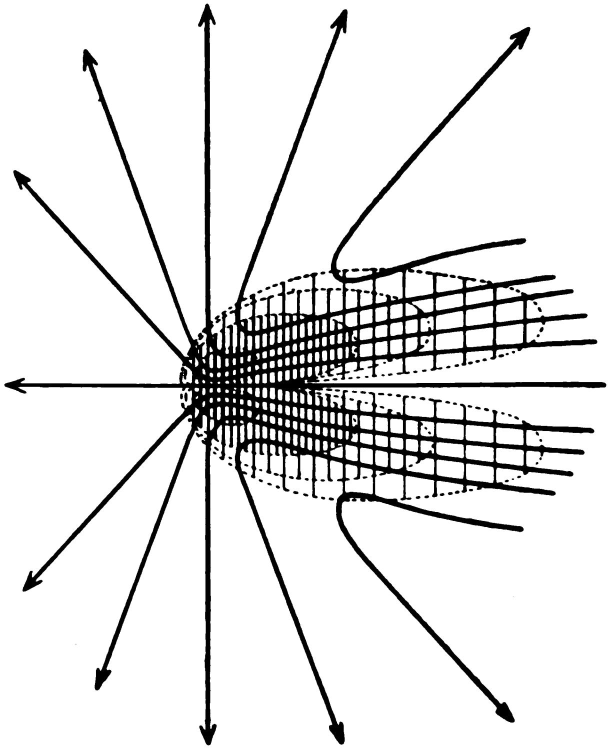


Fig. 11. Flow caused by a sphere for $R = 0,4$ according to OSEEN and LAMB. Distribution of the vorticity and absolute flow. Radius of the sphere in this figure = ca 0,2 mm.

§ 6. OSEEN's *limiting current for* $R \rightarrow \infty$.

In different papers OSEEN has investigated the properties of the general solution of equation (10) and of the corresponding equations for the velocity **v** and in two publications of 1914 and 1915 he determined the limit to which the solution approaches when the friction becomes zero and R therefore infinite [2]).

In this case the diffusion vanishes and as the only cause of the

[1]) The current given by (12) is no exact solution of the equations used (see LAMB, l.c. p. 598). A more exact approximation has been given by R. W. BURGESS, American Journal of Mathematics **38**, p. 81, 1916. — LAMB's method has been applied to ellipsoids by B. PAL, Bull. Calc. Math. Soc. **X**, p. 81, 1919.

[2]) C. W. OSEEN, Zur Theorie des Flüssigkeitswiderstandes, Nov. Acta R. Soc. Scient. Upsaliensis Ser. IV, Vol. **4**, 1914.

motion of the vortex elements is the translational current (according to equation (10)) there will be formed in front of the body an infinitely thin vortex layer (of finite total intensity) which extends itself backward in a *cylindrical sheet*, parallel to the direction of the current, which sheet surrounds the body along the parallel circle of largest diameter. This is sketched in fig. 12, where the vortex sheet has been indicated by a thick line. Outside the cylinder the motion is *irrotational*, inside it generally not. Along the cylindrical surface the x-component of the velocity changes abruptly.

For the *stationary motion* the following solution is found [1]) (the formulae refer to the absolute flow; the system of coordinates x, y, z moves with the body, the x-axis is in the direction of motion):

Let φ be a potential function, then we have outside the considered cylindrical space:

$$\mathbf{v} = \nabla \varphi, \quad \ldots \ldots \ldots \quad (13^a)$$

and inside it

$$\mathbf{v} = \nabla \varphi + \mathbf{U} - \mathbf{v}^*(y, z), \quad \ldots \ldots \quad (13^b)$$

where $\mathbf{v}^*(y, z)$ indicates the value of $\nabla\varphi$ at the point at the back of the body with the y- and z-coordinates: y, z. Here $\varphi(x, y, z)$ is defined by the following conditions (that follow from the equation of continuity):

$$\Delta \varphi = 0; \quad \ldots \ldots \ldots \quad (14^a)$$

further at the front of the body:

$$\left(\frac{\partial \varphi}{\partial n}\right)_\bullet = U \cos(n\, x), \quad \ldots \ldots \ldots \quad (14^b)$$

and at the back:

$$\nabla \cdot \mathbf{v}^* \equiv \frac{\partial v_y^*}{\partial y} + \frac{\partial v_z^*}{\partial z} \equiv \left(\frac{\partial}{\partial n}\frac{\partial \varphi}{\partial x}\right)_0 = 0 \quad \ldots \quad (14^c)$$

At the back of the body we have

$$\mathbf{v} = \mathbf{U} \quad \text{(parallel to the } x\text{-axis)} \quad \ldots \quad (15)$$

Here the fluid sticks to the body. At the front on the contrary only

---

C. W. Oseen, Beiträge zur Hydrodynamik I, Ann. de Phys. **46**, p. 231, 1915. In the following papers (p. 623 and 1130) Oseen treats the properties of the solution of the non-simplified equations. These are written in the form:

$$\varrho \frac{\partial \mathbf{v}}{\partial t} + \nabla\left(p + \frac{1}{2}\varrho\, v^2\right) - \mu \Delta \mathbf{v} = \mathbf{A}$$

Then the vector $\mathbf{A} = \rho\,(\mathbf{v} \times \mathbf{w})$ is treated as "external force". In the paper of p. 231 **A** has not been considered.

[1]) C. W. Oseen, Ann. d. Phys., l.c. p. 249.

the normal component of the velocity of the fluid has to correspond to the normal component of the velocity of the body; the tangential velocity is not bound, so that generally the fluid will slip along the body. Here a boundary layer exists, and vorticity is formed [1]).

I have tried to represent this solution for the case of the two-dimensional flow along a cylinder with circular section. The radius of the cylinder and the velocity U have both been taken $=1$. The vortex domain lies therefore between $y=+1$ and $y=-1$. Equation (14c) gives then: $\partial v_y^*/\partial y = 0$; and as on the x-axis $v_y^*=0$ (because of the symmetry) we have everywhere $v_y^*=0$. On the line $y=+1$ v changes abruptly by the amount: $1-v^*$; between $y=+1$ and $y=-1$ the vorticity is:

$$w=\frac{dv^*}{dy} \quad \ldots\ldots\ldots\ldots \quad (16)$$

In order to find an approximate value for φ, I have put:

$$\varphi = A_0 \lg r - \sum_1^N \frac{A_n \cos n\theta}{n r^n} \quad \ldots\ldots\ldots \quad (17)$$

($\theta=0$ is the point most in front of the circle, where $x=+1$; at the opposite point $\theta=\pi$). Then the boundary conditions (14b) and (14c) become:

$$-\frac{\pi}{2} \leq \theta \leq +\frac{\pi}{2}: \qquad \sum_0^N A_n \cos n\,\theta = \cos\theta \quad \ldots \quad (18^a)$$

$$+\frac{\pi}{2} < \theta < +\frac{3\pi}{2}: \qquad \sum_0^N (n+1)\, A_n \cos (n+1)\,\theta = 0 \quad . \quad (18^b)$$

By means of the method of least squares a solution has been sought that for a given value of N satisfies as well as possible (18a) and (18b). For $N=3$ is found in this way:

$$A_0 = +\,0{,}374$$
$$A_1 = +\,0{,}375$$
$$A_2 = +\,0{,}248$$
$$A_3 = +\,0{,}086$$

With the aid of these numbers fig. 13 has been drawn for the absolute flow and fig. 14 for the relative flow.

---

[1]) L.c. p. 252; further also p. 623 and 1144.

OSEEN considers moreover the following simple cases (l.c. p. 249/250):

a. Body illimited in backward direction (the thickness has no maximum at a finite distance). Everywhere outside the body we have irrotational motion, defined by (13a), (14a) and (14b).

b. Body illimited in front direction. Then a solution of (14a) and (14c) is given by: $\boldsymbol{\varphi}=0$ (we have no front, so that (14b) vanishes). Outside the cylinder $\mathbf{v}=0$; inside $\mathbf{v}=\mathbf{U}$.

Fig. 12—14. Flow caused by a cylinder for $R = \infty$, calculated with the formulae of Oseen.

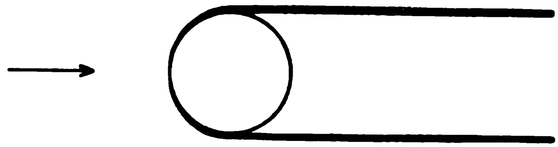


Fig. 12. Vortex layers.

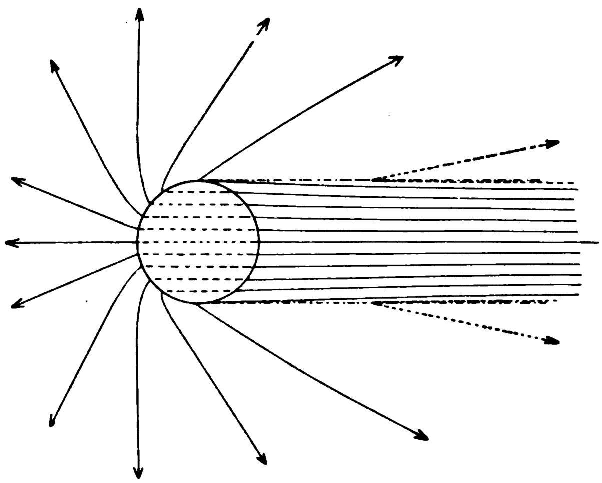


Fig. 13. Absolute flow.

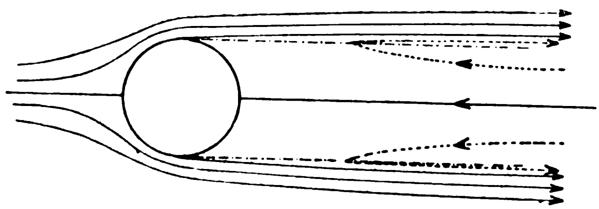


Fig. 14. Relative flow.

At a great distance from the cylinder we have a radial current, which displaces totally: $2\pi A_0 = 2,35$; this amount comes back in the wake stream between $y = +1$ and $y = -1$. In the wake at a great distance from the cylinder $\nabla\varphi$ may be neglected, so that $v = 1 - v^*$; and

$$2\int_0^1 dy\,(1 - v^*) = 2,36.$$

The value of v^* proves to be < 0, so that the velocity in the wake is > 1. In the image of the relative flow a *forward current* is therefore formed. The stream line for $\psi = 0$ follows the x-axis for $x > 1$, the circle until about $\theta = \pm 90°$, then it leaves this and approaches asymptotically to the lines $y = \pm 1{,}18$.

As further $dv^*/dy < 0$ for $y > 0$ the direction of rotation of the vorticity is here opposite to that in the boundary layer $y = +1$ [1]).

§ 7. *Application of a method of calculation of* BOUSSINESQ.

The limiting flow of OSEEN we discussed above is very different from the flow which usually takes place in a fluid with infinitesimal friction. The most typical particularity in the case of OSEEN is the discontinuity of $\mathbf{v}$ at a cylindrical surface that surrounds the body. The form of the surface of discontinuity is determined by the way in which the convection of the vortex motion has been calculated: the vorticity moves in the direction of the x-axis with the translational current U only, so that it extends from the thickest part of the body backward in an infinitesimally thin layer (fig. 12). When however we attend to the elementary description of the initial development motion of the (§ 3) [2]) we should expect for a great velocity and a small ν that the thin vortex layer at the surface of the body is slept backward by the current *along that surface* viz. towards the point where

[1]) The degree of inaccuracy of this solution may be estimated by calculating the value of the stream function $\psi_{abs.}$ for $r = 1$ and $\theta = 90°$; it has the value 0,933 instead of 1,000. Further the values of v_y are for $\theta = 90°$, 120°, 150° and 180° resp.: 0,126, 0,074, 0,034 and 0, instead of all being 0. In order to avoid difficulties the figures have been a little shaped.

An approximation to $N = 10$, gave for A_0 till A_{10} resp.:

$+\ 0{,}366;\ +\ 0{,}419;\ +\ 0{,}240;\ +\ 0{,}024;\ -\ 0{,}059;\ -\ 0{,}019;$

$+\ 0{,}028;\ +\ 0{,}020;\ -\ 0{,}011;\ -\ 0{,}021;\ -\ 0{,}010.$

From this follows:

$$\psi_{abs}\ \text{for}\ r = 1,\quad \theta = 90°\ :\ 0{,}977$$

$$v_y^*\ \text{for}\ \theta = 90°\ (\text{therefore}\ y = 1)\ :\ 0{,}018.$$

In this case we have:

$$\pi A_0 = 1{,}149;\qquad 2\pi A_0 = 2{,}30.$$

The general character of the motion therefore does not change by this closer approximation.

[2]) Also in the limiting case of OSEEN the original flow, when it starts from the state of rest, is the *ordinary irrotational motion*. According to Ann. d. Phys., l.c. p. 246/247 the flow outside the space described by the body is everywhere irrotational; the potential φ is determined by: $\Delta\varphi = 0$; $\dfrac{\partial\varphi}{\partial n} = U \cos(nx)$ at

the current originally closes, and that it moves further in the direction of the axis of the current behind the body. Approximately it will be confined to a small paraboloid round that axis and the whole vortex region may be roughly represented by fig. 15.

In the limiting case $R=\infty$ the vortex layer at the body will be infinitely thin and the paraboloid will contract to the axis (see fig. 16). The vortices from opposite parts of the surface of the body having opposite signs, they will soon vanish in the axis. In the limiting case $R=\infty$ we have therefore everywhere *outside the body* irrotational motion, while at the surface of the body an infinitesimal vortex layer is found [1]). The potential φ of the current is determined by the ordinary condition:

$$\frac{\partial\varphi}{\partial n}=V_n=U\cos(nx) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

for the absolute flow along the whole surface of the body. When we wish to obtain a corresponding representation of the vorticity distribution and yet to use as in OSEEN's solution a *linear* equation, equation (4) must be replaced by:

$$\frac{\partial \mathbf{w}}{\partial t}=\nu\Delta\mathbf{w}-(\mathbf{v}_0\cdot\nabla)\mathbf{w} \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

where $\mathbf{v}_0$ is written for a known current, which at a great distance from the body approaches the parallel current $\mathbf{U}$, following however the surface of the body in its immediate neighbourhood.

For $\mathbf{v}_0$ we may e.g. take the *ordinary irrotational current.* In this case (20) changes into an equation applied by BOUSSINESQ in the calculation of the transport of heat by a moving fluid [2]). It is

---

the front; $\frac{\partial}{\partial n}\frac{\partial\varphi}{\partial t}=\frac{\partial U}{\partial t}\cos(nx)$ at the back of the body. Behind the body we therefore have:

$$\lim_{\tau=0}\frac{\partial\varphi}{\partial n}=\int_0^\tau dt\,\frac{\partial U}{\partial t}\cos(nx)=U(\tau)\cos(nx).$$

As soon as the body is set into motion we shall have over the whole surface $\frac{\partial\varphi}{\partial n}=U\cos(nx)$, so that φ is the ordinary potential; the space outside the region described by the body is then the whole space outside the body, hence everywhere $\mathbf{v}=\nabla\varphi$.

[1]) Perhaps the x-axis behind the body must be regarded as a singular line in the flow.

[2]) J. BOUSSINESQ, Journal de Liouville (6) 1, p. 285, 1905. See also: A. RUSSELL, Phil. Mag. (6) 20, p. 591, 1910, and L. V. KING, Phil. Trans. London A **214**, p. 373, 1914.

evident that neither this choice of $\mathbf{v}$ nor the method of OSEEN gives the true convection of the vorticity along the surface, as the convection velocity must sink to zero there, which is not the case with the value of $\mathbf{v}_e$ taken above.

BOUSSINESQ has shown that for the stationary two dimensional flow equation (20) takes a simple form, when we take as coordinates the stream function and the potential of the flow $\mathbf{v}_e$ (here the relative flow image is used). When as in the notation of BOUSSINESQ $U\alpha$ is written for the stream-function, $U\beta$ for the potential, then the equation

$$0 = \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) - U\left(\frac{\partial \beta}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial \beta}{\partial y}\frac{\partial w}{\partial y}\right) \quad . \quad . \quad . \quad (20^*)$$

becomes:

$$0 = \nu\left(\frac{\partial^2 w}{\partial \alpha^2} + \frac{\partial^2 w}{\partial \beta^2}\right) - U\frac{\partial w}{\partial \beta},$$

or:

$$\underline{\frac{\partial w}{\partial \beta} = \frac{\nu}{U}\left(\frac{\partial^2 w}{\partial \alpha^2} + \frac{\partial^2 w}{\partial \beta^2}\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

A difficulty in the solution of this equation is that the boundary conditions are expressed in v and not in w. The limiting case however treated by BOUSSINESQ himself in the problem of heat transport[1]) is simple: ν/U is very small (this involves a very great R), so that the vortex motion is confined to a very thin boundary layer and the derivative of a quantity with respect to α (viz. in a direction perpendicular to the boundary layer) will be much greater than the derivative with respect to β (in the direction of the boundary layer).

Then we may assume:

$$\underline{\frac{\partial w}{\partial \beta} = \frac{\nu}{U}\frac{\partial^2 w}{\partial \alpha^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

and also:

$$w = -\frac{\partial v_s}{\partial \alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

where v_s is the (true) velocity in the boundary layer parallel to the surface.

By means of these formulae we can calculate the distribution of the vorticity and the current in the boundary layer, when we suppose the velocity outside the boundary layer to be known. In analogy

[1]) J. BOUSSINESQ, l.c. p. 295/296. — BOUSSINESQ also treats the problem for a solid of rotation (p. 305). In thecalculation of the heat transport KING uses the complete equation (21) (l.c.)

Fig. 15—18. Flow produced by a cylinder for a high value of R, calculated according to the method of § 7 and § 10.

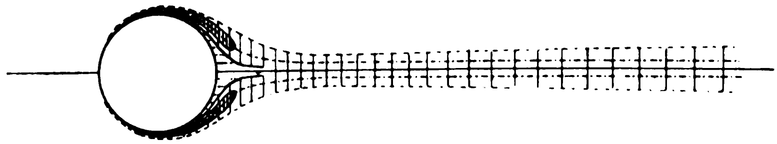


Fig. 15. Distribution of the vorticity.

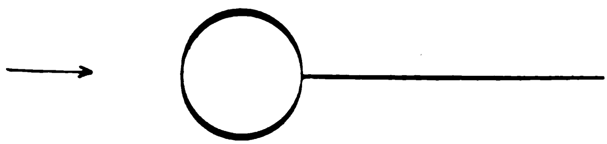


Fig. 16. Limit of fig. 15 for $R = \infty$.

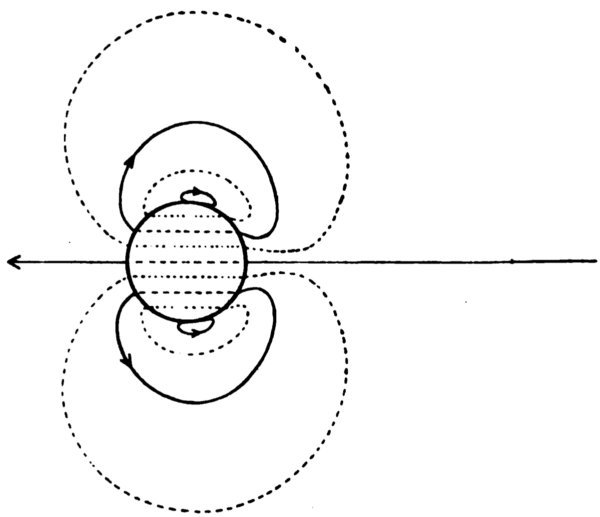


Fig. 17. Absolute flow.

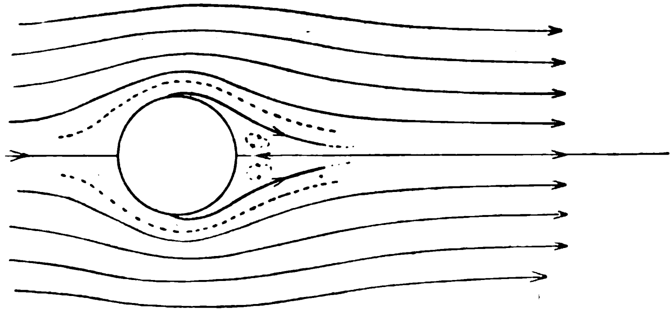


Fig. 18. Relative flow.

with the above we shall assume an ordinary irrotational current to exist outside the boundary layer.

From the calculation we see that an *reversion of the direction of the flow* may take place in the boundary layer, when the outside motion is *retarded*. In this case a counter current will be formed behind the body and the current coming from the front leaves the surface at a certain point. The place of this point depends on the form of the body but not on R. The thickness of the layer, in which these currents take place proves to be proportional to $\sqrt{\nu d/U}$, where d is a dimension of the body; the relative thickness is therefore of the order: $\frac{1}{d}\sqrt{\frac{\nu d}{U}} = \frac{1}{\sqrt{R}}$. In these two points there is a qualitative agreement with the exact method of PRANDTL; there is however no quantitative agreement.

For the details of the calculation see § 10. The distribution of the vorticity has been represented schematically in the above mentioned fig. 15; fig. 17 gives a sketch of the image of the absolute flow; fig. 18 of that of the relative flow.

§ 8. *The method of* PRANDTL.

The method of PRANDTL and his collaborators is the only method of calculation in which equation (4) is not reduced artificially to a linear equation, but where directly a solution of the *quadratic equation* is sought for [1]). A detailed discussion of this method cannot be given here; a few remarks only may find place:

a. The method has been worked out for the two-dimensional and for the axial-symmetrical three-dimensional flow, for high values of R, so that the boundary layer is thin.

b. Because of this last circumstance PRANDTL simplifies EULER's equation to:

$$\frac{\partial v_1}{\partial t} + v_1 \frac{\partial v_1}{\partial x} + v_2 \frac{\partial v_1}{\partial y} = -\frac{1}{\varrho}\frac{dp}{dx} + \nu \frac{\partial^2 v_1}{\partial y^2} \quad . \quad . \quad . \quad . \quad (24)$$

where the x-axis has been taken parallel to the surface of the body, and the y-axis perpendicular to it. The pressure p is given by the state outside the boundary layer and can be treated in this

[1]) L. PRANDTL, Ueber Flüssigkeitsbewegung bei sehr kleiner Reibung, Verhandl. d. III. internat. Mathematiker-Kongresses, Heidelberg 1904, p. 484.

H. BLASIUS, Zeitschr. f. Math. u. Phys. 56, p. 1, 1908 (dissert. Göttingen 1907) en ibidem 58, p. 225, 1910.

E. BOLTZE, Dissert. Göttingen 1908.

K. HIEMENZ, Dinglers Polytechn. Journal 326, p. 321, 1911.

layer as independent of y, v_1 and v_2 are connected by the equation of continuity.

For the vorticity $w = -\frac{\partial v_1}{\partial y}$ we therefore have:

$$\frac{\partial w}{\partial t} = \nu \frac{\partial^2 w}{\partial y^2} - \left(v_1 \frac{\partial w}{\partial x} + v_2 \frac{\partial w}{\partial y}\right) \quad . \quad . \quad . \quad . \quad . \qquad (25)$$

c. From the solutions of the equations (which for the greater part must be obtained by numerical approximation) we find that in the case of a *retarded* flow outside the boundary layer, there is a point in this layer where the *direction of the flow* is *reversed*, so that counter currents are formed and the current coming from the frontside *leaves* the surface. The place of this point is independent from R.

The dimensions of the boundary layer in the direction of y are given by the form of the body and are proportional to $\frac{1}{\sqrt{\ }}$ times the dimensions of the body.

Thus we find by both these methods (of § 7 and § 8) that for high values of R a vortex sheet leaves the surface of the body at both sides (eventually round the body) and emerges into the fluid as has been indicated by Oseen. For increasing values of R however this sheet evidently does not only become *thinner* so that it approaches to a surface of discontinuity, but it fits *more closely round the body* so that the irrotational motion outside gradually extends over a greater space and finally becomes the ordinary irrotational flow [1]).

§ **9**. *Remark on the motion for mean values of R.*

The calculation according to the methods of § 6—8 teaches that for high values of R there is formed behind the body a forward current opposite to the original direction of the current, which is in agreement with the experimental results [2]). For $R \rightarrow 0$ this does

---

1) The considered bodies must not have sharp edges (as was demanded in § 1). Therefore the above remark does not say anything against Helmholtz's discontinuous flow along a plate with sharp edges.

2) In the image of the absolute flow the velocity immediately behind the body is greater than U (the fluid "overtakes" the body), the stream-lines must therefore be packed more closely together than in the original parallel flow. As to the distribution of the vorticity: at the back of the body we have a layer of "opposite" vortex motion (see above § 3, II).

Fig. 19—21. Sketch of the flow produced by a cylinder for a mean value of R.

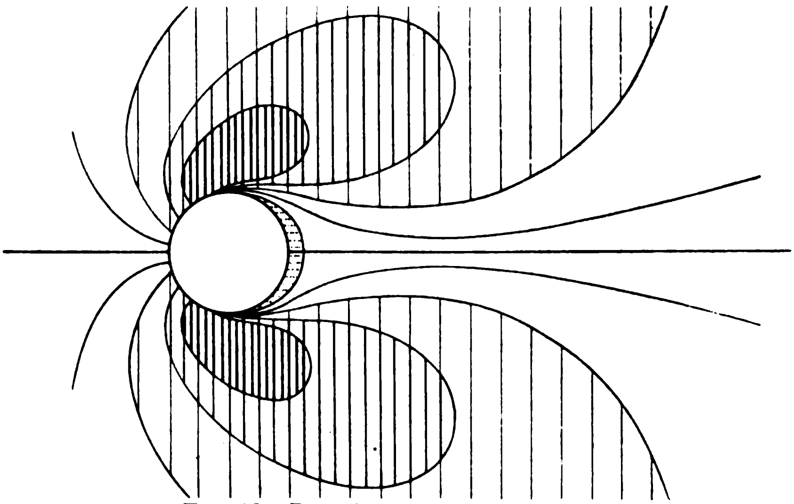


Fig. 19. Distribution of the vorticity.

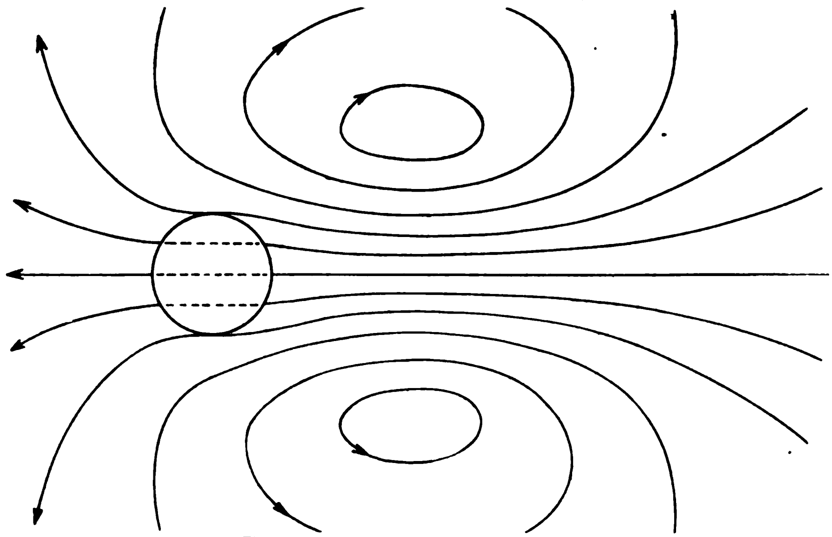


Fig. 20. Absolute flow.

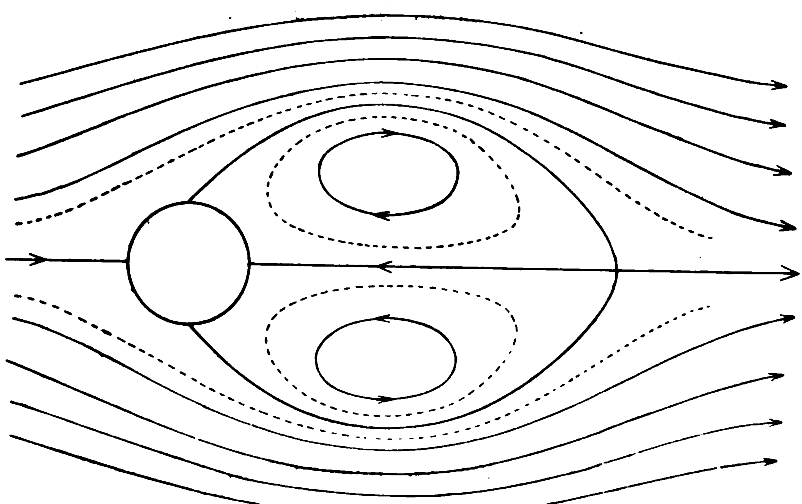


Fig. 21. Relative flow.

Fig. 22—24. Sketch of the flow produced by a cylinder for a mean value of R (but higher than for fig. 19—21).

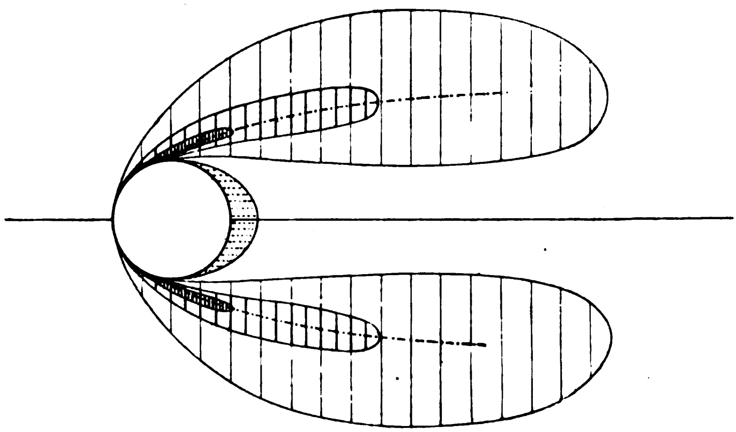


Fig. 22. Distribution of the vorticity.

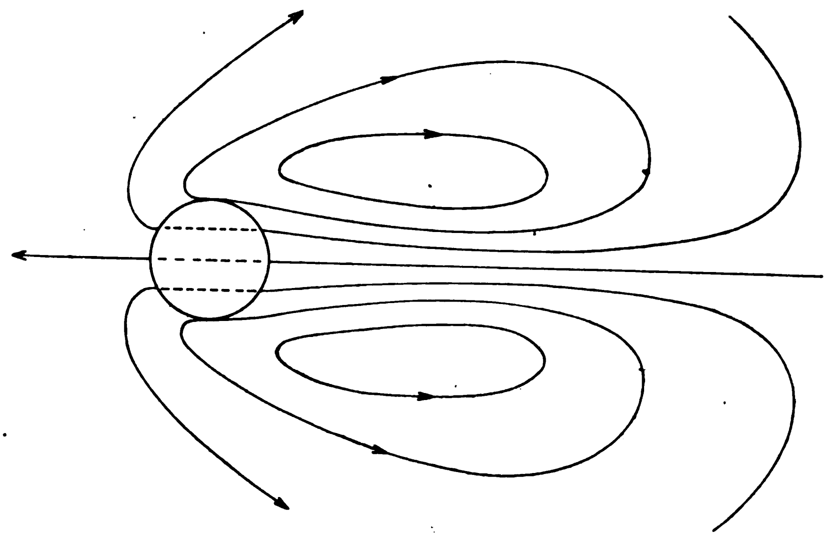


Fig. 23. Absolute flow.

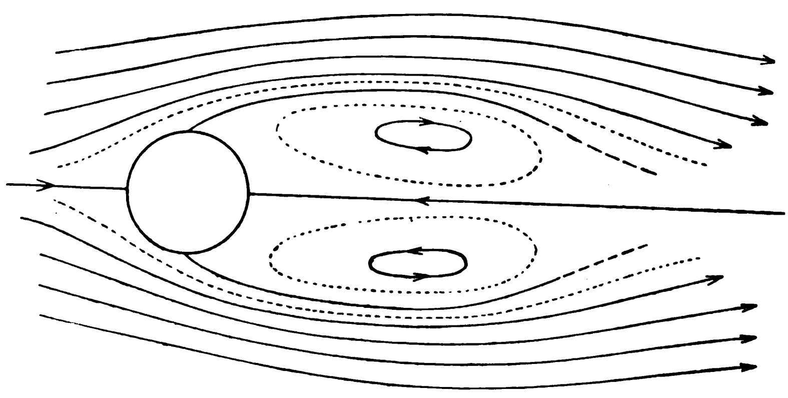


Fig. 24. Relative flow.

not occur [1]), so that the question arises: when does this reversion of the flow begin, is it connected with a definite value of R? Further, how far does the region of the counter currents extend; is it at first a thin layer which becomes thicker with increasing R and which afterwards diminishes again? That the length is finite, is proved by the consideration that the differences in velocity are finally quite dissolved by the friction, so that in the axis the current must reassume the original direction.

A second question is the following: in the image of the absolute flow the stream lines are not closed according to STOKES and OSEEN - LAMB, but they are for the ordinary irrotational motion (the limiting current of §§ 7 and 8). Where is the passage from one case to the other?

The figures 19—24 are meant as a possible interpolation between the considered limiting cases (they have been sketched for the two-dimensional case). Of course they can by no means claim the name of approximation [2]).

Such flows are observed at the beginning of the motion. Afterwards they change into a more or less irregular motion. It is not known for which value of R the lability of the laminar one begins [3]).

§ **10**. *Application of the method of § 7 to the calculation of the diffusion and the convection of the vorticity in the irrotational motion.*

I. *Notation.* U=velocity of the indisturbed parallel current; $v_0 U$= velocity of the irrotational motion, vU=velocity of the fluid in the

---

[1]) In the following cases: flow of STOKES round a sphere; of OSEEN round a sphere and round a cylinder; and of BURGESS round a sphere (see the quotations of § 5) we see from the formulae given by the authors that in the image of relative flow the velocity v on the axis behind the body has the same value as that of the original current U.

[2]) The distribution of the vorticity given in the figures has not been derived by calculation from the distribution of the velocities; they were only sketched on view. Fig. 22 gives the beginning contraction of the vortices to a vortex sheet. In Fig. 24 the length of the domain of the counter currents has been left undetermined.

Theoretical and experimental investigations on the flow produced by a sphere for medium values of R in a space *bounded by solid walls* have been made by W. E. WILLIAMS, Phil. Mag. (6) 29, p. 526, 1915. In the paper several photo's and drawings are represented.

[3]) The "coagulation" of the vorticity cannot increase or decrease its quantity. Therefore even in the real, turbulent flow the "wake" must remain of finite length. The figures 22—24 may be said to represent the "mean state" of the fluctuating current.

boundary layer (parallel to the surface), $wU =$ vorticity. Further αU is written for the stream function of $v_0 U$, and βU for the potential.

The stream line $\alpha = 0$ is the line of symmetry of the motion; in front of the body it splits into two branches. These two unite again at the back of the body. At the branch points the values of β are β_1 and β_2. For a circular cylinder we have f. i. (when the radius of the section is a):

$$\alpha = y\left(1 - \frac{a^2}{r^2}\right) \quad ; \quad \beta = x\left(1 + \frac{a^2}{r^2}\right).$$

The line $\alpha = 0$ consists of the x-axis ($y = 0$) and the circle ($r = a$); at the points of intersection $x = \pm a$, so that:

$$\beta_1 = -2a \quad ; \quad \beta_2 = +2a.$$

II. The differential equation for the vortex motion in the boundary layer is:

$$\frac{\partial w}{\partial \beta} = \frac{\nu}{U} \frac{\partial^2 w}{\partial \alpha^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

which is a shortened form of equation (21). When the latter was derived from equation (20*) it has been divided by:

$$\left(\frac{\partial \beta}{\partial x}\right)^2 + \left(\frac{\partial \beta}{dy}\right)^2 = v_0^2,$$

which quantity is different from zero everywhere but at the points β_1 and β_2, where the stream line $\alpha = 0$ splits. When at these points (21) is satisfied, then necessarily the original equation (20*) is also satisfied. In the neighbourhood of these two points — at least in that of β_2 — the boundary layer may no longer be treated as infinitely thin, so that there the simplification of (21) is not allowed; in the determination of v too we find here a difficulty. With increasing R however the allowable limit of $|\beta - \beta_2|$ decreases.

III. As a solution of (22) we may take for $\beta_1 < \beta < \beta_2$ and for $\alpha \geqq 0$ (that is for the left side of the surface of the body):

$$w = -\int_{\beta_1}^{\beta} d\xi \frac{A(\xi)}{\sqrt{\pi k (\beta - \xi)}} \cdot exp \frac{-\alpha^2}{4k(\beta - \xi)} \quad . \quad . \quad . \quad . \quad (I)$$

where $k = \nu/U$. Because of the symmetry the same expression may hold for the right side of the surface, with the opposite sign.

$A(\xi)$ determines the quantity of vorticity that leaves the surface in the neighbourhood of the point $\beta = \xi$ in unit of time; it diffuses in the direction perpendicular to the surface and at the same time it is washed backward in the direction parallel to the sur-

face. A is determined by the boundary condition for v. When the boundary layer is sufficiently thin (as has heen assumed), we have at all its points, except for the nearest neighbourhood of β_1 and β_2:

$$dn = \frac{da}{V_0}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (II)$$

where n is the normal to the surface, while $V_0 U$ denotes the velocity of the potential current at the foot of the normal, which is a function of β. Then the velocity in the boundary layer is:

$$v = -\int_0^n w\, dn = -\frac{1}{V_0}\int_0^\alpha w\, da = \frac{1}{V_0}\int_{\beta_1}^{\beta} d\xi\, A(\xi)\,.\, Erf\frac{\alpha}{2\sqrt{k(\beta-\xi)}}\,. \quad (III)$$

At the surface ($\alpha = 0$), $v = 0$; when α increases indefinitely, we come outside the boundary layer and we may put $v = V_0$, so that,

$$V_0^2 = \int_{\beta_1}^{\beta} d\xi\, A(\xi), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (IV)$$

whence A is given by:

$$A = \frac{dV_0^2}{d\beta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (V)$$

IV. Formulae (V), (I) and (III) roughly describe the flow in the boundary layer (the velocity in the direction perpendicular to the surface must be determined with the aid of the equation of continuity). From (III) we can immediately derive the occurring of the reversed flow. The values of A at points near β namely are here multiplied by a greater factor than at points situated more towards the front, which is especially obvious for small values of α. At the points β_1 and β_2 $V_0 = 0$; between these V_0 has a maximum at a point β_m, so that according to (V) A is positive for $\beta < \beta_m$, negative for $\beta > \beta_m$. For $\beta > \beta_m$ in (III) these negative values of A will count more than the positive ones; from a certain value of β they will predominate, so that the *sign of V is inverted*.

When α approaches zero, we have:

$$v \cong -\frac{\alpha}{V_0} w_0 = \frac{\alpha}{\sqrt{\pi k}\, V_0}\int_{\beta_1}^{\beta} d\xi\, \frac{A(\xi)}{\sqrt{\beta-\xi}};$$

so that the point β_S where the current leaves the surface is given by

$$\int_{\beta_1}^{\beta_S} d\xi\, \frac{A(\xi)}{\sqrt{\beta_S - \xi}} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (VI)$$

From this equation we see that β_S is defined by the function $A(\xi)$, viz. by the form of the body; in this equation R does not occur.

For the circular cylinder we have: $V_0^2 = 4 - \frac{\beta^2}{a^2}$; $A = -\frac{2\beta}{a^2}$ and

$$w_0 = -\frac{8}{3\sqrt{\pi k}}\frac{(a-\beta)\sqrt{\beta+2a}}{a^2}.$$

This expression is zero for $\beta_S = a$; i.e. 120° from the most forward point of the circle. This is rather far backward; the experiments and the calculations according to PRANDTL's theory give for this distance somewhat less than 90°. This difference is caused by the calculation of the convection, which keeps here a finite velocity up to the surface so that β_S is slept along too far by the flow.

According to (I) the order of magnitude of the thickness of the boundary layer is of the order of $\alpha =$ of the order of $\sqrt{k\beta} =$ of the order of $\sqrt{\frac{\nu a}{U}} =$ of the order of $\frac{a}{\sqrt{R}}$.

V. Values of w and v for $\beta > \beta_2$.

When we suppose, that (I) and (V) may be used up till $\beta = \beta_2$, the distribution of w for $\beta > \beta_2$ is found by the diffusion into each other of the two distributions that exist in $\beta = \beta_2$ for positive and negative values of α respectively (which are equal and opposite). This gives:

$$w = -\int\limits_{\beta_1}^{\beta_2} d\xi \frac{A(\xi)}{\sqrt{\pi k(\beta-\xi)}} \cdot exp\,\frac{-\alpha^2}{4k(\beta-\xi)} \cdot Erf\,\alpha \sqrt{\frac{\beta_2-\xi}{4k(\beta-\beta_2)(\beta-\xi)}}. \quad (VII)$$

The distribution of the velocity is then found from:

$$v_x = V_0 - \frac{1}{V_0}\int\limits_{\alpha}^{\infty} w\,d\alpha \; . \; . \; . \; . \; . \; . \quad (VIII)$$

where $V_0 U$ is the velocity of the irrotational motion in the direction of the x-axis (i.e. along the line $\alpha = 0$).

With the aid of these formulae the distribution of the vorticity and the flow in the boundary layer have been calculated by graphical integration. In order to obtain everywhere abstract numbers, we have put:

$$\beta = Ba \;\; ; \;\; \xi = Xa \;\; ; \;\; \alpha = 2C\sqrt{ka} \;;$$

Then (I) and (VII) take the form

$$w = -\frac{1}{\sqrt{ka}}\int dX \,.\, \varphi\,(X, B, C) = -\frac{1}{\sqrt{ka}} \,.\, f(B,\, C)$$

where φ and f are numerical quantities.

In fig. 25 $f(C)$ has been represented graphically for several values of B; C has been set out in vertical direction, f each time from the C-line in horizontal direction (viz. parallel to the B-axis). The most outward broken line gives the value $f =$ ca. 0,01. The "opposite" values of f occur for small values of C between $B = 1$ and $B =$ ca. 3.

In fig. 26 these distribution curves are given along the circumference of the cylinder and along the x-axis behind it. Here k/a has been taken equal to 0,000625, i. e. $R = 3200$, though this makes the boundary layer already too thick to allow the approximation. The thick lines indicate the distribution of the velocity for the same values of B. The region of negative velocities extends from $B = 1$ till beyond $B = 5$; the limit has not been calculated, as the determination of the integrals by means of the planimeter has not been made with sufficient accurateness thereto.

Fig. 27 gives a sketch of the vortex region and of the stream lines of the relative flow.

Fig. 28 is a sketch of the dimensions of the vortex region for $R = 80000 = 25 \times 3200$; the values of a (and therefore the thickness of the layer) are then divided by 5.

VI. When we use the complete equation (21) in stead of (22), (I) is replaced by an expression with a Besselianfunction under the sign of integration (see the quoted paper of L. V. King).

Physics. — "*On geodesic precession.*" By Prof. J. A. SCHOUTEN. (Communicated by Prof. H. A. LORENTZ).

(Communicated at the meeting of February 26, 1921).

In a preceeding communication I[1]) have demonstrated geometrically, that a system of axes, moved geodesically along a closed curve in a non-euclidean V_3, will show a deviation when returned to its starting point. For the special case that the linear element of the V_3 is the spacial part of the linear element of SCHWARZSCHILD and that the curve is a circle round the sun with a radius equal to the mean radius of the orbit of the earth, this deviation is 0.013″ after one revolution.

Now if firstly the fourdimensional problem of the motion of a material point in a static gravitational field, neglecting as usual quantities of order $\frac{\alpha^2}{R^2}$, could be reduced to a problem of classical mechanics (mechanics with the fundamental theorem: force = mass $\times$ geodesic acceleration) in a threedimensional non-euclidean space, and if secondly we could demonstrate that a geodesically moving system of axes may be regarded in first approximation as an inertial-system, than we might conclude for the earth to a deviation of the ordinary precession to the amount of 0.013″.

In the mean time FOKKER[2]), starting with the complete linear element of SCHWARZSCHILD, has demonstrated with a fourdimensional calculation, that, apart from other relativity-corrections on the ordinary precession, a geodesic precession exists, that is exactly $1^1/_2 \times 0.013''$.

Now we can show that this difference is caused by the fact, that the fourdimensional problem can be reduced then and only then to a threedimensional one, when the square of the velocity is of order $\frac{\alpha^2}{R^2}$, the square of the real occuring velocity in general being of order $\frac{\alpha}{R}$.

The world-line of a material point is given by the equation:

$$\delta \int ds = 0. \qquad (1)$$

[1]) Proc Kon. Akad., XXI 1918, p. 533—539.
[2]) Proc. Kon. Akad. XXIII, 1921, p. 729.

Now if ds^2 has the form

$$ds^2 = \left(1-\frac{\beta}{r}\right)dt^2 - dl^2 = \left(1-\frac{\beta}{r}\right)dt^2 - \left(1+\frac{\alpha}{r}\right)dr^2 - r^2 \sin^2\theta\, d\varphi^2 - r^2 d\theta^2, \quad (2)$$

than (1) can be replaced by

$$\left.\begin{aligned} 0 &= \delta\int_{t_2}^{t_1} \sqrt{\left\{\left(1-\frac{\beta}{r}\right)-\left(\frac{dl}{dt}\right)^2\right\}}\, dt = \\ &= \delta\int_{t_2}^{t_1}\left\{1-\frac{\beta}{2r}-{}^1/_2\left(\frac{dl}{dt}\right)^2-\frac{1}{8}\frac{\beta^2}{r^2}-\frac{1}{4}\frac{\beta}{r}\left(\frac{dl}{dt}\right)^2-\frac{1}{8}\left(\frac{dl}{dt}\right)^4\right\} dt = \\ &= -\delta\int_{t_2}^{t_1}\left\{\frac{\beta}{2r}+\frac{1}{8}\frac{\beta^2}{r^2}+{}^1/_2\left(\frac{dl}{dt}\right)^2+\frac{1}{4}\frac{\beta}{r}\left(\frac{dl}{dt}\right)^2+\frac{1}{8}\left(\frac{dl}{dt}\right)^4\right\} dt. \end{aligned}\right\} \quad . (3)$$

In this equation the second term and the two last terms only then can be neglected with respect to the other terms, when $\frac{\beta}{r}$ and consequently $\left(\frac{dl}{dt}\right)^2$ is of order $\frac{\alpha^2}{r^2}$. Then the equation changes into

$$\delta\int_{t_2}^{t_1}\left\{\frac{\beta}{2r}+{}^1/_2\left(\frac{dl}{dt}\right)^2\right\} dt = 0 \quad . \; . \; . \; . \; . \; . \; . \quad (4)$$

But this is the equation of classical mechanics in a threedimensional space with the linear element dl and a potential function $U=\frac{\beta}{2r}$. (4) is equivalent to

$$\frac{\partial U}{\partial x^\mu} = g_{\varkappa\mu}\frac{d^2x^\varkappa}{dt^2} + \begin{bmatrix}\lambda\nu\\ \mu\end{bmatrix}\frac{dx^\lambda}{dt}\frac{dx}{dt} \quad . \; . \; . \; . \; . \; . \; . \quad (5)$$

If $\frac{\beta}{2r}$ and consequently $\left(\frac{dl}{dt}\right)^2$ is of order $\frac{\alpha}{r}$, which in particular holds for the linear element of SCHWARZSCHILD, for which $\beta=\alpha$, the reduction to a threedimensional problem is not possible, at least not in this way [1]).

Now we will demonstrate, that in the threedimensional problem

[1]) Hence the equations derived Proc. Kon. Akad. XXI 1918, 1176—1183 on p. 1178—1180 hold only for velocities of order $\frac{\alpha}{r}$.

a geodesically moving system of axes is under certain conditions an inertialsystem. Therefore we firstly write out the equations (4) for the linear element dl.

Since

$$\left.\begin{array}{lll}\begin{bmatrix} rr \\ r \end{bmatrix} = -\tfrac{1}{2}\dfrac{\alpha}{r^2}, & \begin{bmatrix} \varphi\varphi \\ r \end{bmatrix} = -r\sin^2\theta, & \begin{bmatrix} \varphi r \\ \varphi \end{bmatrix} = +r\sin^2\theta, \\ \begin{bmatrix} \varphi\varphi \\ \theta \end{bmatrix} = -r^2\sin\theta\cos\theta, & \begin{bmatrix} \varphi\theta \\ \varphi \end{bmatrix} = +r^2\sin\theta\cos\theta, & \\ \begin{bmatrix} \theta\theta \\ r \end{bmatrix} = -r, & \begin{bmatrix} \theta r \\ \theta \end{bmatrix} = +r, & \end{array}\right\} \quad . \quad (6)$$

the other symbols of CHRISTOFFEL being zero, we have

$$\left.\begin{array}{l} -\tfrac{1}{2}\dfrac{\beta}{r^2} = \left(1+\dfrac{\alpha}{r}\right)\ddot{r} - \tfrac{1}{2}\dfrac{\alpha}{r^2}\dot{r}^2 - r\sin^2\theta\dot{\varphi}^2 - r\dot{\theta}^2 \\ 0 = r^2\sin^2\theta\ddot{\varphi} + 2r\sin^2\theta\dot{r}\dot{\varphi} + 2r^2\sin\theta\cos\theta\dot{\varphi}\dot{\theta} \\ 0 = r^2\ddot{\theta} - r^2\sin\theta\cos\theta\dot{\varphi}^2 + 2r\dot{r}\dot{\theta} \end{array}\right\} \quad . \quad (7)$$

A motion, satisfying these equations is the circular motion:

$$r = R = \text{constant}, \quad \dot{\varphi}^2 = \omega^2_0 = \frac{\beta}{2R^3}, \quad \theta = \frac{\pi}{2} \quad . \quad . \quad . \quad . \quad (8)$$

When we consider only motions, deviating little from this circular one, we can put $\sin\theta = 1$, $\cos\theta = \frac{\pi}{2} - \theta$ and neglect in the first equation $\dot{r}^2$ and $\dot{\theta}^2$, in the second one $\cos\theta\,\dot{\theta}$ and in the third one $\dot{r}\,\dot{\theta}$. Then these equations pass into:

$$\left.\begin{array}{l} -\tfrac{1}{2}\dfrac{\beta}{r^2} = \left(1+\dfrac{\alpha}{r}\right)\ddot{r} - r\dot{\varphi}^2 \\ 0 = r\ddot{\varphi} + 2\dot{r}\dot{\varphi} \\ 0 = \ddot{\theta} - \cos\theta\,\dot{\varphi}^2 \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Now we introduce the variables x, y and z by the equations:

$$\left.\begin{array}{l} r = R + x\left(1 - \dfrac{\alpha}{2R}\right) \\ \varphi = \omega_0 t + \dfrac{y}{R}, \quad \omega_0^2 = \dfrac{\beta}{2R^3} \\ \dfrac{\pi}{2} - \theta = \cos\theta = \dfrac{z}{R} \end{array}\right\} \quad . \quad . \quad . \quad . \quad (10)$$

x, y, z form a rectangular system of axes moving with a velocity

$R\omega_0 t$ along the orbit $r=R$, the axis of x having always the direction of the radius and the axis of y the direction of the motion. Then the equations pass into

$$\left.\begin{aligned}
0 &= \left(1+\frac{a}{2R}\right)\ddot{x} - 2\omega_0\dot{y} - 3\omega_0^2 x\left(1-\frac{a}{2R}\right)\\
0 &= \ddot{y} + 2\omega_0\left(1-\frac{a}{2R}\right)\dot{x}\\
0 &= -\ddot{z} - \omega_0^2 z
\end{aligned}\right\} \quad . \quad . \quad (11)$$

or

$$\left.\begin{aligned}
\ddot{x} &= 2\left(1-\frac{a}{2R}\right)\omega_0\dot{y} + 3\left(1-\frac{a}{R}\right)\omega_0^2 x\\
\ddot{y} &= -2\left(1-\frac{a}{2R}\right)\omega_0\dot{x}\\
\ddot{z} &= -\omega_0^2 z.
\end{aligned}\right\} \quad . \quad . \quad . \quad (12)$$

We further pass to a system of axes x', y', z', which revolves with respect to x, y, z around the axis of z with an angular velocity ω in the sense of y to x, the axis of z' coinciding with the axis of z:

$$\left.\begin{aligned}
x &= x'\cos\omega t + y'\sin\omega t\\
y &= -x'\sin\omega t + y'\cos\omega t\\
z &= z'.
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Then the equations pass into:

$$\left.\begin{aligned}
\ddot{x}' &= -2\left\{\omega-\omega_0\left(1-\frac{a}{2R}\right)\right\}\dot{y}' + 3\omega_0^2\left(1-\frac{a}{R}\right)(x'\cos^2\omega t + y'\sin\omega t\cos\omega t) + \\
&\qquad + \left\{\omega^2 - 2\omega\omega_0\left(1-\frac{a}{2R}\right)\right\}x'\\
\ddot{y}' &= +2\left\{\omega-\omega_0\left(1-\frac{a}{2R}\right)\right\}\dot{z}' + 3\omega_0^2\left(1-\frac{a}{2R}\right)(x'\cos\omega t\sin\omega t + y'\sin^2\omega t) - \\
&\qquad + \left\{\omega^2 - 2\omega\omega_0\left(1-\frac{a}{2R}\right)\right\}y'\\
\ddot{z}' &= -\omega_0^2 z'.
\end{aligned}\right\} (14)$$

Being given a spherical body with centre in the origine of the system x, y, z and so small, that the squares of its dimensions may be neglected. Then, supposing the body with this neglection to be rigid, in the expressions of the moments $\Sigma\, m\,(y'\ddot{z}' - z'\ddot{y}')$ cycl. the terms with x', y', z' will all contain an inertial product or a difference of two equal inertial moments and consequently this terms will vanish. The terms with $\dot{x}'$, $\dot{y}'$ and $\dot{z}'$ then and only then vanish for

every kind of motion with respect to x', y' and z' if ω be choosen in such way that the terms with $\dot{x}'$, $\dot{y}'$ and $\dot{z}'$ vanish in the equations (14), i.e. if:

$$\omega = \omega_0 \left(1 - \frac{\alpha}{2R}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

But in this case x', y', z' is extactly a geodesically moving system of axes as I demonstrated in the publication referred to on the first page. After one revolution this system has turned over $\frac{\pi a}{R}$.

This can of course also be calculated in the fourdimensional way. Starting with the linear element (2) we find a precession passing for $\beta = 0$ (velocity approaching to zero) in the above calculated value $\frac{\pi a}{R}$ and for $\beta = a$ in $1^1/_2 \times$ this value.

It is worth observing the ordinary precession gets possibly also another value in relativistic mechanics than in classical mechanics, a possibility pointed out by DE SITTER. By means of the equations given by FOKKER it will be possible to calculate the deviation caused by this, at least so far as it is not influenced by forces caused by the mutual attractions of the parts of the planet.

Physics. — *Mutual Influence of Neighbouring Fraunhofer Lines"* [1]). By Prof. W. H. JULIUS.

(Communicated at the meeting of January 29, 1921.)

If the hypothesis holds good that the darkness of Fraunhofer lines is *not* a pure absorption effect — as it is commonly supposed to be — but chiefly due to anomalous dispersion (showing itself both in molecular diffusion and irregular ray-curving), we may expect on theoretical grounds [2]) that neighbouring Fraunhofer lines will, as a rule, seem to repel each other. If, now, such a mutual influence is actually found to exist, a mighty support will thus be given to the said interpretation of the solar spectrum, as long as it remains impossible to explain that phenomenon on the basis of the current view that one is dealing with mere absorption lines.

In a communication on "The general relativity theory and the solar spectrum" [3]) we have made use of the already reliable and striking results obtained in a preliminary research on the manifestations of mutual influence of Fraunhofer lines as appearing in the limb-centre displacements measured by ADAMS [4]) about the year 1910. At my request Dr. P. H. VAN CITTERT and Dr. M. MINNAERT have, however, once more examined the same data with the utmost care, using still more rigorously defined criteria, in order that every trace of bias might be avoided in selecting the lines. Besides, the investigation has been extended over the observation son limb-centre displacements published by EVERSHED, NARAYANA AYYER and ROYDS [5]) in 1914—1916. It will appear that this extension of the field has led to a considerable corroboration of the former inferences, so as to put the existence of mutual influence practically beyond doubt.

Care has been taken, of course, that during the act of selecting lines that would probably be influenced, one was ignorant of the observed displacements. Basing ourselves on the conception how,

[1]) This paper is an abstract of an ampler article that has since appeared in the Astrophysical Journal **54**, 92 (1921). (Note, added January 1922).

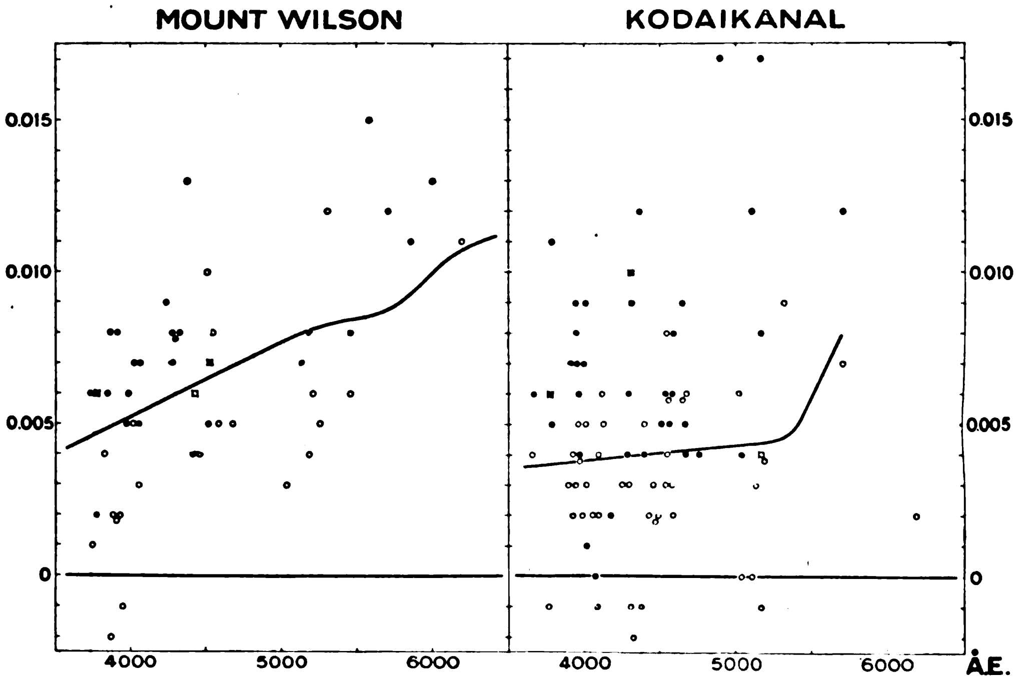
[2]) Cf. Astroph. Journ. **43**, 49—53 (1916).

[3]) W. H. JULIUS and P. H. VAN CITTERT. These Proc. **23**, 522 (1920).

[4]) W. S. ADAMS, Astroph. Journal **31**, 30 (1910); Mt. Wilson Contrib. No. 43.

[5]) EVERSHED and ROYDS, Kodaik. Bull. **39** (1914); NARAYANA AYYER, Kodaik. Bull. **44** (1914); ROYDS, Kodaik. Bull. **53** (1916).

according to the dispersion theory, the asymmetry of the Fraunhofer lines originates, we drew up a scheme of criteria which a line and its surroundings would have to answer for giving reason to expect either an increase or a decrease of its displacement towards the red, owing to neighbour lines. In the tables containing the selected lines (which will be printed in the Astrophysical Journal) we have mentioned the criteria and considerations applied with each line, as well as the direction of the expected effect. Next to these data will be found the actually observed displacements, under one of the headings σ_v or σ_r according as a repulsion from the violet or from the red side (i.e. an increased or a decreased amount of displacement towards the red) was expected.



In the diagram abscissae represent wave-lengths, ordinates displacements, positive when towards the red. The values of σ_v are characterized by full dots, the values of σ_r by circlets.

We have kept the Mount Wilson data separated from te Kodaikanal data in order to ascertain whether these two mutually independent series of observations would yield similar results.

The outcome is very convincing: in both cases the full dots average decidedly higher than the circlets, which means that the displacement towards the red is generally greater for lines with companion on the violet side than for lines with companion on the red side.

We need not wonder at finding some full dots yet to lie low and

a few circlets high. Indeed, taking any dispersion line for itself, the amount of its displacement towards the red will be determined by the shape of the dispersion curve; this, however, does not depend exclusively on the positions and intensities of the *nearest* companion lines, but also on the value n_0 which the index of refraction of the medium would have for light of the small spectral region under consideration, if this where free from lines. Since n_0 varies along the spectrum, there will be a corresponding fluctuation of the values of the displacements, on which the influence of close neighbouring lines superposes itself. Thus we conceive that the two swarms of black dots and circlets must partially penetrate each other.

Let us next consider numerically, to which extent the displacement of a line towards the red is modified, on the average, by the presence of close companions.

For that purpose we make use of the two curves shown in the diagram, which represent the general increase of the displacement with wave-length. They are derived for Mount Wilson [1]) from 450, for Kodaikanal from 392 measured displacements. (The second curve lies sensibly lower than the first; this may be due to the accidental fact that in the Kodaikanal material a greater number of very weak lines, showing small displacements, have been included). These curves define for every region in the spectrum an average or normal displacement δ_m. Now we have calculated for each influenced line the value of the expression

$$\frac{\delta_v - \delta_m}{\delta_m} = D_v \quad \text{or} \quad \frac{\delta_r - \delta_m}{\delta_m} = D_r$$

which may be denominated "relative departure".

With lines having a companion on the violet side, these departures are for the greater part positive, with lines having a companion on the red side they are mainly negative, so that the first group gives a positive, the second group a negative "sum of departures".

From the Mount Wilson data we derive (after applying a correction explained in the original paper):

$$\Sigma D_v = +\ 7{,}09 \text{ (25 lines)} \quad \text{and} \quad \Sigma D_r = -\ 7{,}09 \text{ (23 lines)} \quad . \quad (1)$$

and from the Kodaikanal data

$$\Sigma D_v = +\ 19{,}15 \text{ (36 lines)} \quad \text{and} \quad \Sigma D_r = -\ 19{,}16 \text{ (44 lines)} \quad . \ (2)$$

Hence the *mean* value of a relative departure (positive or negative) is:

$$\overline{D} = \frac{7{,}09 + 7{,}09 + 19{,}15 + 19{,}16}{128} = 0{,}410,$$

[1]) Cf. Julius and van Cittert, l.c.

which indicates that the limb—centre displacement of a line is augmented or diminished by as much as $^{2}/_{5}$ (on an average) of its normal amount if another line is near.

This comparatively great influence can only be explained as arising from the modification which the neighbour line brings about in the refracting power of the medium. The phenomenon thus proves the efficiency of anomalous dispersion in the sun; it strongly suggests that limb—centre shifts in general may be chiefly conditioned by the shape of the dispersion curve of the gaseous mixture; and this inference again vindicates our hypothesis that the distribution of the light in Fraunhofer lines is dominated by anomalous dispersion.

Taking for granted the validity of this interpretation of the solar spectrum we should expect, moreover, that in the spectrum of the centre of the sun's disk the Fraunhofer lines will also be generally displaced towards the red with respect to the positions of their *cores* (which are determined by the values of the free periods *on the sun*), and that these shifts will be comparable in magnitude with the limb—centre displacements.

There is strong reason, therefore, to ascribe the observed centre-arc displacements perhaps wholly, but at any rate for a considerable part, to anomalous dispersion — an explanation, indeed, confirmed by the fact that the principal characteristics of these displacements are very similar to those of the limb—centre shifts [1].

If we now imagine the observed centre—arc displacements to be reduced by substracting from them the purely solar displacements of the centre lines with respect to their cores (as mentioned above), the remaining shifts — if any — will be so small, that the existence of a *gravitational* displacement of the solar line-cores with respect to the terrestrial arc lines (expected to amount to from 0,008 Å to 0,014 Å in the visible spectrum) appears highly improbable.

Let us finally try to express numerically with how much confidence we may assert that the observations really indicate the existence of a mutual influence of Fraunhofer lines.

One might indeed suggest it to be an effect of mere chance that the black dots in our diagram average so much higher than the circlets. The probability of that supposed casual event can be calculated according to the rules of the theory of errors.

From the equations (1), relating to the Mount Wilson measurements, it follows that a line having a companion on the violet side shows a mean relative departure

[1]) Cf. JULIUS and VAN CITTERT. These Proceedings 23, 530 (1921).

$$\overline{D}_v = \frac{+\ 7{,}09}{25} = +\ 0{,}284,$$

and a line having a companion on the red side

$$\overline{D}_r = \frac{-\ 7{,}09}{23} = -\ 0{,}308.$$

Supposing, on the other hand, that out of the 48 cases we had chosen 25 cases without any guiding principle, entirely at random, then the probable departure of the mean of those 25 cases would have been $r_0 = \frac{r}{\sqrt{25}} = 0{,}066$ (in which r, the probable departure for a single line, depends on the "precision" of the entire group, and proved to be $= 0{,}329$).

The mean relative departure $\overline{D}_v$ actually found is, therefore, $\frac{0{,}284}{0{,}066} = 4{,}30$ times as great as the "probable" departure r_0 would have been in case of random choice.

Conformably we find for a line with companion on the red side (r'_0 being $= 0{,}068$).

$$\overline{D}_r = -\ 0{,}308 = -\ 4{,}50\ r'_0.$$

The probability that a mean departure $\overline{D}$, derived from cases selected *without* guiding principle, would be included between $+\ 4{,}30 r_0$ and $-4{,}50 r'_0$, or, what is very nearly the same, between $+\ 4{,}40 r''_0$ and $-4{,}40 r''_0$ (putting $r''_0 = 0{,}067$) amounts to [1])

$$\frac{2}{\sqrt{\pi}} \int_0^{4{,}4\rho} e^{-t^2}\, dt = 0{,}997,$$

so that only 0,003 in left for the probability that, by mere chance, $\overline{D}$ would lie beyond those limits.

Applying the same argument to the equations (2) derived from the Kodaikanal measurements, which include a greater number of influenced lines, we find an even much smaller value for the probability that the observed considerable separation of the two swarms of dots and circlets would be purely accidental, namely 0,00001.

Since these latter data have been obtained independently of the Mount Wilson measurements, we may value the probability of the concurrence of those two casualities at $0{,}003 \times 0{,}00001$.

It has been established, therefore, with a probability of more than 10^7 to 1 that the guiding principle used in selecting the lines is

---

[1]) Cf. CHAUVENET, Spherical and practical astronomy, Vol. II, Table IX A.

physically significant, and that neighbouring Fraunhofer lines really seem to repel each other.

We have of course also considered the idea, that this phenomenon, although undoubtedly involved in the observational results, might be caused by systematic errors in the methods, applied in making, or judging, or measuring the photographs.

The chance, however, for such errors to have appreciably affected the result, is very small especially in the case of limb-centre displacements, because these displacements are usually derived from a comparison between photographs, the density and general appearance of which have been chosen as similar as possible. If, therefore, in estimating the distance between the members of a close pair of lines, a systematic error is made owing to their proximity, that error will be very nearly the same in the limb spectrum as in the centre spectrum, and will thus be eliminated in the limb-centre differences.

As the distance between neighbouring lines is nevertheless found to be greater in the limb spectrum than in the centre spectrum, it is safe to say that influencing each other is a true property of Fraunhofer lines. This property seems only explainable from the point of view of the dispersion theory.

The skilful collaboration of Dr. VAN CITTERT and Dr. MINNAERT in this research is highly appreciated.

Utrecht, January 1921. *Heliophysical Observatory.*

Anatomy. — "*On the Homology of the M. marsupialis and the M. pyramidalis in Mammals*". By D. VAN VUGT. (Communicated by Prof. L. BOLK).

(Communicated at the meeting of January 29, 1921).

The name of M. pyramidalis designates in the Literature of Anatomy a muscle found in the lower part of the anterior abdominal wall of various classes of mammals. In the following classes of mammals a M. pyramidalis has never been met with: Edentata, Glires, Galeopithecidae, Ungulata, Sirenia and Cetaceae; while there may be one in Monotremata, Marsupialia, Insectivora, Chiroptera and Primates. A M. pyramidalis has also been described in Hyrax and in Hyena [1]. Furthermore ELLENBERGER and BAUM [2] describe a M. praeputialis in the dog, which they look upon as the thoracic portion of the M. pyramidalis.

This muscle being always designated by the same name, it is evident that it is considered homologous in all the classes mentioned. This, in fact, has been emphatically acknowledged by some writers. EISLER [3] e.g. says: "Der M. pyramidalis ist ein typischer und groszer Muskel bei den Säugern, die einen Beutelknochen besitzen". In BRONN's Klassen und Ordnungen [4] we read: "Dieser zwischen der ventralen Wand der Rectusscheide und dem Rectus gelegene Muskel ist als der Muskel des Beutelknochens anzusehen und ist deshalb bei den aplacentalen Säugern am stärksten entwickelt". WIEDERSHEIM maintains likewise: "Der Pyramidalis ist der eigentliche Muskel des Beutelknochens" [5] and "Bei den aplacentalen Säugethieren d.h. bei Monotremen und Beutelthieren ist der M. pyramidalis im Anschlusz an die Beutelknochen gewaltig entwickelt" [6]. Of the same opinion are, amongst others, also GEGENBAUR, BARDELEBEN, RAUBER and TESTUT. Only LOTH [7] records that CHUDZINSKY holds another opinion, without mentioning, however, what it actually is.

[1] BRONN's Klassen und Ordnungen. Mammalia II. pg. 788 sqq.

[2] ELLENBERGER und BAUM. Anatomie des Hundes, p. 167.

[3] EISLER. Die Muskeln des Stammes, p. 585.

[4] BRONN l.c.

[5] WIEDERSHEIM. Vergleichende Anatomie der Wirbeltiere, p. 244.

[6] " . Der Bau der Menschen als Zeugnis für seine Vergangenheit.

[7] LOTH. Muskelsystem des Negers. p. 98. Studien und Forschungen zur Menschen- und Völkerkunde IX 1912.

Be this as it may, no evidence has as yet been brought forward for the homology of this muscle in placental and aplacental mammalia, i.e. of M. pyramidalis and M. marsupialis: the mere fact that the M. marsupialis of the aplacental mammalia occupies a place similar to that of the M. pyramidalis of the other mammalia, and that both are enclosed in the rectus-sheath, can hardly be deemed sufficient evidence for the homology.

In this paper we publish the results of an inquiry into the validity of this homology-hypothesis.

The M. marsupialis generally arises from the medial border of the os marsupiale, the lower fibers of the muscle running more or less transversally, while those situated more proximally slope upward towards the linea alba. This muscle lies before the M. rectus, being separated from it only by a thin layer of loose connective tissue. However, it seems that the M. marsupialis does not always lie before the M. rectus, as W. VROLIK [1]) discovered that in Dendrolagus inustus the M. pyramidalis was covered by the right abdominal muscle.

The other abdominal muscles of the marsupials run as follows: The M. obliquus externus always passes along in front of the M. rectus and in front of the M. marsupialis. The course of the M. obliquus internus, however, varies just as that of the M. transversus. A general view of it is given in the diagrams in Fig. 1, which

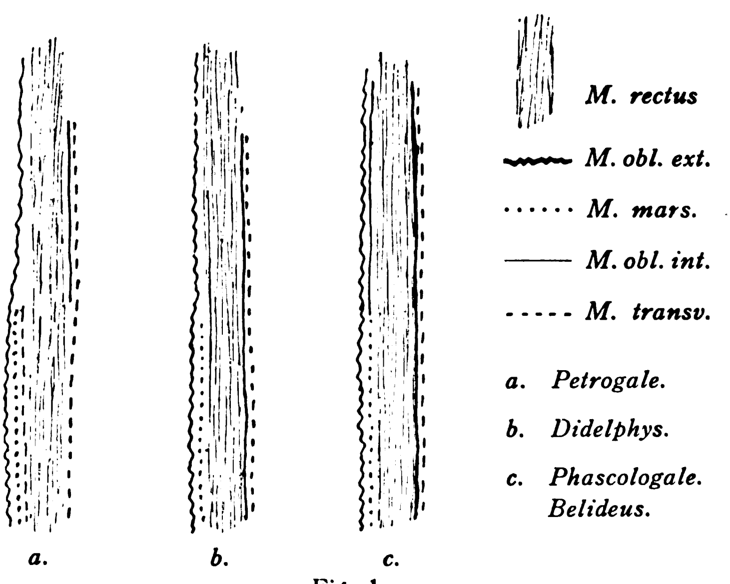


Fig. 1.

[1]) W. VROLIK. Ontleedkundige nasporingen omtrent Dendrolagus inustus. Kon. Akad. v. Wet. A'dam Afd. Nat. Deel V.

show the succession of the various muscles in a longitudinal section of the inferior part of the abdomen.

With Phascologale penicillata (fig. 1*c*) the following relation exists: The inferior portion of the M. obliquus internus inserts itself into the lateral border of the Os marsupiale; the superior portion, unable to attach itself to the Os marsupiale, extends along the upper border of the M. marsupialis and is inserted into the linea alba. Moreover a leaf is split from the aponeurosis, which passes with the aponeurosis of the M. transversus behind the M. rectus. With Belideus ariel the course of the M. obliquus internus and M. transversus was the same as with Phascologale. Here also the M. marsupialis lies in front of the M. rectus, the muscle-fibers, however, follow a more oblique, proximal course than in the case of Phascologale.

Between M. marsupialis and M. rectus of Petrogale penicillata (fig. 1*a*) runs the anterior leaf of the aponeurosis of the M. transversus; this splitting into an anterior and a posterior leaf takes place only in the inferior portion of the M. transversus. More towards the cranium the M. transversus continues unsplit behind the M. rectus. The M. obliquus internus of this animal does not continue between M. marsupialis and aponeurosis of the M. transversus, but follows the latter muscle behind the M. rectus. In Petrogale xanthopus PARSONS [1]) found nearly the same condition: "The internal oblique is inserted into the last three ribs, dorsal to the lateral line of the body it is fleshy, while ventrally it becomes aponeurotic and blends with the transversalis. The transversalis .... passes in the anterior two thirds of the abdomen deep to the rectus, in the posterior third it splits to enclose that muscle".

On the M. obliquus internus and M. transversus of Didelphys virginiana ELLIOTT COUES [2]) writes: "The lower border (of the internal oblique) is fleshy and stretches nearly horizontally inward from Poupart's ligament to the upper part of the marsupial bone, a stout bundle of fibres being inserted into the tip of that bone. The rest of the muscle passes more and more directly upward, till its posterior part is vertical. Its anterior margin ends along a linea semilunaris by blending the aponeurosis with that of the transversalis". Of the latter muscle COUES says: "There is no splitting of the aponeurosis to get outside the rectus below" (fig. 1*b*).

---

1) PARSONS. On the anatomy of Petrogale xanthopus. Proceedings of the Zoological Society of London. June 16, 1896.

2) ELLIOTT COUES. The osteology and myology of Didelphys Virginiana. Memoirs of the Boston Society of Natural History, Vol. II. Part. I 1872.

GEGENBAUR[1]) writes of the M. pyramidalis — also of that of the Monotremata and of the Marsupials: — "allein seine Lage nicht nur, sondern vielmehr sein Anschlusz an den Rectus so wie der Einschlusz in eine mit dem Rectus gemeinsame Scheide macht seine Entstehung aus dem Rectus wahrscheinlich und verweist auf die Thatsache, dasz bereits bei Amphibien mehrfache Rectus-bildungen vorkommen, von welchen die oberflächliche der Metamerie entbehrt, gleich dem Pyramidalis der Säugethiere, welcher auch nicht mit Unrecht als vorderer Rectus unterscheiden ward". Also in ELLIOTT COUES's inquiry, quoted above, we find a differentiation between a M. rectus externus and internus. This view is not plausible as regards the M. marsupialis. As appears from the diagram of the abdominal wall of Petrogale (fig. 1*a*) a leaf of the aponeurosis of the M. transversus passes between M. rectus and M. marsupialis. This renders it highly improbable that the M. marsupialis should arise from the M. rectus. The diagrams of Phascologale and Belideus (fig. 1*c*) also show that the M. marsupialis is not exactly invested by the rectus-sheath — which according to GEGENBAUR would speak for its arising from the M. rectus — but that it rather constitutes a part of the sheath itself. The M. obliquus internus does not play a part in the formation of the frontal leaf of the vagina musc. recti, owing to its insertion into the lateral border of the Os marsupiale; now it is just the M. marsupialis which completes that same part of the rectus-sheath. Again, the fibres of the inferior part of the M. marsupialis run at a right angle to the M. rectus which, though this is not a cogent proof of the reverse, renders it by no means plausible they should arise from the M. rectus.

The fibre-course of the M. marsupialis corresponds much more to that of the M. obliquus internus. The fact that the M. marsupialis completes, as it were, the M. obliquus internus, and the location of this muscle between the M. obliquus externus and M. transversus — which appears above all in Petrogale (fig. 1*a*) — renders it more probable that the M. marsupialis is a part of the M. obliquus internus. This conception helps us to realise the condition described by VROLIK that the M. rectus overlies the M. pyramidalis of Dendrolagus inustus, such being not at all extraordinary for a part of the M. obliquus internus[2]).

This has been clearly demonstrated in a Didelphys marsupialis (fig. 2).

[1]) GEGENBAUER. Vergleichende Anatomie, I. 1898 p. 664.

[2]) For the course of the abdominal muscles relative to the M. rectus, we refer to W. A. MIJSBERG: "Over den bouw van den musculeuzen buikwand der Primaten." Kon. Akad. v. Wet. A'dam 14 Mei 1915.

The lower portion of the M. obliquus internus is inserted into the lateral border of the Os marsupiale; the lower portion of the M. marsupiale arises from the medial border of this bone. The aponeurosis of the middle part of the M. obliquus internus does not continue as far as the linea alba, but blends with the upper portion of the M. marsupialis, so that this part of the aponeurosis constitutes an intermediate tendon between the two muscles. The upper part of the M. obliquus internus is inserted into the linea alba and the lower ribs. Here, then, there is a close relation between the two muscles, so that it might be called a Pars marsupialis musculi obliqui interni. But in most cases the relation between the two muscles is less distinct. In the description VROLIK gives us of Dendrolagus inustus, alluded to above, we read that "Zij (de buidelspier) hecht zich aan de witte lijn en slaat zich achter het marsupiaalbeen om ten einde zich te vereenigen met de peesplaat van M. obliquus internus en transversus." True, this description falls short of clearness, but it is not impossible that in Dendralogus a condition occurred such as I observed in Didelphys marsupialis.

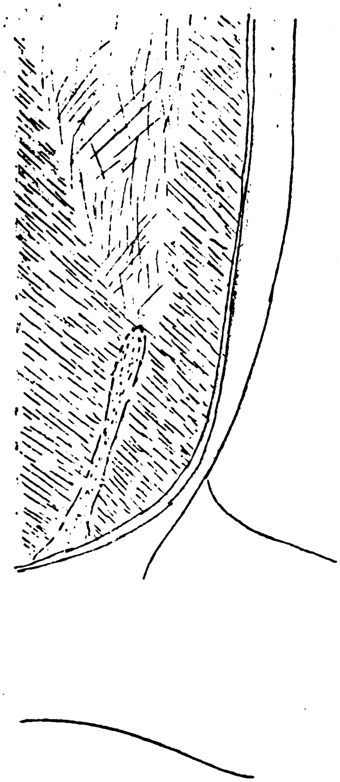


Fig. 2.

Various opinions are prevalent concerning the Ossa marsupialia. FLOWER [1]) considers them as "Verknöcherungen der inneren Sehne der äusseren schrägen Bauchmuskel selbst, oder doch innig mit ihr verbunden, und sie fallen daher unter der Kategorie der Sesam beine". KATZ [2]) speaks of "Ossificationen in einer hintern sehnig gedachten Partie des M. Pyramidalis". Since WIEDERSHEIM [3]) undertook his extensive inquiries into the pelvic girdle, it is generally received that the Oss marsupiale is a strongly developed Epipubis. In the matter of homology of M. pyramidalis and M. marsupialis, the significance of the Os marsupiale is immaterial.

[1]) FLOWER. Einleitung in der Osteologie der Säugethiere, 1888 p. 298.

[2]) KATZ. Zur Kenntniss der Bauchdecke und der mit ihr verknüpften Organe bei den Beutelthieren. Zeitschrift für wiss. Zoöl. Bd. 36. 1882.

[3]) WIEDERSHEIM. Die Phylogenie der Beutelknochen. Zeitschrift für wiss. Zoöl. Bd. 53—1882. (Suppl.).

Now concerning the M. pyramidalis the following facts may be pointed out.

The M. pyramidalis in man is a small triangular muscle in the anterior abdominal wall, arising from the pubic crest in front of the rectus muscle. It is directed obliquely upwards, to be inserted for a variable distance into the linea alba. Some superficial fibers are also inserted into the posterior side of the ventral leaf of the rectus-sheath [1]). This muscle is lodged right in front of the M. rectus and is separated from it only by a thin layer of loose connective tissue. Yet the connective tissue between the two muscles sometimes seems to become a solid membrane, for we read in EISLER's [2]) work: "In der Regel findet sich zwischen Pyramidalis und Rectus nur eine dünne Schicht lockeren Bindegewebes, doch schiebt sich gelegentlich von latera her ein aponeurolisches von der ventralen Rectus scheidewand im besonderen von der Aponeurose des M. transversus abdominis abgespaltenes Blatt zwischen beide, ohne aber eine vollständige Abschlieszung des Pyramidalis herzustellen".

KRAUSE [3]) maintains even that the latter condition is the rule. Since, however, the obturation (Abschlieszung) as EISLER remarks, is never complete, and occurs only exceptionally, anyhow is not constant by far, — witness the different opinions prevailing in this respect — it does not seem probable that the obturation is effected by a true aponeurosis.

The M. pyramidalis is lacking bilaterally in ± 16 °/₀ of the Europeans, in half the other cases it occurs only unilaterally. Furthermore the M. pyramidalis is absent in approximately 10 °/₀ of the Negroes and 4 °/₀ of the Japanese.

The following remarks are still given by SANTORINI, CRUVEILHIER, QUAIN and GEGENBAUR. In the absence of the M. pyramidalis the caudal end of the M. rectus is broader and stronger; conversely, MACALISTER [4]) reports that the insertion of the M. rectus is narrow when the M. pyramidalis is strongly developed. We shall see that these relations are of some consequence.

In man the aponeurosis of the M. obliquus internus continues as far as the linea alba, without any junction between this aponeurosis and the M. pyramidalis, as in the case of Didelphys. The direction of the fibres of M. pyramidalis and of M. obliquus internus differs rather much; moreover the aponeurosis of the M. transversus is

1) EISLER. Die Muskeln des Stammes. p. 572.
2) EISLER l.c.
3) KRAUSE. Handbuch der menschlichen Anatomie, 1879 Bd II, p. 242.
4) EISLER l.c. p. 574.

lying between the two muscles. For these several reasons we cannot consider also the M. pyramidalis as a portion of the M. obliquus internus, as the M. marsupialis is. From this it follows again that the M. pyramidalis and the M. marsupialis cannot be homologous.

What then is the significance of the M pyramidalis of the Monodelphys? An answer to this question will be found in the following record of an investigation into the mode of insertion of the M rectus abdominis in some mammalia. In Tarsius spectrum the Os pubis presents only a small place for insertion, which is narrower than the breadth of the muscle. The insertion is nevertheless effected over the whole breadth of the muscle because the inferior part of the M. rectus bends round anteriorly and laterally at the linea alba, so that a small triangular portion of the muscle is disposed in front of the larger unbent portion of the M rectus. This portion bears a close resemblance to a M. pyramidalis; however, it has not an insertion of its own into the linea alba, but at the linea alba its fibres merge directly into those of the M. rectus. We see then that in Tarsius the M. rectus has a U-shaped insertion. However in case, through some cause or other, such a triangular piece of muscle should become independent and obtain an insertion of its own into the linea alba, a M. pyramidalis is originated. In Insectivores such a case is encountered in the crossing of the two Mm recti, as described by Leche [1]). In the embryo the crossing commences in the most caudal part. The Mm recti draw near to the middle line, the right M. rectus splits up in two in order to allow the left M. rectus to pass (at Talpa europea) or otherwise both Mm. recti split and in this way a more complicate network is formed. The right M. rectus then inserts itself into the left Os pubis and the left M. rectus into the right Os pubis. As the embryo develops, the process continues proximally, until the full-grown condition is reached. In an investigation of embryos of Talpa I detected likewise a bending of the medial border of the M. rectus anteriorly and laterally; now when the unbent parts of the rectus cross, the triangular anterior layer must of necessity be disconnected from the rest of the M. rectus and gets an insertion of its own, in other words it becomes a M. pyramendalis. The impossibility of a connection between bent and unbent pieces of the M. rectus is dictated here by the circumstance that the medial border of the M. rectus, along which the bending took place anteriorly and laterally, is shifted, after the crossing, towards the opposite side. In embryos, in which the

[1]) Leche. Zur Anatomie der Beckenregion der Insectivera. Kon. Svenska Vet. Akad. Handl. Bd 20, N°. 4, 1883.

crossing of the Mm. recti began but was not entirely accomplished, the bending of the M. rectus is discernible at a certain level, while at a lower level, where the crossing had already been partly accomplished, the connection between the two pieces of the M. rectus has been abolished.

How it is that in the Simiae and in man a bent piece of the M. rectus is liberated as a M. pyramidalis, is not so easy to understand as it is in Insectivora. But, also in other regions of the human system it occurs that the portion of a muscle with U-shaped insertion, the insertion being one of the arms of the U, becomes independent. This e.g. may be the case with the pars abdominalis of the M. pectoralis major.

With this exposition of the origin of the M. pyramidalis in man several facts are in perfect harmony. First of all the fact that SANTORINI and other authors have established that the insertion of the M. rectus into the Os pubis is narrow, when the M. pyramidalis has a broad origin in this skeletal bone and conversely. In agreement with this is also the fact that some fibres of the M. rectus are regularly inserted into the inferior part of the linea alba, as is described by NICAISE [1]), as the rectus must of necessity obtain an insertion, along the previous line of flexion, into the linea alba, when the same happens with the M. pyramidalis.

Because the function of the M. pyramidalis is very inconsiderable this muscle disappears in many cases either on one side or on either side, the only indication of its earlier existence and mode of origin then being the insertion of fibres of the M. rectus into the inferior part of the linea alba.

The question may be asked why in some animals the insertion of the M. rectus is U-shaped. In this respect there may be relationship between the breadth of the M. rectus on the one side and on the other side the dimensions, the form and the size of the pelvis i.e. the space for insertion. This, however, requires further investigation. We only wish te observe that among the half-apes the M. rectus has a simple, recti linear insertion in the Lemurinae and that in the other half-apes the M. rectus presents a more complicated mode of insertion. This, no doubt, has something to do with the fact recorded by WEBER [2]) that the Lemurinae have a wide pelvis: „Gegenüber dem weiten Becken der Lemurinae, haben die nicht-madagassichen Prosimiae ein enges Becken."

[1]) EISLER. Die Muskeln des Stammes, p. 565

[2]) M. WEBER. „Die Säugetiere". p. 747.

Physics. — "*On the deviations of liquid oxygen from the law of* CURIE". By Prof. W. H. KEESOM. (Communication N°. 8 from the Laboratory of Physics and Physical Chemistry of the Veterinary College at Utrecht). (Communicated by Prof. H. KAMERLINGH ONNES).

(Communicated at the meeting of February 26, 1921).

§ 1. *Introduction.* It will be well-known that the magnetic susceptibility of gaseous oxygen [1]) follows the law of CURIE

$$\chi T = C \quad \dots \dots \dots \quad (1)$$

but that for liquid oxygen [2]) we have the relation

$$\chi (T + \Delta) = C \quad \dots \dots \dots \quad (2)$$

where Δ is a constant viz. $\Delta = 71$. An explanation of this phenomenon has been sought in different ways.

We may for this liquid base our considerations on the validity of the fundamental idea of LANGEVIN's theory. Then the deviation from the law of CURIE might be caused by: —

1. a change of the number of elementary magnets *e. g.* by polymerisation of the oxygen molecules (KAMERLINGH ONNES and PERRIER, Leiden Comm. N°. 116). More recent experiments having shown that similar deviations also occur in the case of solids, for which such a polymerisation, changing with the temperature, can hardly be assumed, we should prefer to find another explanation for oxygen also.

2. a decrease of the magnetic moment of the oxygen molecules or atoms, either by a slower circulation at lower temperatures of the electrons which give rise to the magnetic moment or, in the case of liquid oxygen, by the influence of the thermic pressure (KAMERLINGH ONNES and PERRIER, Leiden Comm. N°. 116). More recent experiments on the susceptibility of gaseous oxygen down to 147° K. on the one hand (KAMERLINGH ONNES and OOSTERHUIS l. c.) and of

[1]) P. CURIE. Ann. chim. phys. (7) 5 (1895), p. 289. H. KAMERLINGH ONNES and E. OOSTERHUIS. Leiden Comm. No. 134*d*.

[2]) H. KAMERLINGH ONNES and A. PERRIER. Leiden Comm. No. 116. These Proceedings Vol. XII 1910, p. 799. H. KAMERLINGH ONNES and E. OOSTERHUIS. Leiden Comm. No. 132*e*. These Proceedings Vol. XV 1913, p. 965. For liquid mixtures of oxygen and nitrogen comp. A. PERRIER and H. KAMERLINGH ONNES. Leiden Comm. No. 139*d*. These Proceedings Vol. XVI 1914, p. 901.

liquid mixtures of oxygen and nitrogen (KAMERLINGH ONNES and PERRIER l. c.) on the other are not in favour of these explanations.

3. a change of the field of force, in which the separate oxygen molecules are to be found, by an oppositely acting negative molecular field (KAMERLINGH ONNES and PERRIER, Leiden Comm. N°. 139*d*) [1]).

4. a change in the heat motion. We might for instance suppose that this motion no longer follows the laws of equipartition, but that, in agreement with the quantum-theory with assumption of a zeropoint-energy, at low temperatures it neutralizes the magnetisation to a higher degree than would be the case if the equipartition laws were still valid. (KEESOM, Leiden Comm. Suppl. N°. 36*c*) [2]).

Without further discussing the arguments for or against the two latter ways of explanation, we shall in this paper only treat a question put by Prof. KAMERLINGH ONNES on occasion of discussions held at Leiden on magnetic problems. This question may be formulated in the following way. Might it not be possible that the occurrence of Δ in (2) was caused by what is left, when we take the statistic mean, from a directing action of the couples of force, exerted by the oxygen molecules on each other when, in a magnetic field, the molecules come very near to each other? In our considerations we shall assume that with sufficient approximation the forces exerted by the oxygen molecules on each other may be treated as forces exerted by electric quadrupoles, as has been proved to be

1) As to R. GANS, Ann. d. Phys. (4) 50, p. 163, 1916, comp. p. 1134, note 1.

2) This hypothesis has first been tried by OOSTERHUIS, Leiden Comm. Suppl. No. 31, These Proceedings Vol. XVI (1913), p. 432, for paramagnetic salts, afterwards by KEESOM, Leiden Comm. Suppl. No. 32, These Proceedings October 1913, p. 454 and 468, for ferromagnetic substances. More recently v. WEIJSSENHOFF (Ann. d. Phys. (4) 49, p. 149, 1916) and especially REICHE (Ann. d. Phys. (4) 54, p. 401, 1917) have worked out the hypothesis by the methods of the quantum theory which had then been developed much further. They found a good agreement with the observations on the susceptibility of paramagnetic salts, comp. also A. SMEKAL, Ann. d. Phys. (4), 57, p. 376, 1918. LANGEVIN too (Procès-Verbaux et Résumé des Communications de la Soc. franç. de physique 1919, p. 18) adheres to this way of explaining the deviations from the law of CURIE. Especially from the results of the investigations on crystal structure with the aid of Röntgen rays we can hardly think any longer that in solids molecules rotate, in the heat motion, as a whole like elementary magnets in the sense of the "magnetic molecule", a definition of which, in accordance with the ideas of WEISS, has been given in Leiden Suppl. No. 32*a*, p. 11, note 3. Thus we shall have to consider parts of the molecule, eventually atoms or parts of atoms, comp. O. STERN, Zs. f. Phys. 1, p. 147, 1920. Comp. also W. LENZ, Physik. Zs. p. 613, 1920, P. EHRENFEST, Leiden Comm. Suppl. No. 44*b*, These Proceedings Vol. XXIII (1921), p. 989.

permissible for the explanation of the molecular attraction in the equation of state. In this paper it will be shown that very probably the answer to the above question must be in the negative.

§ 2. *Introductory considerations.*

When in the gaseous state two oxygen molecules come so near to each other that there arise mutual forces between them, these forces will generally form a couple, so that the molecules have a directing influence on each other. When we take the statistical mean and when we treat the electric field of the molecule to the first approximation as that of a quadrupole, it is due to this directing action that there remains an attraction between the molecules, which becomes manifest in the equation of state as the attraction term introduced by VAN DER WAALS. [1])

When we desire to investigate what influence this directing action will have on the susceptibility, we must take into consideration the relative position of the electric quadrupolar axis and the magnetie dipolar axis in the oxygen molecules. In doing this we shall assume these two axes to have a fixed position in the molecule.

Then we may distinguish the following three cases:

a. the magnetic dipolar axis coincides with the quadrupolar axis; *b.* the magnetic dipolar axis is perpendicular to the quadrupolar axis and *c.* they form an arbitrary angle with each other.

In this paper only the two extreme cases viz. *a* and *b* will be treated.

First we shall consider case *a* viz. where magnetic axis and quadrupolar axis coincide. This assumption would seem the more preferable, for all our considerations in this paper will be based on the validity of the laws of equipartition; and in that case only assumption *a* gives the right value for the specific heat in the gaseous state.

First we shall investigate the influence on the susceptibility of the mutual directing action, which the molecules also exert on each other because of their quadrupolar forces, when the gas is placed in a magnetic field. We may then imagine, that by these attracting actions all molecules would be united to form double molecules as has been represented in fig. 1 viz. that one of the two electric poles of one molecule lies against the equator of the other molecule (either one magnetic axis, or both may also have directions opposite to those indicated in the figure).

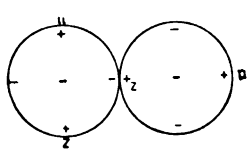


Fig. 1.

[1]) Comp. W. H. KEESOM. Comm. No. 6*b*. These Proceedings 23, p. 943.

The magnetic moment of the double molecule will then be $\mu_1 \sqrt{2}$, when that of a single molecule is μ_1. As in the formula of LANGEVIN for the susceptibility

$$\chi = \frac{1}{3} \frac{n\mu^2}{kT} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

the number of magnetic molecules is now reduced to half its original value, the susceptibility is proved to have remained unchanged.

It is only when we take into consideration that the directing action of the quadrupoles, though the most considerable one for small distances between the molecules, is not the only directing influence and that at the same time we must consider the directing action of the external field or that of the action of the magnetic dipoles on each other, that we find a change in the susceptibility. The calculation and discussion of this change will be given in §§ 3 and 4 under the assumption that the quadrupolar axis and the magnetic dipolar axis coincide. In §§ 5 and 6 the case will be treated in which these axes are perpendicular to each other.

§ 3. *Spherical quadrupolar molecules, having a magnetic dipolar moment in the direction of their quadrupolar axis, in a magnetic field.*

We shall suppose the density of the gas to be such that we have only to consider pairs of molecules and single molecules, while collisions of three or more molecules are so rare that they may be neglected.

In fig. 2 let OA be the direction of the line connecting the centres of the two molecules of a pair, which line we shall draw in the direction from the second molecule to the first one. Let OQ_1 and OQ_2 be the directions of the quadrupolar axes of the first and the second molecule, chosen in the sense that OQ_1 and OQ_2 at the same time indicate the directions of the magnetic axes. Then a pair of molecules is characterized and its orientation with respect to the magnetic field is defined by the coördinates:

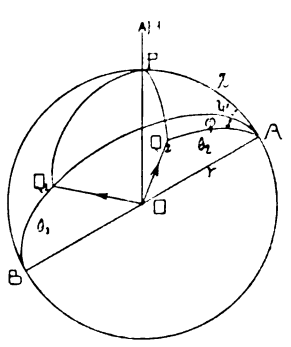


Fig. 2.

$$\chi, r, \psi, \theta_1, \varphi, \theta_2.$$

Here χ may vary between 0 and π, ψ between 0 and 2π, while for r, θ_1, θ_2 and φ we may refer to Leiden Comm. Suppl. N°. 39a § 2 [1]).

The potential energy of the pair of mole-

[1]) These Proceedings 18, p. 636, 1915.

cules due to the quadrupolar forces is then, according to the above mentioned communication, [1]):

$$u_q = v_q \frac{\sigma^5}{r^5} \Psi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

in which

$$\Psi = A + B \cos \varphi + C \cos 2\varphi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and

$$\left.\begin{aligned} A &= 2\,(1-3\cos^2\theta_1)\,(1-3\cos^2\theta_2) \\ B &= 16 \sin\theta_1 \cos\theta_1 \sin\theta_2 \cos\theta_2 \\ C &= \sin^2\theta_1 \sin^2\theta_2 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Here

$$v_q = \frac{3}{4}\frac{\mu_2^2}{\sigma^5} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

(μ_2 = quadrupolar moment) is the potential energy due to the quadrupolar forces, when the molecules are in contact while the two quadrupolar axes are perpendicular to each other and to the line connecting the centres [2]).

The potential energy of the pair of molecules due to the magnetic dipoles is given by [3])

$$u_m = v_m \frac{\sigma^3}{r^3} \Phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

in which

$$v_m = \frac{\mu_1^2}{\sigma^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

(μ_1 = magnetic dipolar moment), and

$$\Phi = 2 \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos\varphi \quad . \quad . \quad . \quad . \quad (10)$$

For the potential energy of the magnetic dipoles in the magnetic field we find

$$u_H = H \mu_1 \Omega, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

in which

$$\Omega = \cos\chi\,(\cos\theta_1 - \cos\theta_2) - \sin\chi\,\{\sin\theta_1 \cos\psi + \sin\theta_2 \cos(\psi + \varphi)\}. \quad (12)$$

---

[1]) Comp. also these Communications No. 6*b* § 3, These Proceedings 23, p. 943, 1920. As, in this paper, we have only to do with pairs of molecules and not with groups of three and more, we shall simplify the notations for the energy by writing for example for the potential energy of a definite pair of molecules u, while until now we have written u_{bl}.

[2]) In this paper we shall neglect the action of induction between the molecules. In Comm. No. 6*b* it has been proved that this action is of no importance compared with that of the quadrupoles.

[3]) Leiden Comm. Suppl. No. 24*b* § 6. These Proceedings June 1912, p. 256.

The total potential energy of the pair of molecules is then:

$$u = u_q + u_m + u_H \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

The two molecules of the pair together contribute to the magnetic moment in the direction of the field H the amount:

$$-\mu_1 \Omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

The number of pairs of molecules in the element

$$dr\, d\chi\, d\theta_1\, d\theta_2\, d\psi\, d\varphi$$

of the $x, \chi, \theta_1, \theta_2, \psi, \varphi$-hyper-space is, according to Leiden Comm. Suppl. N°. 24*b* equation (49):

$$\frac{1}{16\pi}\frac{n^2}{v} e^{-hu} r^2 \sin\chi \sin\theta_1 \sin\theta_2\, dr\, d\chi\, d\theta_1\, d\theta_2\, d\psi\, d\varphi \quad . \quad . \quad (15)$$

These pairs contribute to the magnetic moment in the direction of H the amount:

$$-\frac{1}{16\pi}\frac{n^2}{v}\mu_1 \int_\sigma^\tau\int_0^\pi\int_0^\pi\int_0^\pi\int_0^{2\pi}\int_0^{2\pi} e^{-hu}\,\Omega\, r^2 \sin\chi \sin\theta_1 \sin\theta_2\, dr\, d\chi\, d\theta_1\, d\theta_2\, d\psi\, d\varphi . (16)$$

Here n has been written for the number of molecules in the volume v of the gas and $h = \frac{1}{kT}$.

The mean contribution of each single molecule is, when we consider only the first two terms in the development:

$$\frac{1}{3}\mu_1 . hH\mu_1 - \frac{1}{45}\mu_1 . (hH\mu_1)^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Their number is found by diminishing n by twice the expression (15), integrated over all variables. The contribution of the single molecules added to (16) gives:

$$\left.\begin{array}{l} \frac{1}{3} n\mu_1 . hH\mu_1 \left\{1 - \frac{1}{15}(hH\mu_1)^2\right\} - \\ -\frac{1}{16\pi}\int_\sigma^\tau\int_0^\pi\int_0^\pi\int_0^\pi\int_0^{2\pi}\int_0^{2\pi} e^{-hu}\left\{\Omega + \frac{2}{3}hH\mu_1 - \frac{2}{45}(hH\mu_1)^3\right\} r^2 \\ \sin\chi \sin\theta_1 \sin\theta_2\, dr\, d\chi\, d\theta_1\, d\theta_2\, d\psi\, d\varphi . \end{array}\right\} . \quad (18)$$

If the mutual action of the quadrupoles and of the dipoles were neglected, then these molecules would give a magnetic moment that might be obtained from (18) by substituting e^{-hu_H} for e^{-hu}. This mutual action thus gives rise to an increase of the magnetic moment of the gas. This increase has the value:

$$\left.\begin{aligned}\Delta M = -\frac{1}{16\,\pi}\frac{n^2}{v}\mu_1 \int_\sigma^\infty\!\int_0^\pi\!\int_0^\pi\!\int_0^\pi\!\int_0^{2\pi}\!\int_0^{2\pi} \left\{e^{-h(u_q+u_m)}-1\right\} e^{-huH}\left\{\Omega+\frac{2}{3}hH\mu_1\right.\\ \left.-\frac{2}{45}(hH\mu_1)^2\right\} r^2 \sin\chi \sin\theta_1 \sin\theta_2\, dr\, d\chi\, d\theta_1\, d\theta_2\, d\psi\, d\varphi,\end{aligned}\right\} \quad (19)$$

in which we may take ∞ as upper limit for r. For the calculation of (19) we develop the expression in series using (4), (8), and (11). Viz:

$$\left.\begin{aligned}-e^{huH}\left\{\Omega+\frac{2}{3}hH\mu_1-\frac{2}{45}(hH\mu_1)^2\right\} = \Omega+\frac{1}{3}hH\mu_1(2-3\Omega^2)-\\ -\frac{1}{6}(hH\mu_1)^2(4\Omega-3\Omega^3)-\frac{1}{90}(hH\mu_1)^3(4-30\Omega^2+15\Omega^4)\,..\end{aligned}\right\} \quad (20)$$

$$\left.\begin{aligned}e^{-h(u_q+u_m)}-1 = -hv_q\frac{\sigma^5}{r^5}\Psi - hv_m\frac{\sigma^3}{r^3}\Phi+\tfrac{1}{2}(hv_q)^2\frac{\sigma^{10}}{r^{10}}\Psi^2+\\ +hv_q\,.\,hv_m\,.\frac{\sigma^8}{r^8}\Psi\Phi+\tfrac{1}{2}(hv_m)^2\,.\,\frac{\sigma^6}{r^6}\Phi^2-\frac{1}{6}(hv_q)^3\frac{\sigma^{15}}{r^{15}}\Psi^3-\\ -\tfrac{1}{2}(hv_q)^2\,.\,hv_m\frac{\sigma^{13}}{r^{13}}\Psi^2\Phi-\tfrac{1}{2}hv_q.(hv_m)^2\frac{\sigma^{11}}{r^{11}}\Psi\Phi^2-\frac{1}{6}(hv_m)^3\frac{\sigma^9}{r^9}\Phi^3\ldots\end{aligned}\right\} \quad (21)$$

We may then calculate the different terms of (19) separately.

The term Ω in (20) gives no contribution.

Only the term: $\frac{1}{3}hH\mu_1(2-3\Omega^2)$ in (20) can give a contribution proportional with H.

We can easily prove that $\Psi^l(2-3\Omega^2)$ where l is a positive integer, gives 0 when integrated with respect to χ and ψ.

From this it is evident, that the directing action of the quadrupolar forces on the susceptibility has no influence that might be expressed in (2) by a Δ independent of H.

Of the terms written in (21) only $\Psi^2\Phi$ and Φ^3 give contributions when multiplied by $2-3\,\Omega^2$; viz when we put

$$M=\frac{1}{3}nhH\mu_1^2 \quad \ldots\ldots\ldots\ldots \quad (22)$$

$$\Delta M_{qm}=\frac{128}{2205}\frac{n}{v}\frac{4}{3}\pi\sigma^3(hv_q)^2\,.\,hv_m\,.\,M \quad \ldots\ldots \quad (23a)$$

and

$$\Delta M_m=\frac{1}{75}\frac{n}{v}\frac{4}{3}\pi\sigma^3(hv_m)^3 M \quad \ldots\ldots\ldots \quad (23b)$$

Further Ψ^3 gives when multiplied by $4-30\,\Omega^2+15\,\Omega^4$:

$$\Delta M_q = \frac{128}{11025} \frac{n}{v} \frac{4}{3} \pi \sigma^3 (h v_q)^2 (h H \mu_1)^2 M \quad . \quad . \quad . \quad (23c)$$

Of these contributions ΔM_q is due to the directing action of the quadrupoles, ΔM_m to that of the magnetic dipoles, ΔM_{qm} to the combined action of the quadrupoles and dipoles. They form the first terms of developments in ascending powers of h, hence of T^{-1}.

§ 4. The values derived in the preceeding § for the influence of the above mentioned mutual directing actions on the magnetisation become manifest in the susceptibility by the introduction into (2) of the terms Δ_{qm}, Δ_m, Δ_q. These then become:

$$\Delta_{qm} = -\frac{128}{2205} \frac{n}{v} \frac{4}{3} \pi \sigma^3 \left(\frac{v_q}{kT}\right)^2 \frac{v_m}{kT} T \quad . \quad . \quad . \quad . \quad (24a)$$

$$\Delta_m = -\frac{1}{75} \frac{n}{v} \frac{4}{3} \pi \sigma^3 \left(\frac{v_m}{kT}\right)^2 T \quad . \quad . \quad . \quad . \quad . \quad . \quad (24b)$$

$$\Delta_q = -\frac{128}{11025} \frac{n}{v} \frac{4}{3} \pi \sigma^3 \left(\frac{v_q}{kT}\right)^2 \left(\frac{\mu_1 H}{kT}\right)^2 T \quad . \quad . \quad (24c)$$

The sign of none of these Δ's agrees [1]) with that of the Δ found experimentally for liquid oxygen (§ 1). Further Δ_{qm} and Δ_m prove to be proportional to T^{-2}, Δ_{qm} to T^{-3} and also to H^2, while the observations give no indications of such a dependence of Δ on T and H.

To form us a better judgment of the magnitude of the influence that might be exerted on the susceptibility by the above mentioned mutual directing actions we shall calculate, for special circumstances, the values of the above derived Δ's for oxygen. For this purpose we may take from Comm. N°. 6*a* § 2: $v_q = 5{,}7 \times 10^{-14}$ and $\sigma = 2{,}65 \times 10^{-8}$; further from the same Comm. § 3: $\mu_1 = 2{,}6 \times$ $\times\ 10^{-20}$, so that according to (9): $v_m = 3{,}7 \times 10^{-17}$.

Let us consider oxygen in the gaseous state at 90° K. under the pressure of 1 atm., then $\frac{n}{v}\frac{4}{3}\pi\sigma^3 = 0.0064$ and (with $k = 1{,}34 \times$ 10^{-16}) : $\frac{v_q}{kT} = 4{,}7$, $\frac{v_m}{kT} = 0.0030$. By this we obtain $\Delta_{qm} = -0{,}0022$, $\Delta_m = -2 \times 10^{-10}$.

When further we put $H = 55000$, then $\frac{H\mu_1}{kT} = 0{,}12$, which gives $\Delta_q = -0{,}002$.

[1]) That the mutual action of the magnetic dipoles has an effect in the opposite direction, has already been remarked by LANGEVIN l.c. note 2 p. 1128. This result does not agree with that of GANS l.c. note 1, p. 1128.

From these results we see that even in highly compressed oxygen and, as regards to Δ_q, in the strongest fields, these Δ's will be negligible.

When we take into consideration that the contributions found in (23a) and (23c) are only the first terms in series of ascending powers of $h\nu_q$ (§ 4 the end) and that in our case $h\nu_q$ is about 4.7, then it is evidently quite possible that the further terms in that development will preponderate. Even then those terms might be almost negligible, but further the value of Δ that we should obtain by using those terms would show a still greater dependence on T than was the case with the values given in (24) (unless over a certain temperature interval a very special compensation might occur). Then it would still less agree with a value of Δ that may be regarded as constant throughout a certain temperature region.

Thus an explanation of the experimental deviation from CURIE's law, based on the directing action of the electrostatic forces between the molecules seems to be excluded, unless the directing action occurring in the liquid state might preponderate to a special degree.

In the following §§ will be shown, that this conclusion remains valid also when we suppose the magnetic dipolar axis to be perpendicular [1]) to the quadrupolar axis.

§ 5. *Spherical quadrupolar molecules, with a magnetic dipolar moment in a fixed direction perpendicular to the quadrupolar axis, in a magnetic field.*

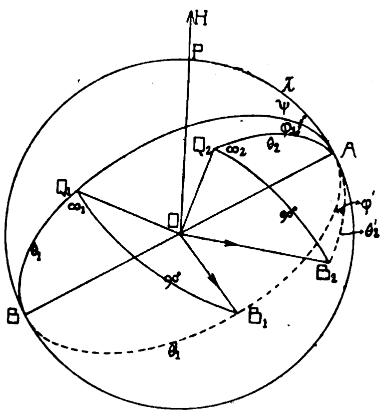


Fig. 3.

In Fig. 3 OQ_1 and OQ_2 again represent the directions of the quadrupolar axes of the two molecules. The directions of the magnetic dipolar axes have now been indicated by OB_1 and OB_2, fixed by the angles ω_1 and ω_2 (from 0 to 2π) and $Q_1B_1 = Q_2B_2 = 90°$.

We shall now follow step by step the calculations of § 3, of the formulae given there we have only to change the following ones. We now have the coördinates:

$$\chi, r, \psi, \theta_1, \varphi, \theta_2, \omega_1, \omega_2.$$

The potential energy of the pair of molecules due to the mag-

[1]) Comp. W. PAULI Jr., Physik. Zs. 21, p. 615, 1920.

netic dipoles is now given by (8) and (9) if we merely substitute in (10) θ_1', θ_2' and φ' (fig. 3) for θ_1, θ_2 and φ.

We thus find:

$$\left.\begin{aligned}\Phi = 2\cos\omega_1\cos\omega_2\sin\theta_1\sin\theta_2 + \cos\varphi\,(\sin\omega_1\sin\omega_2 + \cos\omega_1\cos\omega_2\cos\theta_1\cos\theta_2) + \\ + \sin\varphi\,(\cos\omega_1\sin\omega_2\cos\theta_1 - \sin\omega_1\cos\omega_2\cos\theta_2)\end{aligned}\right\} \quad (25)$$

Instead of (12) we obtain

$$\left.\begin{aligned}\Omega = \cos\chi\,(\cos\omega_1\sin\theta_1 - \cos\omega_2\sin\theta_2) + \sin\chi\,\{\cos\omega_1\cos\theta_1\cos\psi + \\ + \sin\omega_1\sin\psi + \cos\omega_2\cos\theta_2\cos(\psi+\varphi) + \sin\omega_2\sin(\psi+\varphi)\}\end{aligned}\right\} \quad (26)$$

(15) is multiplied by $\frac{1}{4\pi^2}\,d\omega_1\,d\omega_2$, while (16), (18) and (19) undergo similar changes, where the integrations with respect to ω_1 and ω_2 have to be extended from 0 to 2π.

The term Ω in (20) again gives zero.

This is also the case with $\Psi^l\,(2-3\,\Omega^2)$, l being a positive integer, so that the conclusion drawn in § 3 remains valid here also.

Again we find contributions due to $2-3\,\Omega^2$ multiplied by $\Psi^2\Phi$ and Φ^2 viz.

$$\Delta M_{qm} = -\frac{16}{147}\,\frac{n}{v}\,\frac{4}{3}\,\pi\sigma^3\,(h\nu_q)^2\quad h\nu_m\,.\,M, \quad . \quad . \quad . \quad (27a)$$

while for ΔM_m (23b) is found again, as might have been expected for this term is independent of the quadrupolar forces and hence the situation of the dipolar axis with respect to the quadrupolar axis is without influence on ΔM_m.

Further Ψ^2 when multiplied by $4-30\,\Omega^2+15\,\Omega^4$ gives a contribution:

$$\Delta M_q = -\frac{4}{125}\,\frac{n}{v}\,\frac{4}{3}\,\pi\sigma^3\,.\,h\nu_q\,.\,(hH\mu_1)^2\,.\,M. \quad . \quad . \quad (27b)$$

§ 6. The values of Δ_{qm} and Δ_q corresponding to the ΔM_{qm} and ΔM_q found in § 4, may now easily be written down. Both have now the sign agreeing with the observations.

For the circumstances chosen in § 4 we have now (for oxygen): $\Delta_{qm} = 0.0041$, $\Delta_q = 0.0013$. From this we see that the conclusions of § 4 are also valid assuming that the dipolar axis is perpendicular to the quadrupolar axis.

Geology. — "*Quaternary and Tertiary Limestones of North-New Guinea between the Tami-, and the Biri-river basins*". By Dr. L. RUTTEN. (Correspondent of the Academy).

(Communicated at the meeting of February 26, 1921).

In arranging the rocks, collected by the New Guinea-Expedition of 1903, it appeared that in the coastal region of North-New Guinea between the Tami-river and Walckenaers-bay, folded deposits of tertiary age and limestones, belonging to a quaternary transgression, are widely spread along with old basic eruptive rocks and scarce mesozoic sediments [1]).

The tertiary deposits have afterwards been found in the region of the 141[th] meridian comparatively far into the interior: limestones with Lepidocyclina were met with in the Upper course of the Bewani-river (Basin of the Tami) and of the Keerom-river, about 66 km. from the coast [2]). It seems that tertiary deposits play an important part in this zone, which lies between the coast and the large central plain, through which the affluents of the Idenburg-river are flowing.

As known, the Mamberamoriver equally breaks — about 300 km. to the West — through a young folded mountainrange, called the Van Rees-mountains, which are built up chiefly of young tertiary sandstones, shales and limestones with dykes of eruptive rocks. Likewise it is known that older, basic eruptiva and mesozoic sediments are of rare occurrence here [3]). The strike in the Van Rees-mountains is S. 65° to S.E., so it might be expected that the tertiary folded mountains could also be found in the region between the Van Rees-mountains and the Tami-river, while the a priori conclusion might be made that here indications were to be found of the occurrence of older basic eruptiva.

For the "Bataafsche Petroleum-Maatschappij" I recently examined a collection of limestones and marls, collected by Dr. W. VAN HOLST

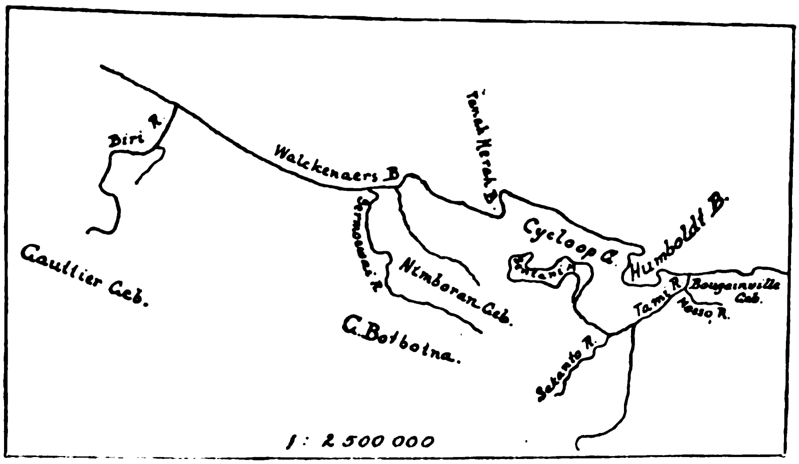
[1]) A. WICHMANN. Nova Guinea. IV. 1917.
L. RUTTEN. Nova Guinea. VI. 1914.

[2]) L. SCHULTZE. Mitteilungen aus den Deutschen Schutzgebieten. Ergänz. Heft 11. Berlijn 1914.

[3]) J. VAN GELDER. Jaarb. Mijnw. Nederl. Indië. 1910. Wetensch. Gedeelte. p. 87—112.

Pellekaan in the territory between Tami- and Biri-river, and originating from localities, at the farthest some 50 km. from the shore. The results of this investigation, the publication of which was so liberally permitted by the "Bataafsche Petroleum-Maatschappij", evidenced that we were right in supposing that tertiary and especially neogene sediments are widely spread over the whole coastal mountain range between the Tami-, and the Biri-river, while also indications were present of the occurrence of older, basic eruptive rocks.

What has been stated above will appear from the following description of a number of thin sections.



Previous investigations tended to show that eocene rocks are of rare occurrence in the coastal region of North-New Guinea. Hitherto we know only boulders of eocene reeflimestone from the Tawarin-river [1]), which however cannot be derived from the present riverbasin. Wichmann suspects that their mother-rock is to be looked for in the territory of the Sermuwai, which rises much farther in the interior [2]).

This hypothesis tallies with the fact that the collection Holst Pellekaan also comprises only two eocene limestones, which were found in the river Nanggoi in the South-Nimboran Mountains i.e. in the basin of the river Sermuwai. They are two blackish-grey rocks of reeflimestone. The one contains Alveolina s.str., the other Alveolina s.str., Lithothamnium, Nummulites cf. Bagalensis Verb. and Orthophragmina; their age is doubtless eocene.

By far the greater number of the collection are of oligomiocene age, and belong to the Lepidocyclina-bearing neogene. They all

---

[1]) Nova Guinea. VI. p. 35.
[2]) Nova Guinea. IV. p. 266—267.

indicate the occurrence of neogene deposits in litoral facies. The material examined did not yield sufficient data to make a subdivision into older and younger levels. The rocks comprised in this collection are the following:

Pure, porous limestone, south of Nafri, Humboldt bay. No terrigenous material. Contains: Lepidocyclina cf. Munieri Lem. et Douv., Globigerina, ? Cycloclypeus, ? Carpenteria and Corals.

Yellowish-grey, highly crystalline limestones from the Singringreng-river, south of the Nimboran mountains. Contains large and small Lepidocyclinae, Rotalidae and Lithothamnium.

Slightly porous limestone from the Sg. Tung, Botbotna mountains with small Lepidocyclinae, Miogypsina, ? Carpenteria conoidea Rutten, Nummulites cf. Cuminghii Carp., Amphistegina, Operculina, ? Cycloclypeus, Rotalidae, Lithothamnium and corals. From the same river a greenish gray reeflimestone, including the same faunula as the preceding and besides these fragments of serpentine-grains, indicative of the occurrence in the vicinity of a subsoil of basic eruptiva.

Two yellowish grey reeflimestones from the river Ohung, right affluent of the river Sermuwai near the village of Sawé, with Lepidocyclina cf. Munieri Lem. et Douv., Operculina, Amphistegina, ? Cycloclypeus, Nummulites cf. Cuminghii Carp., ? Carpenteria, Globigerina, Orbulina, Lithothamnium and corals.

Four limestones from the river Buarim, southwestern affluent of the river Sermuwai with Lepidocyclina cf. Munieri Lem. et Douv., Miogypsina, ? Carpenteria, Amphistegina, Gypsina cf. inhaerens, ? Planorbulina larvata P. a. J., ? Cycloclypeus, Miliola, Globigerina, Lithothamnium, Halimeda and corals.

A greenish calcareous rock from the river Gisé, north of the Botbotna-mountains with an abundance of quartzgrains, and few plagioclase splinters and serpentine grains. It contains small Lepidocyclinae, ? Cycloclypeus, Amphistegina, Globigerina and Lithothamnium.

Different limestone-boulders from conglomerates at the middle course of the Biri-river with Lepidocyclina, ? Miogypsina, Heterostegina, Operculina, Cycloclypeus, Amphistegina, Globigerina, Carpenteria, Miliola, Lithothamnium, Halimeda and corals. One sample contains serpentine grains.

A reeflimestone from the South-river, upper course of the Iwaré, basin of the Biri, contains Cycloclypeus, Lepidocyclina cf. Munieri Lem. et Douv., Heterostegina or Spiroclypeus and corals.

A very finely crystalline reeflimestone from Prauw-bivouac, Biri-

river contains small Lepidocyclinae, Globigerina and Lithothamnium.

Of a number of rocks, which are specially characterized by the occurrence of numerous Globigerinae, the age cannot be established with certainty. Since their localities are often the same as those of the Lepidocyclina-bearing rocks it may be that some of them are of the same age as these rocks and were formed under somewhat different circumstances viz. in somewhat deeper water.

A marly limestone with many Globigerinae comes from the river Urbiahua, Tamibasin.

A greyish green limestone with Globigerinae and a few quartz splinters was found at Sentanilake opposite Dondai.

In the peninsula of Morni, Matterer bay a green marl-lime with quartz- and plagioclase splinters and a few Globigerina was found.

The Ajer Dambé, Demta-bay, produced a grey marl-line rich in grey Globigerina.

Tuffacous lime-sandstones with quartz-, plagioclase-, and serpentine-grains and with scarce Globigerina originate from the vicinity of the Muris-bay.

Brown marl-limes with pyrite-granules, Globigerina, Pulvinulina, and Rotalidae were found in the Sg. Gauw, South Nimboran-mountains.

A limestone rich in pyrite with very many Globigerina, Amphistegina, Operculina, and corals comes from the Sawé-hill, Sermuwai-river. The facies of this rock is intermediate between the reeflimes and the true Globigerina-rocks.

A limestone from the river Gemuwai appeared to be almost quite a breccia of Globigerina and Pulvinulina.

Whereas it is possible that of the above Globigerina-rocks some already belong to the quaternary transgression, this is far more probable for some single limestones which, it is true, do not contain characteristic fossils, but whose habitus gives the impression that they are true "Karang"-samples, "very young reeflimestones". They are porous limestones from the Muris- and Demta-bay.

Of four reeflimestones (Sekanto-river, Middle Biririver, Upper Biririver and Iwarinriver in the Gautier Mountains the exact age cannot be given, but they are certainly of post-cretaceous age.

It appears from the foregoing that all the limestones described belong to the post-cretaceous system, while eocene rocks are only very scarce. Two other rather considerably recrystallized limestones (from village of Semenaré and from the Cyclop mountains) which do not contain recognizable fossils, were the only ones in the collection which probably were of pretertiary age, though they also may be young.

By far the majority of the rocks belong to the Lepidocyclina-bearing tertiary and point to the vast diffusion of neogene deposits in litoral facies.

Finally the assumption is admissible that the material also contains very young, most likely quaternary reeflimestones.

Utrecht, January 15, 1921.

Geology. — "*On the Age of the Tertiary Oil-bearing Deposits of the Peninsula of Klias and Pulu Labuan (N. W. Borneo).*" By Dr. L. RUTTEN. (Correspondent of the Academy).

(Communicated at the meeting of February 26, 1921).

On the geological map of Borneo published by TH. POSEWITZ [1]) a broad band of tertiary deposits is marked along the north-west coast of the island and on the island of Labuan. It extends from South-Serawak to the Northern point of Borneo. It is not known to which subdivisions of the Tertiary these deposits generally belong. T. BELLOT [2]) who was the first to report the existence of coal on the island of Labuan does not say anything about its age. J. MOTLEY [3]) who described the coal formation of Labuan rather minutely, is convinced that it is of tertiary age, but he does not positively say to which subdivision of the Tertiary it belongs. TH. POSEWITZ [4]), however, records that MOTLEY considered the tertiary of Labuan to be eocene. C. SCHMIDT also says that nearly all older writers took the tertiary of Labuan and of the Peninsula of Klias to be eocene and even points to the petrographic analogy between the deposits of Klias and the rocks of Pulu Laut, to the South-east of Borneo [5]). A. V. JENNINGS [6]) described eocene Orbitoids from more Southern territories of North-West Borneo; they were found along the Barram river near Langusan (Batu Gading) and to the South of Barram river (Silungen). NEWTON and HOLLAND [7]) demonstrated that in these southern territories oligomiocene rocks must occur along with eocene rocks; they found in boulders from the Sungei

---

1) TH. POSEWITZ. Borneo. Berlin. 1889.

2) T BELLOT. On the discovery of coal on the island of Labuan, Borneo. Quart. Journ. Geol. Soc. London. 4. 1846. p. 50.

3) J. MOTLEY. Report on the geological phenomena on the island of Labuan etc. Journ. of the Indian Archipelago VI. 1852. p. 555—573. Cf. also: Quart. Journ. Geol. Soc. London 9. 1853. p. 54—57.

4) TH. POSEWITZ. l.c. p. 174.

5) C. SCHMIDT. Ueber die Geologie von NW-Borneo etc. Gerlands Beitr. zur Geophysik. VII. 1905. p. 121—135. 1 pl.

6) A. V. JENNINGS. Note on the orbitoidal limestone of North Borneo. Geol. Magazine. (3) V. 1888. p. 529—532.

7) R. B. NEWTON and R. HOLLAND. On some tertiary Foraminifera from Borneo. Ann. and Mag. Nat. Hist. (7) III. 1899. p. 245—264.

Malinam (Melinan), an affluent of the Sungei Barram, oligomiocene Orbitoides (Lepidocyclina Verbeeki Newt. and Holl.) as well as eocene Orbitoides and Nummulites.

Whereas we know nothing about the nature of the sediments from which the oligomiocene fossils described by NEWTON and HOLLAND originate, we have a fair knowledge of the composition of the deposits of Brunei, Klias and Labuan which are considered to be eocene. According to C. SCHMIDT's (l.c.) descriptions and profiles we have to do here with an oil-bearing formation of several thousands of metres thickness, made up of a highly folded series of sandstones, marls, shales, limestones, conglomerates and coal. Numerous oil-localities and important mud-volcanoes prove this formation to have been rich in oil originally.

In 1914 I received from the late Dr. G. NIETHAMMER some fragments of limestone from the tertiary of the peninsula of Klias. Some time ago Dr. W. HOTZ of Basle sent me a small collection of rocks and boring-samples, some of which had been collected by NIETHAMMER, others by himself. This small collection derives its interest from belonging to different stratigraphic horizons. A fragment of limestone originates from Pulu Burung to the south of Labuan; according to C. SCHMIDT (l.c.) the islet rests upon a syncline, so that the limestone is sure to belong to the more recent horizons of the formation. Another fragment was found by HOTZ on the small island on the westcoast of Klias, which, according to SCHMIDT. was formed during the eruption of a mud-volcano on the 21st of September 1897 on the axis of a deeply folded anticline. Various pieces of limestone were collected on Klias on the surface. From the boring 1 of the Dutch Colonial Oil-Company on Klias I obtained samples, that were brought up from depths of from 773 to 1480 feet. Finally HOTZ collected a fragment of limestone on a cliff to the north of the isle of Tega, situated to the North of Klias.

The rock I received from Dr. NIETHAMMER in 1914 is a yellowish-grey, rather crystalline reef limestone on whose surface with the aid of a loupe Cycloclypeus communis Martin, Heterostegina depressa d'Orb., small Lepidocyclines and Corals can be recognized. In thin sections it can be seen, that some of the Lepidocyclinae are characterized by small dimensions, by scarce thick skeleton-columns centrally arranged, and by median chambers, which become higher towards the periphery, and on this account must be classed along with L. Munieri Lem. et Douv. From one particular specimen we concluded that the fossils are megalospherical and that the embryonic chambers are of the kidney-shaped type. Furthermore the thin sections

reveal Lithothamnia and Lepidocyclina. Another Lepidocyclina being of the shape of a double cone and microspherical, could be removed from the rock; it is closely allied to L. acuta Rutten. From the absence of Nummulites and large Lepidocyclinae on the one side and of Miogypsina on the other, and from the presence of L. acuta we are justified in concluding that the rock belongs to the middle part of the Lepidocycline-bearing tertiary, and, therefore, is most likely middle-old-miocene.

It is not impossible that the rock described is derived from the same finding-place as another limestone of the Hotz-collection, which was found by Niethammer at the Sungei Silico (N°. 499). Here also may be recognized on the surface Heterostegina, small Lepidocyclinae and a few larger, flat Lepidocyclinae. In the thin sections of the rock it can be observed that the limestones are highly crystalline and that they include numerous Lepidocyclinae. Besides these also Cycloclypeus communis Martin, Heterostegina, Operculina, Carpenteria, Rotalidae and Lithothamnium occur. Most of the Lepidocyclinae belong to a small species; the horizontal diameter is from 2—3 mm.; the vertical one from 1—1½ mm. The fossils are megalospherical with kidney-shaped embryonic chambers; there are scarce, but very thick, centrally arranged skeleton-columns; the median chambers are higher towards the periphery, and often the plane of the median chambers is continued out of the proper body of the fossil, forming a collar round the lentoid centre. It is certain, that these fossils must be grouped along with the Lepidocyclina Munieri Lem. et Douv. Beyond these, some slightly larger, microspherical forms occur (horizontal diameter 3—4 mm., vertical diameter 1—1½ mm.) distinguished from the others by a flatter lentoid shape and by the occurrence of numerous skeleton-columns evenly distributed over the whole body.

Lastly there are some sections of a still larger species, whose diameter exceeds 6 mm., whose height, however, is no more than 1½ mm. It is evident, then that the fossils are very flat and the height of the chambers is accordingly insignificant. Probably they are microspherical. True skeleton-columns there are none, but the vertical walls between the lateral chambers are thickened rather considerably. Whereas the two forms first described belong still to the "ordinary" small Lepidocyclina, which characterize the middle-, and the most recent part of the Lepidocyclina-bearing tertiary, the last-described form already begins to show analogies to the group of Lep. formosa Schl. (flatness, larger diameter, absence of skeleton-columns) which characterizes the lowermost part of the Lepidocyclina-bearing tertiary.

The fossils of the limestone of Sg. Silico nevertheless differ from the typical L. formosa in having a much smaller diameter. Also this limestone may just as well be referred to the middle-old-miocene.

The limestone of Pulu Burung south of Labuan, which according to C. SCHMIDT (l.c.) appears in a syncline, is a true reeflime. J. MOTLEY (l.c.), therefore, was wrong in considering the limestones of this island to be the sedimentary products of calciferous sources. The rock is grey, somewhat porous and fairly crystalline. In the thin sections may be recognized Lithothamnium, Halimeda, Amphistegina, Miliola, Textularidae, a very few Miogypsinae and a few small Lepidocyclinae with a diameter of only 1½ mm., of which some still possess strong columns, others are quite devoid of skeletons. The presence of Miogypsina and of small degenerated Lepidocyclinae and the absence of larger Lepidocyclinae points to the fact that the rock must be referred to the youngest part of the Lepidocyclina-bearing tertiary, which conclusion is substantiated by the geological observations in loco.

The rock of the New Island on the west coast of Klias on the other hand certainly originates from the lower part of the sediment-series. It is a brownish-gray Foraminifera breccia impregnated with limonite. The limonitic substance has to some extent filled up the hollows and the pores of the fossils, through which their structure has become very conspicuous. Besides corals, Lithothamnia, Operculina, Heterostegina and Lepidocyclina cf. Munieri Lem. et Douv., here also occur larger, megalospherical Lepidocyclinae of the Eulepidina type. They differ little in size from the typical forms of L. formosa Schl., with which they have in common the considerable flatness, the structure of the embryonic chambers, the absence of columns and the considerable thickness of the vertical walls between the lateral chambers.

Two marly limestones, found by G. NIETHAMMER in the rivulets Napassu and Blanot, do not include any typical fossils.

The samples from the boring N°. 1 of the Dutch Colonial Oil Company are chiefly hard, gray, sometimes somewhat marly clay-shales, very much like the clayshales from the oldest miocene and the oligocene of East-Borneo (samples of 773′, 825′, 830′, 975—980′, 1000′ and 1480′). Sometimes sands or sandstones appear (1415—1420′). On various levels Foraminifera were found in the marly clayshales, of which the samples presented faunulae that differed with the depth at which they were found. At 800′ 1300′, 1308—1312′, 1330—1335′, and at 1342′ we found besides indifferent forms such as Heterostegina and Cristellaria only small Lepi-

docyclinae, which belonged to various species (L. aff. Mumeri Lem. et Douv., L. aff. inflata Provale, L. sp.). I shall not enter into their specific determination or their description, since they are of no special value for stratigraphy. At 1355′—1370′ and at 1412—1415′ however we found along with small Lepidocyclinae also larger individuals, notably the same flat, column-less Eulepidines about 10 mm. in diameter, which also occur in the limestone of New Island, but along with them also other forms characterized by having numerous columns diffused over the whole surface. The former are allied to L. formosa, the latter are related te L. insulaenatalis Jones and Chapman. We see, therefore, that these larger forms begin to appear only in the deeper horizons of the boring.

Whereas all the rocks, described above, certainly belong to one and the same series of sediments, the habitus of the limestone, found by Hotz on a lime reef to the north of Pulu Kalumpunian to the north of Pulu Tega, is different. It is a grey, pseudo-oolitic rock, containing a great many Lithothamnia and some Miliolidae, but in which the typical litoral Foraminifera of the oligomiocene are altogether lacking. This limestone also may belong to the Tertiary, but it is quite impossible to say to which subdivision.

Our examination of the rocks described above shows first of all that the oil-formation of Northwest Borneo and more particularly that of the peninsula of Klias and the island of Labuan is not, as hitherto assumed, of eocene age, but that its anticlinal cores hardly reach the oligocene[1]), the typical large Lepidocyclina which characterize the oligocene being absent here; neither do we find here reticulate Nummulites. (Deeper parts of the boring Klias I, rock from New Island). The youngest rock examined (limestone from P. Burung South of Labuan) — which, however, does not yet belong to the youngest part of the sediments-series — must still be considered to belong to the miocene s.str.; the fossils, however, prove the rock to originate from the topmost part of the Lepidocyclina-bearing tertiary.

It appears then, that a satisfactory stratigraphical concordance exists between the oil-formations of Northwest- and those of East-Borneo.

Both formations originated in the same period of prolonged sedimentation, attended with subsidence of the sedimentation-regions,

[1]) It may be, of course, that the eocene is developed under the anticlinal cores. As to this nothing can be said for certain, nor can anything be surmised on the basis of the examined material.

which probably begins in the oligocene (gas-, and oil-bearing layers from the core of the Sangatta, Bungalun and Sekurau-anticline in East-Borneo[1]) and continues into the pliocene (oil-layers of P. Tarakan).

MOLENGRAAFF[2]) demonstrated a short time ago that the tertiary oil-deposits of the western part of the East-Indian Archipelago in the E., S., and S.W., are marginal to the old "Sunda-land" and suspects that genetically the oil-formation of North-West Borneo, which occupies an analogous situation at the N.W. margin of this land-mass, is closely related to the oil deposits on the Dutch territories. This view appears to be correct. According to MOLENGRAAFF's conception the origin of the material of the neogene sediments of N.W. Borneo must be looked for in the South-East, in the old massif of Borneo. MOTLEY's (l.c.) view was diametrically opposite to this conception. This writer considered the tertiary sediments of Labuan, Brunei and Klias to be partly sedimentations at a south-eastern coast of a South-Asiatic continent, partly delta-formations of an enormous river, which he presumes to have come down from Central-Asia. To this divergence of opinion we shall revert presently.

We will first call attention to the fact that the margins of the oligomiocene "Sundaland" will be marked still better by the neogene reeflimes than by the diffusion of the oil-, and coaldeposits; the former, whose facies agrees with those of the recent-tropical reeflimes, having never been absent from a coastal fringe of any extent, the latter originating only under certain favourable conditions of sedimentation. The most inward oligomiocene reeflime stones, therefore, will mark the nearest limit of the border of the island of Borneo that was washed by the sea during the neogene.

When considering the island from this point of view we see that there must have been periods in the neogene in which only comparatively small portions of Borneo emerged from the sealevel.

Beginning in the north we see that neogene reeflimes are known of the islands of Balambangan and Banguey[3]). Of the vicinity of Kudat I possess a Lepidocyclina-bearing limestone, found there by Dr. W. HOTZ. Of the basin of the Sg. Kinebatangan miocene litoral limes of the G. Gomanton have long been known[4]). Of the Batu Tjinagat the Geol. Institute of Utrecht possesses a Lepidocyclina-

[1]) L. RUTTEN. These Proceedings, XIX. 1917. p. 728.

[2]) G. A. F. MOLENGRAAFF. These Proceedings. XXIII. 1920. p. 440—447.

[3]) L. RUTTEN. Samml. Geol. Reichsmus. Leiden. (1) X. 1915. p. 11—17.

[4]) R. BULLEN Newton and R. HOLLAND l.c.

containing limestone. While these localities are all pretty near to the coast and only little is known of the inland in these northern regions, we see, that farther in the South of East Borneo the litoral neogene limestones appear more and more towards the interior. Of the basin of the Berau river I possess Lepidocyclina-bearing limestones from the rivulets Birang and Lassan, collected by Dr. F. WEBER. As known, on Sangkulirang neogene sediments are widely diffused. Towards the south, certainly as far as South of the Balikpapan Bay, a coastal belt of rather more than 100 km. in breadth is built up of folded neogene rocks. We know that not all these sediments are of marine origin; important and numerous ingressions and regressions must have taken place [1]). That these ingressions have encroached far on the inland once at least, is borne out by the findings of oligomiocene limestone near Udju Halang [2]) and Kiham Halo [3]) at the Upper-Mahakam-river. Contiguous with this towards the south are the old neogene reeflimes of the Middle-Barito-river near Batu Putih [4]) and of the Mahangjongriver in the basin of the Sg. Kapuwas Murang [5]). Still farther to the south I do not know of any occurrence of neogene coastal deposits.

When we cross to the Northwest coast of Borneo, we find farthest into the inland the formation of neogene limestones at the Melinau river, a left affluent of the Barram-river (BULLEN, NEWTON and HOLLAND l.c.). Data produced by J. MOTLEY (l.c.) seem to point to the existence on the Redjang-river of neogene deposits of a litoral character. On the other hand the rocks of Klias and Labuan described above are all lying in the litoral zone.

On the accompanying map we have, on the basis of all these data, indicated very roughly which territories of Borneo were not covered by the sea during the farthest neogene ingression. We see here a central landmass rather narrow in the North and broadening towards the South-West, where it is connected with the old Sunda continent [6]),

This sketchmap enables us to realize the stupendous changes undergone by Borneo in the neogene. After the farthest old-

---

1) L. RUTTEN. These Proceedings l.c. 1916.

2) I. PROVALE. Rivista italiana di Paleontologia. XV. Catania 1909. p. 95.

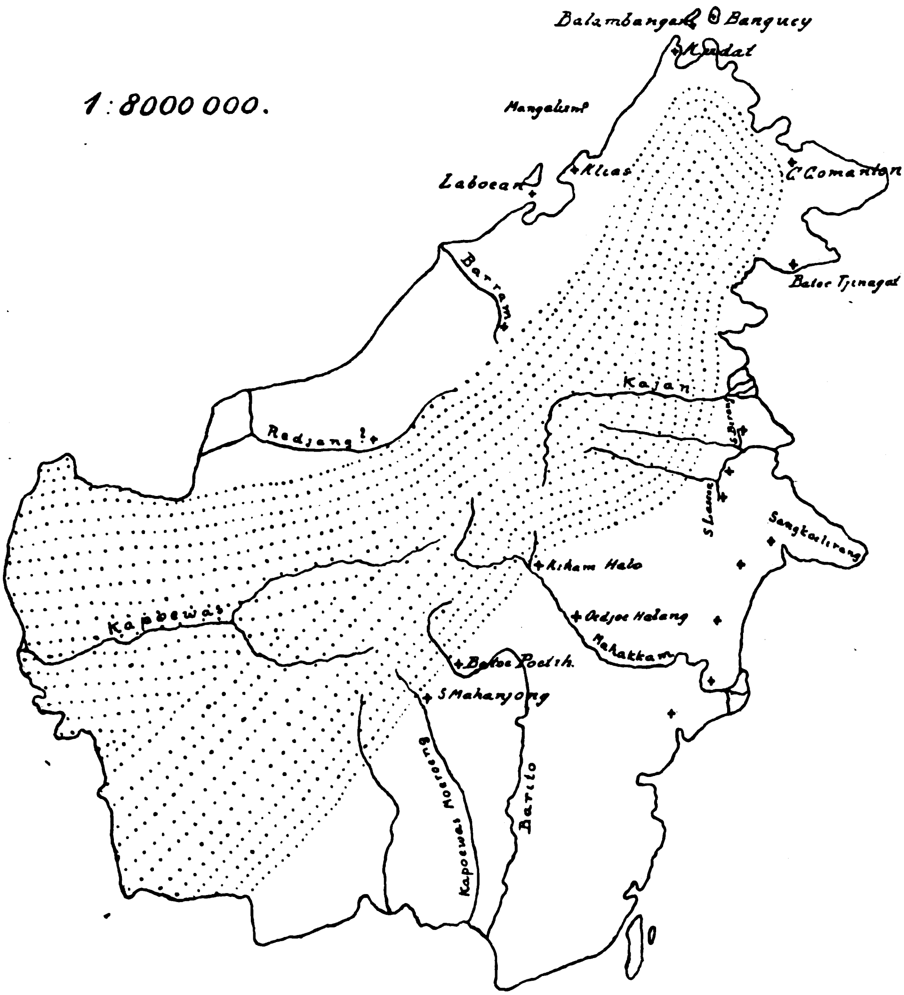
3) N. WING EASTON. Tijdschr. Kon. Ned. Aardr. Gen. (2) 34. 1917. p. 680—695.

4) L. RUTTEN. Samml. Geol. Reichsm. Leiden. (1) IX. 1914. p. 320—322.

5) L. RUTTEN. Samml Geol. Reichsm. Leiden. (1) IX. 1912. p. 213—217.

6) In several places the old central landmass of Borneo was no doubt still smaller than is shown on the map; but no published data are at our disposal. On the other hand it is probable that out of this central core, e.g. in the farthest South East of the island, some territories were not transgressed by the sea.

neogene ingression the central part of the island must have been subjected to continual upheavals, for only in this way could it



+ *Oligomiocene limestones.*

Central core not or only partly covered by the sea during the most significant neogene ingression.

procure the incredible masses of detritus which were required for building up the neogene deposits in the East, the South-East, and the North-West, which are several thousands of metres thick and which during their deposition, when they had not yet been pushed up by the latest folding-process, covered a much larger area than at present. In the neogene period, therefore, the old centre of the land of Borneo was a very pronounced geanticlinal region, whereas the marginal zones in the N.W. and S.E. were true geosynclines.

It is probable that the material of the recent deposits in the eastern syncline came from the West, i.e. entirely from the old Borneo

centre; in fact in Sankulirang the deposits become more and more pelagic, as the sediments lie more eastward, and point to an old marine territory in Macassar Strait (L. RUTTEN, l.c. 1916). It does not seem probable, though, that also the western geosyncline was bounded in the N.W. by a deep sea. First of all there are factors pointing to the smaller significance of this geosynclinal territory than that of the East. The strike of the western geosyncline almost coincided with the northwest coast of the island as is shown on the map of C. SCHMIDT (l.c.). In its south-western elongation nothing is known of a continuation of the subsided area; we find ourselves there in the old landmasses of Sambas [1]) and the Natuna-Islands [2]). One is impressed with the idea that the subsided region, which most likely extended from the Philippines as far as the northwest coast of Borneo, terminated here. There is still something else. In 1914 I obtained from Dr. NIETHAMMER a fragment of limestone from a territory far removed from the Borneo-coast viz. the islet of Mangalum (see sketchmap). It is an Operculina limestone, which, it is true, includes only Operculina complanata Defr., and which on that account may be quaternary as well as tertiary, but it bears a close resemblance to a Lepidocyclina-bearing Operculina-limestone from Pulu Labuansklambu near the northwestern point of Borneo. It cannot be doubted, therefore, but that the the limestone from Mangalum is still met with in litoral facies so far from the coast, so that it seems highly improbable that the Northwestern geosyncline should have been bounded in the North-West by a deep sea. It may be deemed more probable that here lay a subsided area, which at one time was alternately shallow sea, delta-territory or low land; that it was bounded on the one side by the old land-centre of Borneo, on the other by an old continent now transgressed by the Chinese Sea, and formerly perhaps connected with Indo-China, which is also an old continental region. According to this view both J. MOTLEY and MOLENGRAAFF would be right, the former in referring the source of the material of the tertiary formations of Northwest Borneo to the Northwest, the latter in looking for it in the South-east. Moreover this view would also favour the conclusion that the central landmass of Borneo, which had already to contribute so much detritus towards the East and the South-East, was somewhat disburdened as to its contribution towards the Northwest.

Utrecht, 2 Febr 1921.

---

[1]) N. WING EASTON. Versl. Geol. Sectie. Geol. Mijnbk. Gen. I. 1914. p. 179—189.

[2]) P G. KRAUSE. Samml. Geol. Reichsmus. Leiden. (1) V. 1898. p. 221—236.

Chemistry. — *"In-, mono- and divariant equilibria"*. XXI. By Prof. F. A. H. SCHREINEMAKERS.

(Communicated at the meeting of February 26, 1921).

Equilibria of n components in n phases, in which the quantity of one of the components approaches to zero. The influence of a new substance on an invariant (P or T) equilibrium. (Continuation).

In communication XX we have examined the influence of a new substance on the invariant (P or T) equilibrium:

$$E = L_1 + L_2 + \ldots + F_1 + F_2 + \ldots \quad . \quad . \quad . \quad . \quad . \quad (1)$$

With this we have assumed that L_1 $L_2 \ldots$ are liquids and F_1 $F_2 \ldots$ solid substances of unvariable composition. The general form of this equilibrium E is:

$$E = G + L_1 + L_2 + \ldots + M_1 + M_2 + \ldots + F_1 + F_2 + \ldots \quad (2)$$

in which G is a gas and M_1 $M_2 \ldots$ are mixed crystals, which may contain all components or not.

When we know the reaction, occurring in this invariant (P or T) equilibrium, then we may deduce again with the aid of (12) and (15) (XX) which influence has the addition of a new substance. Now we shall consider some special cases of this equilibrium.

We take the unary equilibrium $G + L_1$ viz. an unary liquid in equilibrium with the vapour; this equilibrium is invariant (P or T). As the reaction is $L_1 \rightleftarrows G$, it follows:

$$\Sigma (\lambda x) = x - x_1 \qquad \Sigma (\lambda H) = H - H_1 \qquad \Sigma (\lambda V) = V - V_1$$

in which x, H and V relate to the vapour G.

Now it follows from (12) and (15) (XX):

$$(dT)_P = -\frac{RT(x - x_1)}{H - H_1} = \frac{RT^2 (x_1 - x)}{\Delta W} \quad . \quad . \quad . \quad . \quad (3)$$

$$(dP)_T = \frac{RT(x - x_1)}{V - V_1} = P(x - x_1) \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Herein ΔW is the heat of evaporation of a molecular quantity of liquid and $V - V_1$ the increase of volume at the evaporation of this quantity of liquid. Consequently we refind in (3) and (4) the known formula's. We find, therefore, the rule well-known:

When, at addition of a new substance the concentration of this

substance in the vapour (viz. x) is larger (smaller) than in the liquid (viz. x_1) then the boiling-point under constant pressure is lowered (raised) and the vapour-tension at constant T is raised (lowered);

when the new substance is not volatile (consequently $x = 0$) then the boiling-point under constant P is raised and the vapour-tension at constant T is lowered.

We consider the invariant (P or T) equilibrium

$$E = G + L_1 + L_2 + L_3 + \dots \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

viz. a complex of liquids in equilibrium with their vapour G. We write the reaction which may occur in E:

$$\lambda_1 L_1 + \lambda_2 L_2 + \lambda_3 L_3 \dots \rightleftarrows G \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

When all reaction-coefficients $\lambda_1 \lambda_2 \dots$ are positive, then all liquids take part in the formation of vapour in the ratio $\lambda_1 : \lambda_2 : \lambda_3 \dots$, then we shall say that the vapour has a "average" composition. When, however, in (6) one or more of the reaction-coefficients are negative, then also one or more liquids arise at a time with the vapour; the vapour has then a "non-average" composition.

When we represent by ΔW the heat, wanted to form one quantity of vapour, and the corresponding increase of volume by ΔV, then ΔW and ΔV are positive, unless in very special cases.

Now we may write for (12) and (15) (XX):

$$(dT)_P = -\frac{RT^2 \Sigma (\lambda x)}{\Delta W} \quad \text{and} \quad (dP)_T = \frac{RT \Sigma (\lambda x)}{\Delta V} \quad . \quad . \quad (7)$$

Herein is:

$$\Sigma (\lambda x) = x - (\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \dots) \quad . \quad . \quad . \quad . \quad (8)$$

Now we take a mixture of the liquids L_1 $L_2 \dots$ in such a ratio that it has the same composition as the vapour G. We call this mixture or complex of phases the "reduced mixture". As appears from (6) this mixture has the composition:

$$\lambda_1 L_1 + \lambda_2 L_2 + \lambda_3 L_3 + \dots \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

in which one or more of the coefficients may also be negative. When the vapour has an average composition, then all coefficients in (9) are positive; when the vapour, however, has a non-average composition, then one or more of the coefficients in (9) are negative. When we put now:

$$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \dots = c \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

consequently c is the concentration of the new substance X in the reduced mixture. Instead of (7) we may write now:

$$(dT)_P = \frac{RT^2 (c-x)}{\Delta W} \quad \text{and} \quad (dP)_T = \frac{RT (x-c)}{\Delta V} \quad . \quad . \quad (11)$$

Hence follows the rule[1]):

when, on addition of a new substance the concentration of this substance in the vapour (viz. x) is larger (smaller) than in the reduced mixture (viz. c) then the boiling-point under constant pressure is lowered (raised) and the vapour-tension is raised (lowered) at constant temperature.

Applying those and the following rules and formula (11) we have to consider to following.

When the vapour has an average composition, then c is positive when, however, the vapour has a non-average composition, then c may be as well positive as negative; $x—c$ is then always positive when c is negative.

When we substitute in the rule above "reduced mixture" by "liquid" then we refind the rule, which is true for the addition of a new substance to the unary equilibrium $L + G$.

When the added new substance is not volatile, then $x = 0$; (11) passes then into:

$$(dT)_P = \frac{RT^2 c}{\Delta W} \quad \text{and} \quad (dP)_T = -\frac{RTc}{\Delta V} \quad . \quad . \quad . \quad . \quad (12)$$

When we keep in mind that c may be as well positive as negative, then we find the rule:

when the vapour has an average composition, then, on addition of a new substance, the boiling-point under constant pressure shall be raised and the vapour-tension at constant temperature shall be lowered;

when the vapour has a non-average composition, then for $c > 0$ this rule is true also; for $c < 0$ however an opposite rule is true.

We may for some cases also represent the above results geometrically. Let us firstly consider the addition of a new substance to the binary equilibrium $E = L_1 + L_2 + G$. In figs 1—4 the sides ZX and ZY of the ternary concentration-diagram are partly drawn· a_1 a_2 and a represent the two liquids and the vapour G of this equilibrium E. In figs 1 and 3 this vapour a has an average, in figs 2 and 4 a non-average composition. When, at constant T or P, we add a new substance, then the liquids L_1 and L_2 trace the curves $a_1\, b_1\, c_1$ and $a_2\, b_2\, c_2$; the vapour traces curve $a\, b\, c$. Each of the

[1]) This rule is deduced already formerly for a definite case, viz. the addition of a new substance to the binary equilibrium $L_1 + L_2 + G$. F. A. H. SCHREINEMAKERS. Zeitschr. f. Phys. Chem. **38** (1901) 252.

phases of the ternary equilibrium traces, therefore, a three-phases-curve. Now we assume that on addition of very little of the new substance the two liquids and the vapour are represented by the points $b_1\,b_2$ and b; consequently these points are situated in the immediate vicinity of $a_1\,a_2$ and a; for sake of clearness they are drawn however in the figures on greater distance. Relating to the situation of the vapour-point b with respect to the line b_1 b_2 we shall say that the three-phases-triangle $b_1\,b_2\,b$ turns his vapour-point in figs 1 and 2 away from the side YZ of the components-triangle and in figs 3 and 4 towards this side.

When we represent the reaction by

$$\lambda_1 L_1 + \lambda_2 L_2 \rightleftarrows G \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (13)$$

then is:

$$\Sigma(\lambda x) = x - \lambda_1 x_1 - \lambda_2 x_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (14)$$

in wich λ_1 and λ_2 are defined by

$$\lambda_1 + \lambda_2 = 1 \quad \text{and} \quad \lambda_1 y_1 + \lambda_2 y_2 = y \quad . \quad . \quad . \quad . \qquad (15)$$

so that λ_1 and λ_2 are linear functions of y. When we consider x and y in (14) as running coördinates, then $\Sigma(\lambda x) = 0$ represents

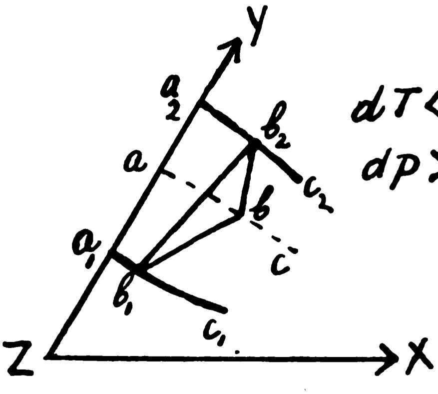


Fig. 1.

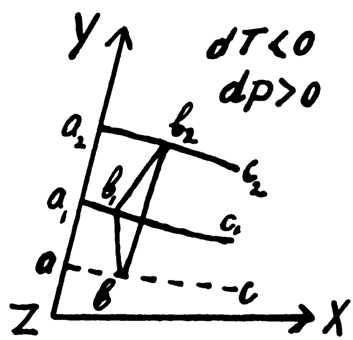


Fig. 2.

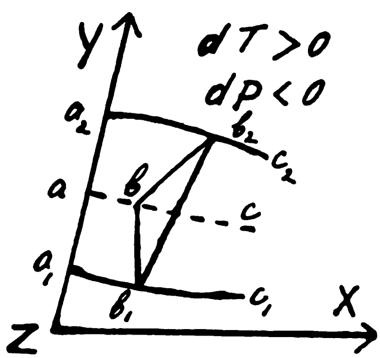


Fig. 3.

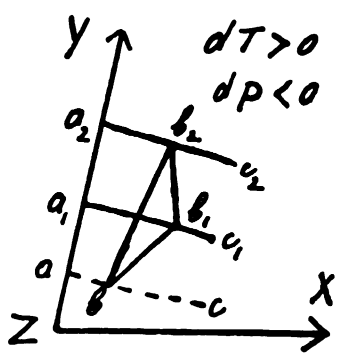


Fig. 4.

the equation of the straight line $b_1 b_2$. When we put viz. $x = x_1$ and $y = y_1$ then follows from (15) $\lambda_1 = 1$ and $\lambda_2 = 0$ so that $\Sigma(\lambda x) = 0$; consequently the line goes through the point b_1. In the same way it is apparent that this line goes through the point b_2. When the point xy is not situated on the line $b_1 b_2$, then $\Sigma(\lambda x)$ is not zero, but positive or negative, in accordance to the situation of this point on the one or on the other side of this line. Now we imagine in the figs 1—4 a horizontal line drawn through a; as b is situated in the immediate vicinity of a, this line goes also, by approximation, through b; we call r the point of intersection of this line with the line $b_1 b_2$. On this horizontal line in r consequently $\Sigma(\lambda x) = 0$, at the right of r $\Sigma(\lambda x)$ is > 0 at the left of r $\Sigma(\lambda x)$ is < 0. Consequently when we trace the line $b_1 b_2$ in the direction from b_1 towards b_2, then $\Sigma(\lambda x)$ is at the right of this line positive and negative at the left of this line.

As in figs 1 and 2 the vapour point b is situated at the right of the line $b_1 b_2$ consequently $\Sigma(\lambda x)$ is positive. Therefore, follows from (7):

$$(dT)_P < 0 \quad \text{en} \quad (dP)_T > 0 \quad . \; . \; . \; . \; . \quad (16)$$

which is also indicated in those figures. Hence follows: when we trace the three phases-curves (viz. abc, $a_1 b_1 c_1$ and $a_2 b_2 c_2$) beginning at their binary terminating-points (viz. $a\, a_1$ and a_2) then the temperature decreases under constant pressure and the pressure increases at constant temperature.

In figs 3 and 4 the vapour-point is situated at the left of the line $b_1 b_2$ so that $\Sigma(\lambda x)$ is negative. Now it follows from (7):

$$(dT)_P > 0 \quad \text{en} \quad (dP)_T < 0 \quad . \; . \; . \; . \; . \quad (17)$$

which is also indicated in the figs 3 and 4. Hence it follows, therefore: when we trace the 3 three-phases-curves, beginning at their binary terminating-points, then the temperature increases under constant pressure and the pressure decreases at constant temperature.

We may summarise those results in the following way:

When we trace the three-phases-curves of the ternary equilibrium $L_1 + L_2 + G$ beginning at their binary terminating-points.

then under constant P the temperature decreases and at constant T the pressure increases, when the three-phases-triangle turns its vapour-point away from the side (with the binary terminating-points) (figs. 1 and 2)

and under constant P the temperature increases and at constant T the pressure decreases, when the three-phases-triangle turns its vapour-point towards the side (with the binary terminating-points) (figs. 3 and 4).

The previous rule is deduced in the supposition that the three-phases-triangle is situated in the immediate vicinity of the side with the binary terminating-points. As however by this is determined the direction in which T and P increase or decrease along the three-phases-curves, the rule remains also true, when the three-phases-triangle moves further away from this side. When the three-phases-triangle passes into a straight line, then on the three-phases-curves a point of maximum or minimum pressure occurs.

The previous considerations are also valid when the added new substance is not volatile. The threephases-curve $a\,b\,c$, which indicates the composition of the vapour, however, then not goes from a into the triangle, but it falls on the side YZ. Of the three-phases-triangle $bb_1\,b_2$ the angle-points b_1 and b_2 are situated, therefore, within the concentration-diagram, but, as is drawn in fig. 5, the point b is situated on YZ in the immediate vicinity of point a.

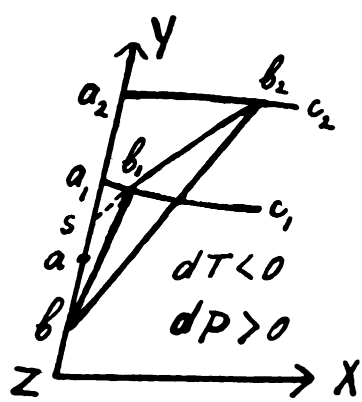


Fig. 5.

Now we shall indicate by s the point of intersection of $b_1\,b_2$ with the side YZ; in fig. 5 this point of intersection is drawn, in the other figures, however, it is not drawn. We may now distinguish two cases, viz.

1° the point a (and consequently also b) is situated on the other side of s as the points a_1 and a_2.

2° the point a (and consequently also b) is situated on the same side of s as the points a_1 and a_2.

The first case is represented in fig. 5, the second in the figs. 3 and 4; we imagine, however, in the two latter figures the point b on the side YZ in the immediate vicinity of point a. Now we call those figures the new figures 3 and 4. In those new figures 3 and 4, the three-phases-triangle $b_1\,b_2\,b$ turns, just as in the old figures its vapour point b towards YZ; in the new figures $(dP)_T$ and $(dT)_P$ have consequently the same sign as is indicated in the old figures.

Notwithstanding that also in fig. 5 the point b is situated on the side YZ, it is yet apparent that we must say here, that the three-phases-triangle turns its vapour-point b away from the side YZ. Consequently $(dT)_P$ and $(dP)_T$ must have the sign, indicated in fig. 5.

Consequently on addition of a new substance, which is not volatile,

in the case of the new figures 3 and 4 the boiling-point is raised under constant P and at constant T the vapour-pressure is lowered;

in the case of fig. 5 under constant P the boiling-point is lowered and at constant T the vapour-pressure is raised.

Easily we see that this all is also in accordance with the deductions from formula (12). In fig. 3 the vapour a has a mediate composition and c in formula (12) is, therefore, positive. In figs 4 and 5 the vapour has a not-mediate composition but c is in fig. 4 still positive, but negative in fig. 5.

The previous considerations are valid still also, when also one of the components of the binary equilibrium $E = L_1 + L_2 + G$ is not volatile. When Y is this component which is not-volatile, then in the figs 1—5 point a coincides with Z. When also the new substance is not-volatile, then the vapour remains always represented by point Z; when the new substance is volatile indeed, then the vapour proceeds along the side ZX. Also in those cases, the boiling-point shall be raised or lowered under constant P and the vapour-pressure shall be lowered or raised at constant T, dependent on the situation of the three-phases-triangle.

In a similar way we may deduce also the influence of a new substance on the ternary equilibrium $E = L_1 + L_2 + L_3 + G$. When we represent the reaction by:

$$\lambda_1 L_1 + \lambda_2 L_2 + \lambda_3 L_3 \rightleftarrows G \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

then is:

$$\Sigma(\lambda x) = x - \lambda_1 x_1 - \lambda_2 x_2 - \lambda_3 x_3$$

and λ_1 λ_2 and λ_3 are defined by:

$$\lambda_1 + \lambda_2 + \lambda_3 = 1 \qquad \lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3 = y \quad . \quad . \quad (19)$$
$$\lambda_1 z_1 + \lambda_2 z_2 + \lambda_3 z_3 = z$$

When we take a regular tetrahedron for concentration-diagram, then the phases of the ternary equilibrium E are represented by four points on the side-plane YZU; we shall call those points a_1 a_2 a_3 and a. When, at constant P, we add a new substance X, then each phase traces a four-phases-curve; we call those curves $a_1 b_1 c_1$, $a_2 b_2 c_2$. $a_3 b_3 c_3$ and $a\,b\,c$.

The four-phases-tetrahedron $b_1\, b_2\, b_3\, b$ turns now its vapour-point b either towards the side-plane YZU or away from that plane. Easily now we find the rule:

when we follow the four-phases-curves of the quaternary equilibrium $L_1 + L_2 + L_3 + G$ starting from their ternary terminating-points, then

under constant P the temperature decreases and at constant T

the pressure increases, when the four-phases-tetrahedron turns its vapour-point away from the side (with the ternary terminating-points)

and under constant P the temperature increases and at constant T the pressure decreases, when the four-phases-tetrahedron turns its vapour-point towards the side (with the ternary terminating-points).

We now take en invarient (P or T) equilibrium, in which occur, besides a liquid L yet also the mixed-crystals M_1 M_2 ...; we represent this equilibrium by

$$E = M_1 + M_2 + M_3 + \ldots + L \quad . \quad . \quad . \quad . \quad . \quad (20)$$

and the occurring reaction by:

$$\lambda_1 M_1 + \lambda_2 M_2 + \lambda_3 M_3 + \ldots \rightleftarrows L \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Of course one or more of the reaction-coefficients may be negative. When all coefficients are positive, so that the liquid has a mediate composition, then (21) represents a congruent melting of the mixed-crystals; when one or more of the coefficients are negative, so that the liquid has a not-mediate composition then (21) represents a conversion of the one of mixed-crystals in the other, with formation of liquid, consequently an incongruent melting.

We shall call a mixture of the mixed-crystals M_1 M_2 ... taken in such ratio, that it has the same composition as the liquid, the "reduced complex of mixed-crystals". Now we have:

$$\Sigma (\lambda x) = x - \lambda_1 x_1 - \lambda_2 x_2 - \lambda_3 x_3 - \ldots = x - c \quad . \quad . \quad (22)$$

in which, therefore, c represents the concentration of the new substance X in the reduced complex of mixed crystals; consequently c may be as well positive as negative.

Therefore, we obtain again formula (11) in which ΔW is the heat, which is necessary for the congruent or incongruent melting, and ΔV the change in volume, occurring with this. We may assume again that in general ΔW is positive; ΔV may be, however, positive or negative.

When we only consider the change of the congruent or incongruent melting-point under constant pressure consequently, then follows the rule:

when, on addition of a new substance, the concentration of that substance in the liquid (*viz* x) is greater (smaller) than in the reduced complex of mixed crystals (*viz* c) than the (congruent or incongruent) melting-point of the mixed crystals under constant pressure decreases (increases).

When the new substance, which is added, does not occur in the mixed-crystals, then, as x_1 x_2 x_3 ... are zero, also $c = 0$. (11) then passes into:

$$(dT)^P = \frac{-RT^2x}{\Delta W} \quad \text{and} \quad (dP)_T = \frac{RTx}{\Delta V} \quad . \quad . \quad . \quad . \quad (23)$$

Hence it follows:

on addition of a new substance, which does not occur in the mixed crystals, the (congruent or incongruent) melting-point of the mixed-crystals under constant pressure is lowered.

We shall briefly consider more in detail the case that a new substance is added to the binary invariant (P or T) equilibrium; $E = M_1 + M_2 + L$. For this we imagine that in the figs 1—4 the points a_1 a_2 and a represent the two mixed-crystals M_1 and M_2 and the liquid L. In the figs 1 and 3 then a congruent melting takes place and in the figs 2 and 4 an incongruent melting.

When we add at constant T or P a new substance, which occurs also in the mixed-crystals, so that ternary mixed-crystals occur, then M_1 and M_2 trace the three-phases-curves $a_1\,b_1\,c_1$ and $a_2\,b_2\,c_2$; the liquid L traces the three-phases-curves $a\,b\,c$. In figs 1 and 2 the three-phases-triangle $b_1\,b_2\,b$ turns its liquid-point b away from the side YZ of the components-triangle, in figs 3 and 4 towards this side.

In a similar-way as with the equilibrium $L_1 + L_2 + G$ we now find that also for the equilibrium $M_1 + M_2 + L$ the sign of $(dT)_P$ must be the same as is indicated in the figs 1—4; this is also valid for $(dP)_T$ when ΔV is > 0; for $\Delta V < 0$ we must take, however the opposite sign for $(dP)_T$.

Consequently we find the rule:

when we trace the threephases-curves of the equilibrium $M_1 + M_2 + L$ starting from their binary terminating-points,

then under constant P the temperature decreases, when the three-phases-triangle turns its liquid-point away from the side (with the binary terminating points) (figs 1 and 2),

and under constant P the temperature increases, when the three-phases-triangle turns its liquid-point towards the side (with the binary terminating-points) (figs 3 and 4).

The reader himself may easely find the rule for the change of pressure at constant T.

When the added new substance does not occur in the mixed-crystals, so that those rest binary ones, then in the figs 1—4 the curves $a_1\,b_1\,c_1$ and $a_2\,b_2\,c_2$ must coincide with the side YZ. The three-phases-triangle $b_1\,b_2\,b$ is then always situated with the angle-points b_1 and b_2 on the side YZ, while the liquid-point b is situated within the concentration-diagram. Therefore, we imagine in the figs

1 and 2 the points b_1 and b_2 on YZ in the vicinity of a_1 and a_2. Then from fig. 1 a diagram arises, such as is drawn in fig. 6.

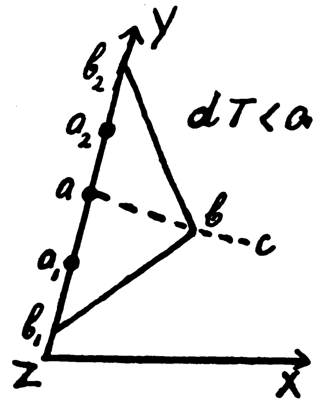


Fig. 6.

As the three-phases-triangle now turns always its liquid-point away from the side YZ, the temperature must, according to the previous rule, decrease under constant P, viz. when we trace the curves starting from their binary terminating-points. This is in accordance also with the first formula (23).

In the previous communication XX we have deduced the relation:

$$(dP)_T : (dT)_P = -\left(\frac{dP}{dT}\right)_{x=0} \quad \ldots \ldots \quad (24)$$

Hence appears: when in the invariant (P or T) equilibrium (which consequently is monovariant) the pressure increases at increase of T, then $(dP)_T$ and $(dT)_P$ have opposite signs; when, however, the pressure decreases at increase of T, then $(dP)_T$ and $(dT)_P$ have the same sign. We now may express this in the following way:

we add a new substance to an invariant (P or T) equilibrium;

when in this equilibrium the pressure increases at increase of temperature, then the influence on the pressure at constant T is opposite to the influence on the temperature under constant P;

when in this equilibrium the pressure decreases at increase of temperature, then the influence on the pressure at constant T is the same as the influence on the temperature under constant P.

It is evident that "influence" means here the sign of the change of pressure of temperature.

Leiden. Inorg. Chem. Lab.

(*To be continued*).

Physiology. — "*On the serological specificity of haemoglobin in different species of animals*". By K. LANDSTEINER. (Communicated by Prof. C. H. H. SPRONCK).

(Communicated at the meeting of January 29, 1921).

The knowledge of species-specificity is based principally on morphological facts, on the phenomena of inheritance and transplantation, and on serological reactions. The first allow the direct inference that species-specificity is a property of all the constituents of the body; serological data, on the other hand — as far as animal organisms are concerned — bear first of all on the proteins of the blood serum.

Concerning these latter there exist numerous results of precipitin tests, of which the greater part were carried out by NUTTALL[1]), and which may be summarised, that serological relationship in general corresponds to the zoological classification and that the difference of the precipitin reactions increase in a measure corresponding to the difference in the scale of animal classification.

For the other constituents of the body an analogous relationship is probable, and is often accepted as a matter of fact; this subject, however has been far less thoroughly investigated than the properties of the bloodserum.

Some proteins differ in a greater degree (protein of the eye-lens[2]) and the horny substance[3])) and others (casein[4])) in a smaller degree from the specificity type mentioned i. e. their specificity is more or less independant of the animal species. In these cases we cannot exclude the possibility that the species-specificic structure although present is concealed by other structures.

Whether the haemagglutinating immune-sera with regard to their group reactions correspond as well to the zoological classification, as the before mentioned precipitins, is indeed still doubtful[5]).

---

1) NUTTALL. Bloodimmunity and bloodrelationship, Cambridge (1904).

2) UHLENHUTH. Festschr. f. Rob. Koch.

3) KRUSIUS. Arch. f. Augenheilk. **67**. (1910).

4) VERSELL. Zeitschr. f. Immunitätsf. **24**. 267. (1915).

5) Cf. LANDSTEINER u. REICH. Zeitschr. f. Hyg. **58**. 227. (1907).
BROCKMANN. Zeitschr. f. Immunitätsf. **9**. 114 (1911).

The same applies to the immune sera against the cells of various organs [1]).

As far as the globulins of milk are concerned it may be accepted as highly probable and experience goes to prove, that they behave quite similarly to the serum-globulin, and the same may be said for all proteins which have a very similar chemical composition as the serum-proteins as regards the kind and quantity of the amino-acids which build up their molecule.

But when one considers proteins such as haemoglobin which in their chemical structures i.e. in their construction out of amino-acids differ widely from the proteins of the serum, a note-worthy problem presents itself.

If these behave in regard to the species specificity similarly to the serum protein, two possibilities present themselves: either the species-structure in such proteins and in serum proteins is realised in wholly different and independent ways, or else there is a relation in this way that among the different proteins of a species one corresponding smaller or larger group (or several groups) of the molecule is the bearer of the species-specificity.

In the latter case the problem would arise of demonstrating this species-specific nucleus. In the other and perhaps less probable case there would be a similarity of chemical structure of two homologous proteins in related animal-species, but no chemical similarity would be found between the different proteins of the same species. From this it appears indicated to investigate the different proteins as to their species-specificity and I have, therefore, undertaken the investigation of the precipitins against haemoglobin. This substance is well suited for the examination of the question raised because the composition of its protein constituent i.e. of globin varies widely from that of serum globulins and serum albumins.

Leblanc [2]), Ide [3]) and Demees [4]) were the first to report on the formation of precipitins against haemoglobin. Thomsen [5]) investigated the question of species-specificity and came to the conclusion that anaphylaxis against erythrocytes is to a certain extent species-specific. The greater number of his experiments, as also the researches of

---

1) Salus. Bioch. Zeitschr. **60**. 1. 1914.

2) La Cellule **18**. 337. (1901).

3) „ „ **20**. 263. (1902).

4) „ „ **24**. 423. (1907).

5) Zeitschr. f. Immunitätsf. **3**. 539. (1909). Cf. Weichardt. Zeitschr. f. Immunitätsf. **14**. 609. (1912). On the specificity of haemoglobin crystals. cf. Reichert a. Brown. Carnegie Institution Publication Washington 1909.

BRADLEY and SANSUM [1]), were carried out with a blood solution and not with isolated haemoglobin. In a second series of experiments with crystallised haemoglobin these authors found that guinea pigs sensitized with dog's haemoglobin reacted strongly (no acute death) on dog's bloodsolution, in a lesser degree with such solutions from pig, and turtle and not at all with those from fowl, calf, horse, goat, rabbit, rat, guinea pig, sheep and man. This result does not allow of a definite conclusion yet.

As the species differences of haemoglobin certainly are due to its protein components, the experiments on globin ought to be mentioned here. BROWNING and WILSON [2]) found a strongly homologous reaction with an immune serum against guinea pig globin, weak complement fixation with rabbit-globin, and none with ox-globin. In a recent publication [3]) these same authors find that an immune serum against ox-globin reacts also on globin of goats, guinea pigs and ducks but not on rabbit-globin. On this point they state, "Thus while evidence of species-specificity exists in certain cases, there is also a wide though not universal, community of antigenic properties shared by the globin of widely separated animal-species".

It ought still to be mentioned that some time ago SCHMIDT and BENNETT [4]) on the ground of some previous (FORD and HALSEY) and of their own negative findings, deny the ability of haemoglobin to produce antibodies and they ascribe the positive results to impurities in the injected materials.

BROWNING and WILSON, who themselves obtained a distinctly active serum against haemoglobin, believe that they can explain the negative results in this way,' that haemoglobin possesses only a weak antigenic property. I am inclined to agree with this view [5]), as in my experiments with 5 animals with intensive immunisation (2 gr. of haemoglobin and more) only one gave a good active serum, 2 others very weak sera, whereas the sera of the remaining two were not active at all. I have no cause to doubt that the reactions obtained are really attributable to the haemoglobin, seeing that the active sera gave only very weak haemolysis with

---

[1]) Journ. Biol. Chem. **18**. 497. (1914).

[2]) Journ. of Path. a. Bact. **14** 174. (1909). Cf. GAY a. ROBERTSON J. Exper. Med. **17**. 535. (1913).

[3]) Journ. of Immunol. **5**. 417 (1920).

[4]) Journ. infect. Dis. **25**. 207. (1919).

[5]) Perhaps alteration occur during the process of isolation, which diminish the antigenic properties.

horse blood and that only a long-continued standing produced a just perceptible turbidity with dilutions of horse-serum [1]).

LEBLANC's [2]) results also are in favour of the antigenic property of haemoglobin, for he found, that the haemoglobin is carried down with the precipitate produced by precipitin-serum.

As I intend further to pursue this investigation I shall perhaps come back to this point later and also describe more fully the results which are only shortly summarised here.

In my experiments I used for the immunisation of rabbits crystallised horse-haemoglobin, prepared in the same manner as described by IDE and DEMEES; i.e. defribinated horse blood was centrifugalised, the sediment washed five times and again centrifugalised, the bloodcorpuscles dissolved in etherised water of double the volume of the original blood, kept in the icechest for one or two days, and during this time repeatedly shaken, and finally decanted to get rid of any sediment which might eventually still be present. In order to free it from stromata and globulin, it was mixed with an equal volume of saturated ammonium sulphate solution, the precipitate filtered off through a folded filter. To the clear filtrate ammonium sulphate crystals were added until the haemoglobin (standing in the cold) was precipitated, the filtered precipitate washed with ammonium sulphate solution, suspended or dissolved in water, some ether added and the mixture dialysed. Finally 1 % of sodium chloride was added.

In the test tube experiments I took solntions which had been prepared from very well washed blood corpuscles with water and ether, the water amounting to twice the blood volume. The solutions were completely freed from susupended matter by filtration through an asbestos filter. In the case of horseblood, moreover, the solution of crystallised haemoglobin was used.

The tests for specificity gave the following result.

Technique of the experiments: The 5 % haemoglobin solution was diluted to 1 : 500 and 0.2 cc. of this taken. To this were added 3 capillary drops of 0.04 cc. each, of immune serum and left to stand one hour at room temperature and the reading taken. Horse-serum diluted 500 times.

Tests were also made with all solutions in dilutions of 1 : 100 and 1 : 2.500. They are not given in detail because they agree with the experiments of the 1 : 500 dilution.

Haemoglobin solutions.													serum
man	dog	horse	ass	pig	ox	sheep	goat	rabbit	rat	mouse	fowl	pigeon	horse
0	0	+++	++±	0	0	0	0	0	0	+	0	0	0

[1]) A solution of the horse haemoglobin reacted in the same manner against an ordinary anti-horse precipitin.

[2]) IDE. l.c. p. 263.

According to these tests, the species-specificity of haemoglobin is certainly not much less marked than that of the serum-protein.

Somewhat striking is the fairly strong reaction with mouse-blood, yet similar reactions are also known to occur in the case of precipitin reactions with serum albumin. It would be interesting to prove, whether the group reaction mentioned also takes place with other precipitin sera against horse-haemoglobin. After the tests had stood about 20 hours some further weak group-reactions were observed.

A reaction for haemoglobin solutions generally succeeds with the help of the inhibition test [1]). If to the tests, containing the quantities of horse-haemoglobin and the corresponding immune-serum mentioned above, 0.05 cc. of a 1 % solution of haemoglobin belonging to different animal species was added, an inhibition of the precipitation took place, which as experiments with the addition of smaller quantities of haemoglobin showed, was with one exception strongest in the case of horse and ass haemoglobin.

Haemoglobin solutions.

horse	ass	dog	ox	sheep	rabbit	guinea-pig	rat	control
0	0	0	0	0	0	+	±	++±

I also endeavoured to investigate by means of the inhibition reaction the question of the presence of a common species-specific group in serum-protein and haemoglobin. From this it appeared that by the addition of 0.05 cc. of normal serum belonging to different animal species, to the mixtures of haemoglobin and their antibodies, the precipitation was inhibited.

When serum from 11 different species of animals were tested simultaneously the inhibition was strongest and equally marked in the case of the serum from the horse, ass and two other kinds of serum. The result, therefore, does not yet permit of any definite conclusions. It is, however, possible, that through further variation (immune bodies against other kinds of haemoglobin) and perhaps through modifications of the method of experiment, a positive result will be obtained.

[1]) LANDSTEINER. Bioch. Z. 93. 115, 104. 280. HALBAN u. LANDSTEINER. Munch. med. Woch. 1902. N°. 12.

Physiology. — "*On Heterogenetic Antigen*". By K. LANDSTEINER. (Communicated by Prof. C. H. H. SPRONCK).

(Communicated at the meeting of February 26, 1921).

It is well known that FORSSMAN [1]) discovered the fact, that by injection of organs of guinea pigs into the rabbit the formation takes place of a strongly active specific hemolysin against sheep's blood. It soon was observed that the same property may be ascribed to the organs of a large number of animals, e.g. to those of the horse, the cat and the fowl, whereas these active substances are absent in other animal species.

These substances are generally called heterogenetic antigens; the hemolysins generated by their injection are termed heterogenetic antibodies.

A complete account of the investigations on this subject and their results will be found in the publications referred to below. In this paper I shall record only the results regarding the heterogenetic antigen which are of special interest for my work.

DOERR and PICK [2]) found that the said substances of the organs are resistant to the influence of alcohol; SACHS and GEORGI [3]) record the use of alcoholic extracts, FRIEDBERGER [4]) (POOR, SUTO, SCHIFF) report the solubility in alcohol of the heterogenetic antigen occurring in the urine (see DOERR and PICK) of animals. SORDELLI and FISCHER [5]) found that the active constituents of the horse's kidney can be split into two fractions by treating with alcohol (and ether), the one of which fractions, soluble in alcohol, combines in vitro with immune bodies, but does not bring about immunization, while the part that is insoluble in alcohol and ether, combines only inappreciably in vitro; however, *it generates immune bodies*. (SORDELLI and PICO [6]) also observed, just as SACHS and GUTH [7]) did, a flocculation

---

[1]) J. FORSSMAN. Biochem. Zeitschr. **37**. pag. 78 (1911).

[2]) R. DOERR en R. PICK. Biochem. Zeitschr. **50**. pag. 129 (1913).

[3]) SACHS en GEORGI. Zeitschr. f. Immun. **21**. pag. 346 (1914).

[4]) FRIEDBERGER. Berl. Klin. Wochenschr. 1913, N. 34. Zeitschr. f. Immun. **18** pag. 269 (1913) **28**. pag. 217 (1919).

[5]) SORDELLI en FISCHER. Revista del Inst. Bacteriol. Buenos-Aires. Vol. I. 1918 pag. 229.

[6]) SORDELLI en PICO. Revista del Inst. Bacteriol. Buenos-Aires Vol. II. 1919. p. 261.

[7]) SACHS en GUTH. Mediz. Klinik 1920 No. 6.

reaction of the alcoholic organ-extracts with the heterogenetic immune serum).

It is evidently difficult to explain these results as it would seem as if specific immune bodies against one substance are formed by the injection of another.

In my experiments, which were performed for the greater part without my having knowledge of the publications of SORDELLI and his coworkers, I obtained results that agree with their observations in main points, as I also found that the *alcoholic extracts* (and the ether-extracts) of a horse's kidney — though in vitro their capacity of combination is very great — *do not bring about any or any appreciable formation of antibodes.* However, further examination induced me to give a modified explanation of this matter.

In my experiments preparations of horse's kidney were injected into rabbits. In one of these the procedure was as follows[1]):

The kidneys were twice ground finely then sifted through a course-mesh, and through a fine-mesh-sieve, emulsified with three times their volume of 0.9 per cent salt solution, placed in the ice-chest for 24 hours, being shaken repeatedly, and finally percolated (I). The turbid extract was digested with six times the volume of 95 % alcohol during two days at room temperature, subsequently after filtration digested once more for 24 hours with 1 vol. of alcohol and the insoluble precipitate was taken up in 0.9 per cent salt solution (II). A third portion was prepared like II, but here the substance that was obtained by evaporating the alcoholic extracts to dryness, was emulsified in 0.9 per cent salt solution, and was added to the suspension of the first precipitate (III). A fourth portion was digested similarly as to II with alcohol at room-temperature, and then the precipitate, obtained after filtration, was boiled for half an hour with one volume of alcohol and the insoluble part was suspended 0.9 per cent salt solution (IV) and finally the aqueous extract or the suspension was heated for half an hour in boiling water (V).

All the suspensions were brought to the original volume of the extract, and 0.25 % phenol was added.

The following table shows the hemolytic action of rabbit's sera, after two intraperitioneal injections (at an interval of a week) of the five extracts described above. The amount of every injection was about equal to 1 grm of kidney-substance; allowance should here be made to the fact that in percolating a large portion of the kidney-substance is left behind.

The dilution of the sera was 1 : 250; 0.5 c.c. of this dilution is added 0.5 c.c. of guinea pig complement 1 : 10 and 0.05 % sheep's blood.

---

[1]) The subject will be discussed at length elsewhere.

In the following table the degree of hemolysis after one hour is indicated as follows:

c = complete; $a.c.$ = almost complete; m = marked; d = distinct; f = faint; tr = trace; o = none;

Injection of preparation	I				II			III				IV				V			
Number of the rabbits	1	2	3	4	7	8	9	11	12	13	14	15	16	17	18	21	22	23	24
	c.	c.	a.c.	c.	m.	f.	o.	d.	m.	d.	m.	o.	o.	tr.	tr.	a.c.	c.	c.	c.

Sera 1, 2 and 4 also dissolved completely in dilutions of 1 : 1000, 1 : 500, 1 : 500. The experiment described here, indicate that the kidney-substance, treated with alcohol at room temperature, has still the power to generate hemolysins but in a much smaller degree than the original emulsion. Under the actual conditions this immunizing capacity was abolished almost completely by boiling with alcohol. The antigenic property is still well preserved after boiling in aqueous solution (DOERR and PICK), although it should be noted that the serum titre was lower than after the injection of the unheated material.

My investigation therefore proved that by treating with alcohol, substances are brought into solution, which react with hemolysin in vitro; that however, they have lost their immunizing capacity, and that the residue which is left after alcohol-extraction, has a much weaker antigenic property, which is all but abolished after heating with alcohol.

It seems to me that the most obvious interpretation of the facts recorded is, that the antigen is made up of a part, that is essential to immunization and is probably built up of protein, and of another part combined with this, which contains the specific reacting groups and is perhaps a lipoid. Furthermore it appears, that the latter can be separated from the former by means of alcohol.

This view is supported chiefly by results obtained in experiments that I published some time ago [1]). These experiments proved that there are substances which probably react specifically in vitro, without being antigens.

As it will be convenient to have a special name for such

[1]) K. LANDSTEINER. Biochem. Zeitschr. **93**. pag. 106 (1919) Biochem. Zeitschr. **104**. pag. 280 (1920).

substances which react specifically with immune serum which however do not immunize and consequently are not antigens, I propose to term them as *haptens*.

As to the chemical composition of these haptens from the horse's kidney, I obtained results from preliminary tests, which in some way correspond with those achieved by WERNICKE and SORDELLI [1]). However, I was not able to prove, whether the data concerning the solubility of this substance might be confirmed in all points.

If it is true that organs of different animal species, which possess heterogenetic antigens, contain compounds of protein and one or similar haptens, it can be understood that material derived from quite different sources should bring about the formation of antibodies producing a similar effect.

The Hague. *From the R. K. Hospital Laboratory.*

[1]) WERNICKE en SORDELLI. Revista del Inst. Bacteriol. Buenos-Aires Vol. II. p. 281 1919.

Physiology. — "*Concerning the Sensitivity to Poisons in Animals suffering from Avitaminosis.*" By W. STORM VAN LEEUWEN and F. VERZÁR. (Communicated by Prof. R. MAGNUS).

(Communicated at the meeting of November 27, 1920).

EIJKMAN found in 1893 that fowls, fed on polished rice, develop polyneuritis, and that this disease could be prevented by an undermilled rice-diet, or by adding to the polished rice the "silverlayers" detached from it. He found, moreover, that in man an abundant diet of polished rice subserved the development of beri-beri, whereas the disease was not produced, or if already produced, was cured when the silverlayers had been added to the rice. Later on it appeared that these findings bear on a special case of a general rule. Not merely in unhusked rice, but also in all sorts of foodstuffs, constituents occur that are essential for the normal growth and the healthy condition of men and animals, even though these foodstuffs contain an adequate amount of the proper, long known nutritious element. These constituents, whose real nature is unknown as yet, are often classed together as "vitamins".

Through the latest achievements in this field, notably of American workers, our knowledge of these things has largely increased. We now know that the term "vitamin" is not applied to a single substance, but that it includes various accessory foodstuffs, the fat soluble *A* and the water soluble *B*, and perhaps a third substance *C*; we are also aware that according as various nutritious elements are wanting in the foodstuffs, the symptoms of a disease may be widely different and that these symptoms may also be very different in different animals. Many experimental data concerning the occurrence of vitamins in various foodstuffs have been brought forward. There is one thing, however, of which we must still admit great ignorance, viz. the causation of the symptoms of the disease, revealing themselves in animals that suffer from a deficiency of vitamins. Some hold that under certain conditions this deficiency brings about a predisposition to infection with certain bacteria, but this supposition does not afford complete satisfaction and certainly does not clarify every case.

The symptoms of avitaminosis that present themselves are disorders

in the innervation of the striated muscles, disorders in the innervation of the unstriated muscular tissue (paralysis of the esophagus and the gastro-intestinal canal in the fowl, among others) and also trophical disorders. These disturbances are no doubt partly of a nervous character; in animals suffering from avitaminoses, e.g. in fowls attacked by polyneuritis, distinct anomalies occur in the peripheral nerves. These anatomic anomalies, however, cannot be the decisive factor in the origin of the avitaminosis, since very often in animals, exhibiting marked symptoms of avitaminosis, an injection of the vitamins concerned may exert a highly curative effect in a very short time, so that the animal may practically be cured. This leads to the conclusion that part of the disorders occurring with avitaminosis are doubtless functional, i.e. the organs of the animal do not react on the stimuli present at that moment, but may recover, or nearly so, their normal function again through the addition of a special substance: vitamin.

It should seem then that with avitaminosis the condition frequently occurs that several striated and unstriated muscles do not indeed react, but that they may be incited to reaction through the addition of a special substance.

The question, therefore, arises: Why do these striped and smooth muscles not react?

In our judgment three possibilities must be considered, anyhow so far as the unstriated muscular tissue is concerned:

Firstly, the organs do not react, because the substance which has to stimulate the organ is not present in an adequate amount.

Secondly: the organs do not react because their sensitivity to stimulating substances, even if present in an adequate amount, is lessened.

Thirdly: the sensitivity of the organs is normal, there is sufficient quantum of stimulating substances, but specific (colloidal) substances are wanting in the body of the animals, which have to facilitate or to promote the action of the stimulating substances through the organs.

An intimation that influences on the sensitivity of unstriated muscular tissue are in operation in the symptoms of avitaminosis, is found in a report by UHLMANN [1]), who showed that in a vitamin-preparation, orypan, there is a substance which plays an influence on unstriated muscular tissue which resembles the influence of pilocarpin.

Without having taken any cognizance of UHLMANN's researches,

---

[1]) FR. ULHMANN. Beiträge zur Pharmakologie der Vitamine. Habilitationsschrift.

VERZÁR and BÖGEL had examined the influence of extracts, which certainly contained fat soluble *A* or water soluble *B* on various surviving organs, and had found this influence to be inappreciable. However, since owing to external circumstances, they were not in a position to examine also the "vitamin"-properties of their extracts, and since their extracts were most likely different from UHLMANN's, UHLMANN's finding is by no means disqualified by their investigation.

We believe that prior to any endeavour to better understand the action of vitamins, and to realize the significance of the observations made by UHLMANN a.o., it is necessary to decide on the three possibilities suggested above.

For this reason we narrowly considered the probability expressed in the second question by trying to ascertain whether in animals suffering from avitaminosis a lessened or anyhow altered reaction on poisons could be demonstrated. Of course, if in this way an altered reaction was found, it still remained for us to decide whether this altered reaction rests on a modification of the sensitivity of the organs (compare sub 2) or would prove to depend on the possibility suggested sub 3.

We experimented with fowls and with cats.

The fowls were fed for some weeks on polished rice. As known, these animals relish this food at first, but their appetite for it gradually diminishes and soon they most often show a disinclination to eat it; then we had recourse to "forced feeding". Their reaction on poisons was not examined in these experiments until marked symptoms of polyneuritus made themselves evident; some animals were already moribund during the experiment. We anaesthetized the animals with ether, registered the bloodpressure and determined the sensitivity to adrenalin, to cholin and to histamin intravenously; we also ascertained how strong the electric current had to be for the vagus-stimulation to yield a distinct lowering of the bloodpressure, and subsequently we endeavoured to determine the quantity of atropin that was required to abolish this influence of the vagus on the bloodpressure.

After this bloodpressure-experiment the animal was killed and the gut, in some cases also the esophagus, was removed, put into Tyrode-solution and the same day or the next we determined the sensitivity of the surviving gut to pilocarpin, to atropin, afterwards also to cholin and to histamin.

Not knowing the sensitivity of normal fowls to the above-mentioned poisons we first examined four normal fowls.

In the four cats which were examined, an avitaminosis was elicited by means of a prolonged meat-diet, the meat being prepared in the manner described by VOEGTLIN [1]). The meat deprived of its fat and made alkaline, was heated to 120° in the autoclave for three hours. This meat, when neutralised by the addition of acid, was relished by the cats.

In our experiments with cats a special inquiry into the reaction of normal animals was not necessary, because we had sufficient data, already obtained in our laboratory, at our disposal.

We wish to call attention to the fact that, although the symptoms, exhibited in our animals, depended for the major part, anyhow in fowls, on a deficiency of water soluble *B*, the food was devoid not only of one but of several vitamins, and there was also a deficiency of other foodstuffs; but this did not matter in our investigation, considering that we only wished first to ascertain whether a deficiency of vitamins would at all result in a difference in sensitivity. Had this inquiry yielded positive results, we still should have had to find the special vitamin, which was the determinant factor here. Seeing that the result was negative a more detailed investigation was no longer needed.

The results of our research will be published in extenso elsewhere. Suffice it to say here that — beyond expectation — in morbid animals the reaction did not in any respect differ from that found in healthy animals. True, there occur rather marked individual deviations in sensitivity to the poisons examined, but these were not greater in the diseased animals than in the normal ones.

When we assume that many of the automatic functions of the unstriated muscles are brought about by chemical stimuli, and when we see moreover that in many unstriated muscles that function has lost much of its activity in animals suffering from avitaminosis, then the result of our researches compels us to believe that in these diseased animals there is presumably a deficiency of stimulating substances, and that the receptive organ is *not* the seat of the disturbance, and also that the decrease in activity is not brought about by a deficiency of (colloidal) substances that promote the action of poisons.

We have already pointed out that UHLMANN has established that a vitamin-preparation (orypan), examined by him, acted pharmacologically in a similar way to pilocarpin. On this finding is based

---

[1]) CARL VOEGTLIN and G. C. LAKE. Experimental Mammalian polineuritis produced by a deficient diet.

the hypothesis that in the case of avitaminosis an impaired function of many organs is caused by a deficiency of a substance, which is supposed to be a constituent of "orypan". We believe that this problem is not yet ripe for solution for the simple reason that only few positive facts are known. We only wish to point out that the above hypothesis migth be supported by the results of our research.

CONCLUSIONS.

When avitaminosis has been elicited in fowls through a polished rice diet, or in cats fed on specially prepared meat, the sensitivity of the animals to adrenalin, histamin, cholin, and atropin, and the sensitivity of the surviving organs of those animals to histamin, pilocarpin, atropin and cholin, is unmodified.

In two experiments it was proved that atropin (in doses of from 0,001 mgr. to 1 mgr. added to 75 c.c. of Tyrode) had no inhibitive effect on the guts of fowls suffering from avitaminosis; these guts performed only faint spontaneous movements. The gut of normal fowls displayed unmistakable inhibition on the application of atropin. In view of LE HEUX's experience of the influence of cholin on the inhibitory or the stimulating effect of atropin on the gut, this would also lend support to the conception that in hens fed on polished rice a stimulating substance in the gut is wanting.

Physics. — "*The rectilinear diameter of hydrogen*". By E. MATHIAS, C. A. CROMMELIN and H. KAMERLINGH ONNES. Communication N°. 154*b* from the Physical Laboratory at Leiden. (Communicated by Prof. H. KAMERLINGH ONNES).

(Communicated at the meeting of January 29, 1921).

§ 1. *Introduction.* This communication forms the continuation of a series of contributions to the knowledge of the density-curves for liquid and saturated vapour and of the diameters, in the case of substances of low critical temperature and simple molecular structure. The investigation by means of the dilatometer-method was started in the Leiden physical laboratory a considerable time ago and has dealt successively with oxygen [1]), argon [2]) and nitrogen [3]). Great importance was attached to the extension of the measurements to hydrogen for the knowledge of its equation of state, especially in connection with previous determinations of the liquid densities between boiling point and melting point [4]), of the critical point [5]) and the various computations of the critical density [5]) [6]).

The research could not be carried out, however, until the experimental difficulties had been overcome as regards the construction of a transparent bath of constant and uniform temperature between the critical point and the boiling point, i.e. between about — 240° C. and — 253° C.

§ 2. The *apparatus* for the compression of hydrogen, the measurement of the liquid and vapour volumes, the determination of the vapour pressures and that of the volume of the gas under

[1]) E. MATHIAS and H. KAMERLINGH ONNES, these Proceedings 13, p. 939, Leiden Comm. N°. 117.

[2]) E. MATHIAS, H. KAMERLINGH ONNES and C. A. CROMMELIN, these Proceedings 15, p. 667, Leiden Comm. N°. 131*a*.

[3]) E. MATHIAS, H. KAMERLINGH ONNES and C. A. CROMMELIN, these Proceedings 17, p. 953, Leiden Comm. N°. 145*c*.

[4]) H. KAMERLINGH ONNES and C. A. CROMMELIN, these Proceedings, 16, p. 245, Leiden Comm. N°. 137*a*.

[5]) H. KAMERLINGH ONNES, C. A. CROMMELIN and P. G. CATH, these Proceedings 20, p. 178, Leiden Comm. N°. 151*c*.

[6]) J. J. VAN LAAR, Chem. Weekblad 16 (1919), p. 1557.

normal conditions were exactly the same, as those used in the case of nitrogen. For these we may therefore refer to the paper on the diameter of nitrogen quoted above.

As regards the cryostat, only a small part of the temperature range of the measurements might have been covered with a liquefied gas, namely with neon (boiling point — 245°.92 C., triple point — 248°.67) [1]). Uniform and constant temperatures over the whole range can only be obtained by means of superheated vapour, as used in the hydrogen vapour cryostat described on a former occasion [2]). But this apparatus, being opaque, could not be used without alteration, since the position of the meniscus in the dilatometer has to be read.

A description of the modified arrangement by which we succeeded in obtaining a transparent bath of superheated hydrogen vapour which answered all requirements, will be given in the next communication [3]).

The hydrogen was freed from all impurities by freezing in liquid hydrogen [4]) and is therefore to be looked upon as having been absolutely pure.

§ 3. The *experiments* were also conducted in the same manner as in the case of argon and nitrogen.

The quantity of liquid or vapour in the dilatometer was found by reading the position of the meniscus on the scales of the upper tube or the appendix at the bottom of the dilatometer; the hydrogen was then blown off into the carefully exhausted volumenometer and the quantity collected was finally measured under normal conditions.

Although a series of vapour pressures of hydrogen between the critical point and the boiling point was available [5]) for the purpose of the necessary corrections, it seemed advisable to make measurements of the pressure regularly during the experiments. For this

1) H. KAMERLINGH ONNES and C. A. CROMMELIN, these Proceedings 18, p. 515, Leiden Comm. N⁰. 147*d*. In this paper on p. 518, line 8 from the bottom, (Leiden Comm. p. 52, l. 10) the pressure 76.00 should be replaced by 75.95 (one atmosphere at Leiden). The temperature of the boiling point — 245°.92 is correct.

2) H. KAMERLINGH ONNES, these Proceedings 19, p. 1049, Leiden Comm. N⁰. 151*a*.

3) H. KAMERLINGH ONNES and C. A. CROMMELIN, these Proceedings 23, p. 1185, Leiden Comm. N⁰. 154*c*.

4) H. KAMERLINGH ONNES, these Proceedings 11, p. 883, Leiden Comm. N⁰. 109*b*.

5) P. G. CATH and H. KAMERLINGH ONNES, these Proceedings 20, pp. 991, 1155, Leiden Comm. No. 152*a* and P. G. CATH, Dissertation, Leiden 1917, p. 103.

purpose we used the open standard manometer [1]), since the pressures were all below 12.80 atm. (critical pressure) and in our closed hydrogen-manometer, which is otherwise much simpler in use, the mercury does not become visible till 20 atmospheres.

To the results of these measurements and the use made of them in the calculation of the corrections we shall return in discussing the calculations.

The temperature was measured and at the same time kept constant by means of two platinum resistance thermometers Pt_{22} and Pt_{24} which were compared with a helium thermometer shortly after the measurements and under exactly the same conditions, i. e. in the cryostat and at the same temperatures as had been used in the diameter-measurements. The agreement of the two thermometers was completely satisfactory. [2])

The temperature to be taken for the glass capillary of the dilatometer was this time separately determined with a simple gas thermometer in the form of a tube which was mounted beside the capillary in question in the lid of the cryostat, and had been similarly used in previous measurements [3]). In this manner the temperature of the capillary is measured in each determination separately, instead of making an estimation or deriving it from previous determinations; the method also simplifies the calculation of the corrections, since only one temperature is dealt with, whereas before the capillary had as a rule to be divided into three portions each with its own mean temperature [4]).

A few days before the commencement of the real measurements a "general rehearsal" was held, chiefly to ascertain the degree of constancy and uniformity of the temperature to be obtained with the new cryostat, and the influence of various methods of illumination on the temperature. The filling of the dilatometer with liquid hydrogen was found to proceed without a hitch.

During the period of the measurements the illumination trials were continued for the first ten days: a metal wire lamp behind a vessel with alum solution was found to give no disturbance; with clear weather diffuse daylight was also sometimes used.

---

1) H. KAMERLINGH ONNES, these Proceedings 1, p. 213, Leiden. Comm. N°. 44.

2) Comp. fig. 2 of the next communication.

3) For instance by P. G. CATH and H. KAMERLINGH ONNES, these Proceedings 20, p. 991, p. 1155, Leiden. Comm. N°. 152*a* and P. G. CATH, Dissertation, Leiden 1917, Figure on p. 101.

4) F. HENNING. Die Grundlagen, Methoden und Ergebnisse der Temperaturmessung (1915) pp. 46 and 47.

The first six experimental days were used for studying one definite temperature, — 243°·10, in order to obtain as much routine and experience as possible, in the management of the cryostat (so different in principle from the ordinary liquid baths), as also in the adjustment of the equilibrium and the taking of the readings. Special attention was given to checking the equilibrium between vapour and liquid. When this is reached, not only the hydrogen meniscus in the dilatometer but also the mercury in the tube of the compression cylinder remains at absolutely constant level, sometimes after considerable intervals of waiting. Both the liquid density and that of the vapour were measured three times; moreover a number of measurements of the vapour pressure were made under various conditions (meniscus in appendix, bulb or tube).

In the discussion of our results it will appear that in a special range of temperatures certain circumstances prevail which affect the measurements of the vapour density and require to be more fully investigated.

For the reduction of the experiments the pressure in the dilatometer has to be accurately known and for this purpose, as mentioned, during the measurements a great number of pressures were determined, referring therefore to a meniscus sometimes in the appendix and sometimes in the tube; moreover a number of readings were taken with the hydrogen meniscus in the bulb of the dilatometer, in which case a proper equilibrium between liquid and vapour can be better counted on. The measurements with the meniscus in the appendix and the tube, i. e. with little or much liquid, also serve to form an opinion as to possible deviations of the temperature low down and high up in the cryostat, since the absolute purity of the hydrogen could be assumed as certain. These pressure measurements we would rather not call "vapour pressure determinations", since for determinations of that kind, when properly conducted, stirring is an absolute necessity. We propose shortly in the same cryostat to measure vapour pressures of hydrogen in a piezometer with electro-magnetic stirrer.

The result of our pressure measurements was as follows: the readings in the bulb, when plotted, lie on a smooth curve, which as it seems to us cannot deviate appreciably from the true vapour pressure curve; the readings in the appendix are a little less regular, but they do not show any systematic deviations from the previous ones; the observations in the tube, on the other hand, at corresponding temperatures consistenly give slightly higher values.

From this we have to infer, that the temperature must have been

slightly higher high up in the cryostat, than half way down and below. We intend to check this result shortly by special measurements in the cryostat. But although thus there appears to be a small difference of temperature below and above, this causes but a very small inaccuracy in our density determinations. The middle of the resistance thermometers, which were mounted parallel to the dilatometer, was about on a level with the middle of the dilatometer; hence the temperature indicated by the thermometers must be very near the mean temperature of the dilatometer and this is the essential quantity.

For the reduction of the "noxious" volumes the pressures were of course used as measured, without any correction.

It may be mentioned that for the whole series of measurements no less than about 170 litres of liquid hydrogen and 400 litres of liquid air were required.

§ 4. *The calculations.* For these we may again in many respects refer to previous communications. The reduction to the normal conditions of the gaseous volumes in the noxious spaces at low temperature gave no difficulties, since the thermal behaviour of hydrogen at the temperatures and pressures of the experiments is accurately known [1]). It was a great convenience, that these measurements have been united in a special reduced equation of state for hydrogen VII. H_2. 3. [2]); by using the coefficients in this equation the various reductions could be accurately computed.

The volumes of the hydrogen menisci were found by using previous measurements in the Leiden laboratory concerning the capillary constants of hydrogen [3]). For the manner in which these calculations were made we refer chiefly to our paper on the diameter of argon.

For the density of hydrogen under normal conditions, i.e. the mass of 1 litre of hydrogen at 0° and 1 atmosphere MORLEY's value [4]) 0.089873 was taken, which appeared to us to deserve most confidence.

As regards the accuracy of our measurements it may be observed that 0.1 of a mm. on the tube of the dilatometer corresponded to an accuracy of the liquid volume of about $^1/_{8000}$; this accuracy

[1]) H. KAMERLINGH ONNES and C. BRAAK, these Proceedings 9, p. 754; 10, p. 204, p. 413, Leiden Comm. Nrs. *97a*, *99a*, *100a*.

[2]) Published by J. P. DALTON, these Proceedings 11, p. 863, Leiden Comm. N°. *109a*.

[3]) H. KAMERLINGH ONNES and H. A. KUYPERS, these Proceedings 17, p. 528, Leiden Comm. N°. *142d*.

[4]) E. W. MORLEY, Zs. f. phys. Chemie 20 (1896), p. 68, 242, 417 and SMITHSONIAN Contr. to knowledge 1895.

corresponds to 1/100 of a degree in the temperature and to this same amount the temperature in the cryostat could be kept constant for a long time, when the conditions were favourable. The accuracy of the measurements in the volumenometer could be made much higher without any difficulty. Taking into account the specially favourable conditions under which the measurements were made, we think we may put the accuracy of our results for the liquid densities at 1/2000. As regards the vapour densities we must suspend judgment, as certain deviations are left as yet unexplained.

§ 5. *Results.* The results are plotted in fig. 1 and contained in the following table. The plot as well as the table also give the previous Leiden measurements of the liquid density between the

	θ	T	$\varrho_{T\,liq.}$	$\varrho_{T\,vap.}$	$y(O)$	$y(C)$	$O-C$ abs.	$O-C$ in %
MATHIAS, CROMMELIN and KAMERLINGH ONNES, Comm. No. 154*b*.	−240°.57	32°.52	0.04316	0.01922	0.03119	0.03128	−0.00009	−0.29
	241.83	31.26	5001	1366	3184	3178	+ 6	+ 19
	243.03	30.06	5402	1081	3241	3225	+ 16	+ 49
	244.30	28.79	5740	806	3273	3275	− 2	− 6
	245.73	27.36	6050	613	3332	3331	+ 1	+ 3
	247.79	25.30	6416	405	3411	3412	− 1	− 3
	249.89	23.20	6724	264	3494	3495	− 1	− 3
KAMERLINGH ONNES and CROMMELIN, Comm. No. 136*a*.	−252.68	20.41	0.07081	0.00135	0.03608	3605	+ 3	+ 8
	253.24	19.85	7137	116	3627	3627	0	0
	253.76	19.33	7192	101	3647	3648	− 1	− 3
	255.19	17.90	7344	64	3704	3704	0	0
	255.99	17.10	7421	49	3736	3735	+ 1	+ 3
	256.75	16.34	7494	38	3766	3765	+ 1	+ 3
	257.23	15.86	7538	31	3784	3784	0	0
	258.27	14.82	7631	20	3826	3825	+ 1	+ 3

θ = temperature on the KELVIN-scale in degrees centigrade.
$\varrho_{T\,liq.}$ and $\varrho_{T\,vap.}$ are the densities of the liquid and vapour respectively, in grammes per cm³.
$y\,(O)$ is the observed ordinate of the diameter, i.e. the mean of the figures in the previous columns.

boiling point and the triple point and the corresponding vapour densities, as found by calculation. These values have also served in the computation of the diameter constants.

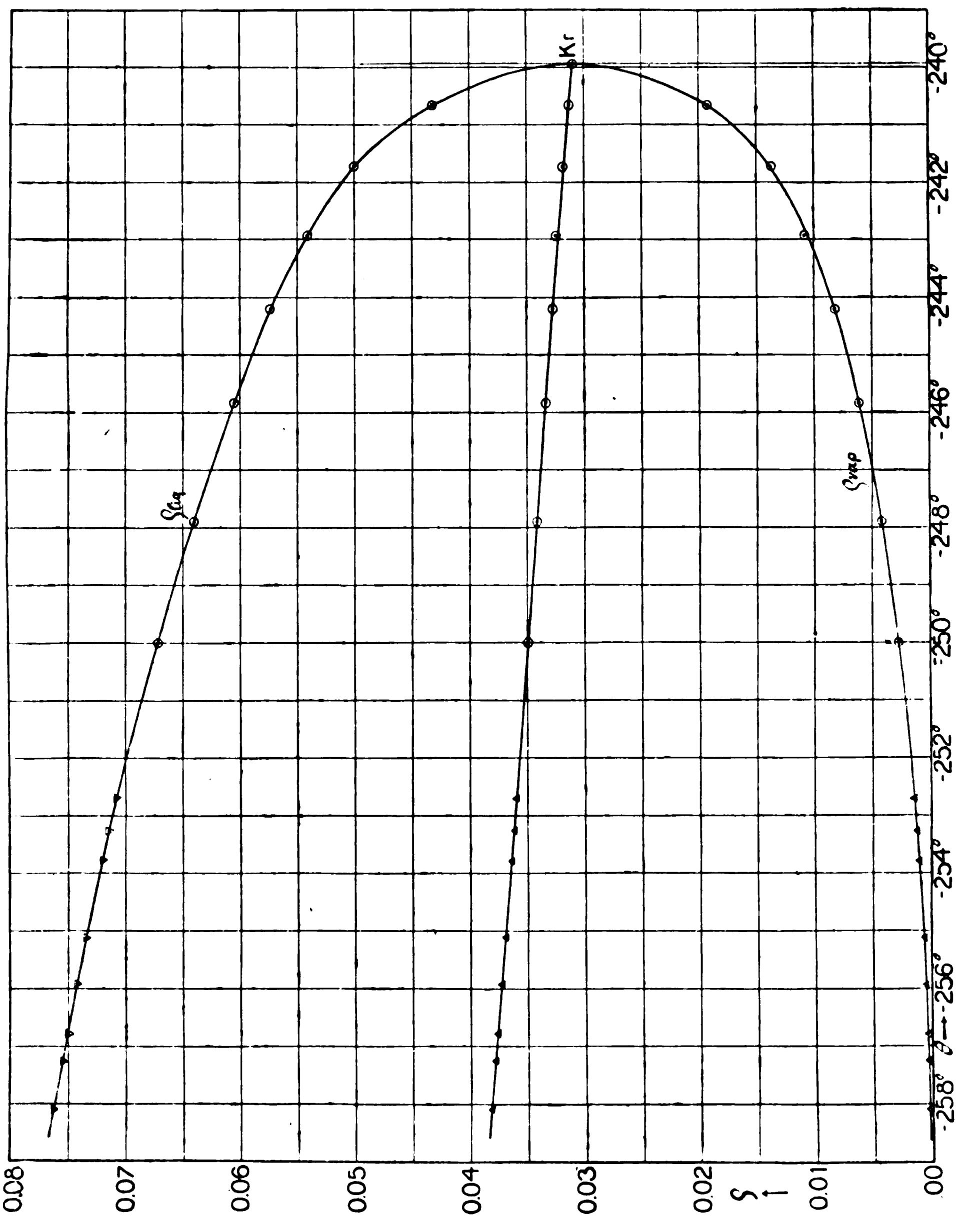


Fig. 1.

From the values of $y(O)$ the equation of the diameter was deduced by the method of least squares; the first three values nearest the critical point were, however, left out of account, since they gave a

somewhat considerable deviation from the straight line, as a preliminary plot has shown. In this manner the rectilinear character of the diameter is duly observed and the critical density is found in a more rational way than otherwise.

The equation of the diameter being written in the form

$$y = a + bt$$

the coefficients are found as follows

$$a = -0{\cdot}063510.$$
$$b = -0{\cdot}00039402.$$

The 6[th], 7[th] and 8[th] column of the table refer to this calculation; $y(C)$ are the calculated values of y, $O - C$ the differences, absolute and in percentages of $y(C)$ respectively.

§ 6. *Discussion.* From the last column it appears that the deviations from the straight line are usually small, less than $^1/_{1000}$, except near the critical point, where they rise to almost $^1/_{200}$. The experimental diameter evidently shows a small hump. A similar hump, although very little marked, was previously found with argon, but on the other hand it was very pronounced with nitrogen. The maximum deviation there amounted to nearly $^1/_{50}$. The magnitude of this deviation surprised us considerably at the time and made us suspect some irregularity in our method of observation or calculation as the cause. Since, however, we have now found a hump of a similar character, although four times lower, with hydrogen, we think we may infer that this result may be due to some systematic error, not yet explained, by which in some manner (condensation on the wall?) a not quite correct vapour density is observed.

The *slope of the diameter* is given by the coefficient

$$b = -0{\cdot}00039402$$

the smallest value so far found for any substance.

The *critical density* as derived from the diameter, using $-239^\circ{\cdot}91$[1]) as the critical temperature, is

$$\varrho_{kd} = 0.03102$$

in complete accordance with the value previously calculated 0·0310.

A comparison of this value with the value to be derived from the isothermals by means of the equation

$$\left(\frac{\partial p}{\partial T}\right)_{v.k.} = \left(\frac{dp}{dT}\right)_{coëx.k.}$$

[1]) These Proceedings 20, p. 178, Leiden Comm. N°. 151*c*.

(as in the case of argon) could not be made, since isothermals in the neighbourhood of the critical point are not available.

The critical coefficient becomes

$$K_{4d} = \frac{p_k v_k}{R T_k} = 3.276$$

the smallest value yet found for any substance.

For a ready comparison with other substances of simple molecular structure we subjoin the following table of the critical values, the diameter slope and the critical coefficient, as far as known at the time with sufficient accuracy.

The numbers concerning xenon are taken from the measurements of PATTERSON, CRIPPS and GRAY [1]), the others from the Leiden measurements.

	θ_k	ϱ_{kd}	b	K_{4d}
X	+ 16·6	1·155	0·003055	3·607
O_2	—118·82	0·4299	2265	3·419
Ar	—122·44	0·5308	2623^5	3·424
N_2	—147·13	0·3110	1958	3·421
H_2	—239·91	0·03102	03940	3·276

The data thus obtained, combined with the values of the vapour pressure, allow a computation of the heat of evaporation L and of the quantity

$$\frac{d}{d \log T}\left(\frac{L}{T}\right) = \frac{dL}{dT} - \frac{L}{T} = m' - m$$

(m' and m being the specific heats of the saturated vapour and the liquid respectively).

In a later paper we hope to communicate the results of these calculations not only for hydrogen, but also for oxygen, nitrogen and argon.

For the investigation described in this paper we have had the benefit of a grant from the BONAPARTE-fund allowed by the "Académie des Sciences" of Paris, for which we wish here to express our profound gratitude.

---

1) H. S. PATTERSON, R. S. CRIPPS and R. W. GRAY, London Proc. R. S. 86 (1912) p. 579.

We gladly thank all who have assisted us in carrying out the measurements, in the first place Mr. F. M. PENNING, who was entirely responsible for the regulation and measurement of the temperatures and the calibration of the resistance thermometers, and made all the calculations involved every time; further Miss H. VAN DER HORST, who assisted in the temperature measurements and carried out and calculated the greater part of the observations with the helium thermometer; finally to Mr. G. J. FLIM, chief mechanic of the cryogenic laboratory and to Messrs. L. and A. OUWERKERK, with whom the far from simple management of the vapour cryostat was in very able hands.

Physics. — "*Methods and apparatus used in the cryogenic laboratory. XVIII. Improved form of a hydrogen vapour cryostat for temperatures between* — 217° *C. and* — 253° *C.*". By H. KAMERLINGH ONNES and C. A. CROMMELIN. Communication N°. 154*c* from the Physical Laboratory at Leiden. (Communicated by Prof. H. KAMERLINGH ONNES).

(Communicated at the meeting of January 29, 1921).

§ 1. *Introduction*. In a previous communication one of us has described a hydrogen vapour cryostat for temperatures between the melting point of oxygen and the boiling point of hydrogen [1]). During the preliminary determinations of the critical points of neon [2]) and of hydrogen [3]), made in this cryostat, it was found that in principle the apparatus properly performed its function, but at the same time we discovered certain faults and deficiencies. When therefore the modification, which had been foreseen all the time and by which it would become possible to take readings on a tube inside the cryostat, became necessary, we resolved with the assistance of Mr. FLIM, chief mechanic of the cryogenic laboratory, to revise the construction in all details and rather than execute the modifications as originally planned to build a new apparatus of somewhat larger dimensions. This new cryostat which in the various measurements has given almost complete satisfaction and may therefore be considered to have approached its final form, will be described in this paper.

The apparatus is shown in fig. 1, partly in section, partly in direct view, as appears easily from the different details. Moreover three sections are given at different heights, a view of the turnable drum and diagramatically the electric connections, as will be further explained later on.

---

[1]) H. KAMERLINGH ONNES, these Proceedings 19, p. 1049, Leiden Comm. N^0. 151*a*.

[2]) H. KAMERLINGH ONNES, C. A. CROMMELIN and P. G. CATH, these Proceedings 19, p. 1058, Leiden Comm. N^0. 151*b*.

[3]) H. KAMERLINGH ONNES, C. A. CROMMELIN and P. G. CATH, these Proceedings 20, p. 178, Leiden Comm. N^0. 151*c*.

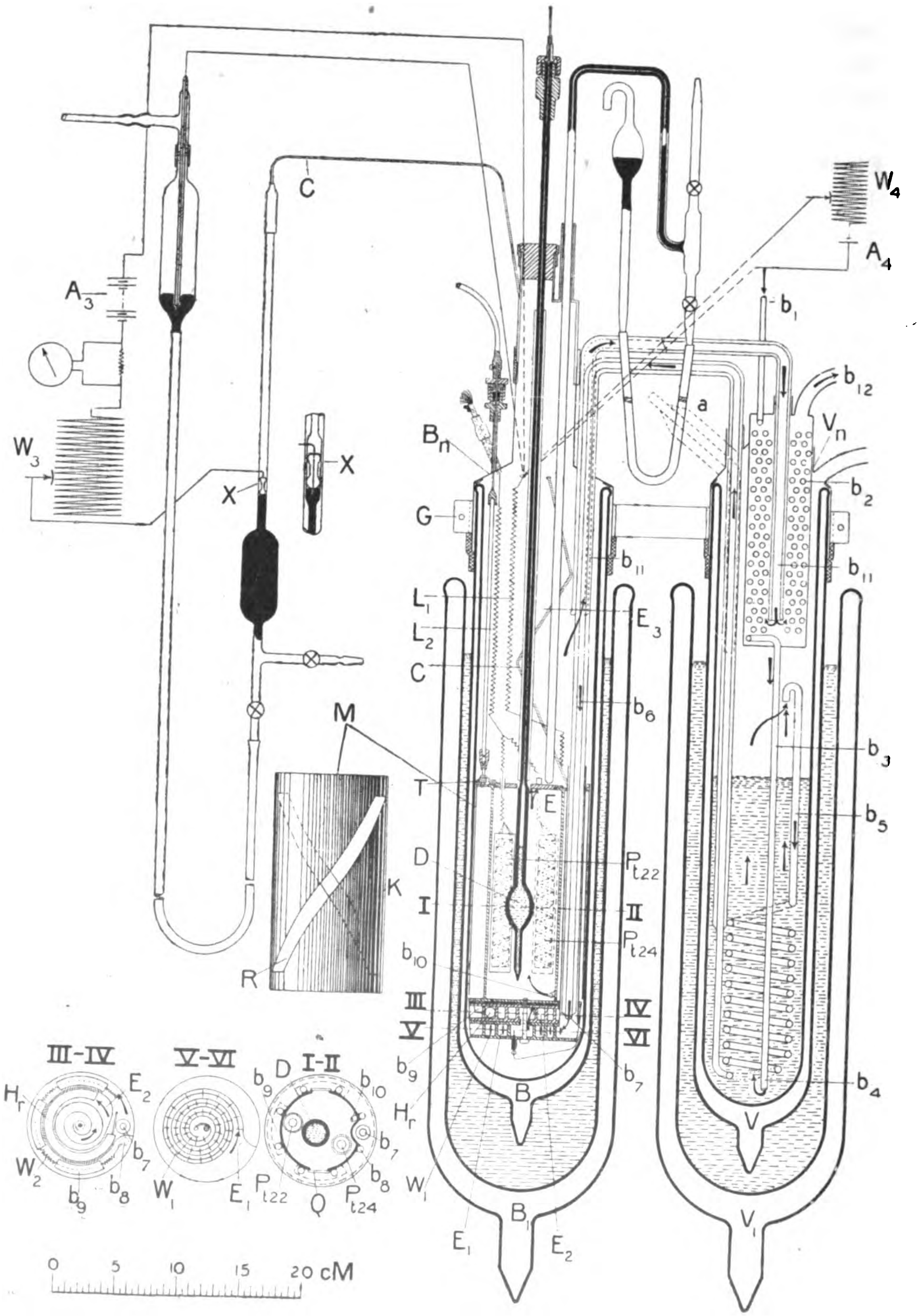

C
W4
A4
A3
b1
b12
a
Vn
Bn
W3
X
X
b2
G
b11
b11
L1
E3
L2
C
b6
M
b3
T
E
b5
D
Pt22
K
I
II
Pt24
b10
R
III
IV
V
VI
b9
b7
b4
B
Hr
V
W1
III-IV
V-VI
I-II
D
b9
b10
Hr
E2
b7
b7
W2
b9
b8
W1
E1
Pt22
Q
b8
Pt24
B1
V1
E1
E2
0
5
10
15
20 cM

Fig. 1.

§ 2. *Description of the cryostat* [1]).

The cryostat consists of two vacuum vessels, viz. an evaporator V and cryostat B, both enclosed in vacuum vessels with liquid air V_1 and B_1 and closed by german silver lids or caps V_N and B_N.

The principle of the apparatus consists in the experimental space E being kept at constant temperature by a current of gaseous hydrogen, the temperature of which is kept at a definite value by automatic regulation of an electric current through a heating wire.

Let us follow the gaseous hydrogen on its way through the apparatus, as shown by the arrows. At b_1 it enters under small excess of pressure. For the manner in which the current is kept constant and regulated, we refer to fig. 6 of the previous paper on the vapour cryostat. It then flows through the copper spiral b_2, which is cooled, as will appear presently, by the waste hydrogen, and the tube b_3 (of german silver in order to minimize the supply of heat to the liquid hydrogen below) from which it escapes at b_4 bubbling up through the liquid hydrogen which fills the evaporator.

In this manner a fairly strong and regular evaporation of the liquid hydrogen is obtained; the hydrogen vapour passes into the tube b_5, which continues as a spiral under the liquid hydrogen, and hence into the double walled vacuum tube b_6 which carries the gas (cooled to about — 253°) to the lower end of the cryostat at b_7.

Here it enters the spiral-shaped heating space E_1 (see also section V—VI) and flows round a heating wire W_1 which heats the gas to about the desired temperature by means of an automatically regulated current. The space E_1 communicates at the centre through a hole with a second space E_2 above it. Here the gas first passes along a copper thermometer bulb H_2 (which serves for the regulation of the temperature, as will be explained further down), then along a resistance W_2 and finally rises through b_8 into a thickwalled copper tube going four times up and down, the horizontal parts of which are seen in section III—IV at b_9, the vertical soldered on the outside against the very thick copper mantle (for the purpose of a uniform temperature) of the experimental space E; this motion up and down serves to communicate the required temperature to the copper mantle. Finally the gas enters the experimental space E with the measuring instruments (in the figure a dilatometer D

---

[1]) We here refer to Leiden Comm. N⁰. 151a (these Proceedings 19, p. 1049) where many constructional details are mentioned and the principle of the apparatus is explained; at the same time the present description is so arranged, that it may be read and understood by itself.

and two platinum resistance thermometers Pt_{22} and Pt_{24} as used for the determination of the diameter of hydrogen [1])) passes through it and by the holes in the lid E enters the space E_2 above; it is then carried through the vacuum tube b_{11} into the space surrounding the spiral b_2, by which the gas inside acquires a preliminary cooling, and finally escapes by b_{12} to a gasometer.

The half *automatic regulation of the temperature* is very simple in principle. By the resistance W_1 the gas is heated to the desired temperature, the current being furnished by the accumulators A_3 (36 volts) and adjusted by the regulating resistance W_3. The gas-thermometer H_r already referred to, which is connected with the manometer through the capillary C, provides for the automatic closing and opening of the current, in a manner easily understood from the figure. As long as the temperature is too low, the manometer being properly adjusted, there is electric contact at D; when the temperature rises above the desired value, the contact is broken. When the cryostat is in action and everything properly regulated, the automatic arrangement is continually in action and the current is opened and closed at regular intervals (of say 10 or 20 seconds).

When this arrangement was tested, it appeared from the readings of the platinum resistance thermometers that owing to some inertia in the automatic action small fluctuations of the temperature in the experimental space of ·02° to ·03° were still present. With a view to a still finer regulation the resistance W_2 was then introduced in the space E_2.

The current through this small resistance is furnished by an accumulator A_4 and is regulated by hand by the observer at the galvanometer by means of the regulating resistance W_4. The deflection of the galvanometer shows the small variations of temperature much more quickly than the automatic regulator can act and the observer is able by a change of W_4 at once to bring back the temperature to the desired value. In this manner it is possible to keep the temperature constant to .01°, sometimes during hours.

The connecting wires of the various resistances are only shown diagrammatically in the figure. Thus L_1 represents the 4 wires of the automatic regulator and the resistance W_4, L_2 the eight wires of two platinum resistance thermometers Pt_{22} and Pt_{24}.

One of the principal desiderata in the previous simpler cryostat was the possibility to make visual observations in the space E. The apertures which had originally been made in the copper mantle for

[1]) E. Mathias, C. A. Crommelin and H. Kamerlingh Onnes, these Proceedings 23, p. 1175, Leiden. Comm. N°. 154*b*.

this purpose had afterwards been closed again by copper plates for safety, in order to avoid fluctuations of the temperature in consequence of the presence of these apertures in connection with the small heat-capacity of the gaseous hydrogen. On the latter account the vapour cryostat requires much greater precautions than an ordinary liquid cryostat and for this reason in our first trials all superfluous complications were avoided. The difficulty which was then feared (correctly, as we found later on) is met in the present model by providing the copper mantle with two diametrically opposite narrow glass windows and surrounding the mantle with a second one *M* of german silver (also shown separately in the figure). In this mantle two screw shaped slits have been cut. Thus a field *K* is left clear for illumination from behind and observation from the front.

The mantle *M* may be turned by means of a glass rod *G* and the cogwheel *T* which works in a rack fitted on the outside of the mantle at the top: by this means *K* may be moved vertically up or down. An experimental tube mounted axially in the space *E* may thus successively be observed from the bottom to the top. The resistance thermometers which do not need to be seen may be mounted excentrally.

In this manner the aperture for visual observation which might produce fluctuations of the temperature by radiation is reduced to a minimum, and moreover the mantle can be closed by turning far enough as soon as an observation is finished.

We have systematically investigated whether the opening did still cause any change of temperature and for this purpose have tried various sources of light, for instance a metal wire lamp with or without alum filter, diffuse daylight, etc. A lamp without a filter immediately produces a rise in the temperature of a few hundredths of a degree. When a heat filter (alum solution) was placed between the lamp and the cryostat, and the light was used with caution, no changes of the temperature of as much as .01° could be observed.

In the silver coating of the vessels B and B_1 narrow slots have naturally also been left open, and similarly in the coating of vessel V_1 in order that the evaporation of the liquid hydrogen may be followed. V is for the greater part unsilvered.

The construction of these vessels, which was particularly difficult in the case of the large outer ones, we thank to the exceptional ability of the chief glassblower Mr. O. KESSELRING. It seems as if with these largest vessels the limit of what can be done with cylindrical vessels has almost been reached. They have to be treated with the utmost care and have to be very slowly cooled, at least

for half an hour, before liquid air may be poured in. Even then it has happened repeatedly that a vessel of this kind burst, specially during the addition of liquid air in the narrow space between the vessels.

To begin with we feared that the very low temperatures near the boiling point of hydrogen would give difficulties and might

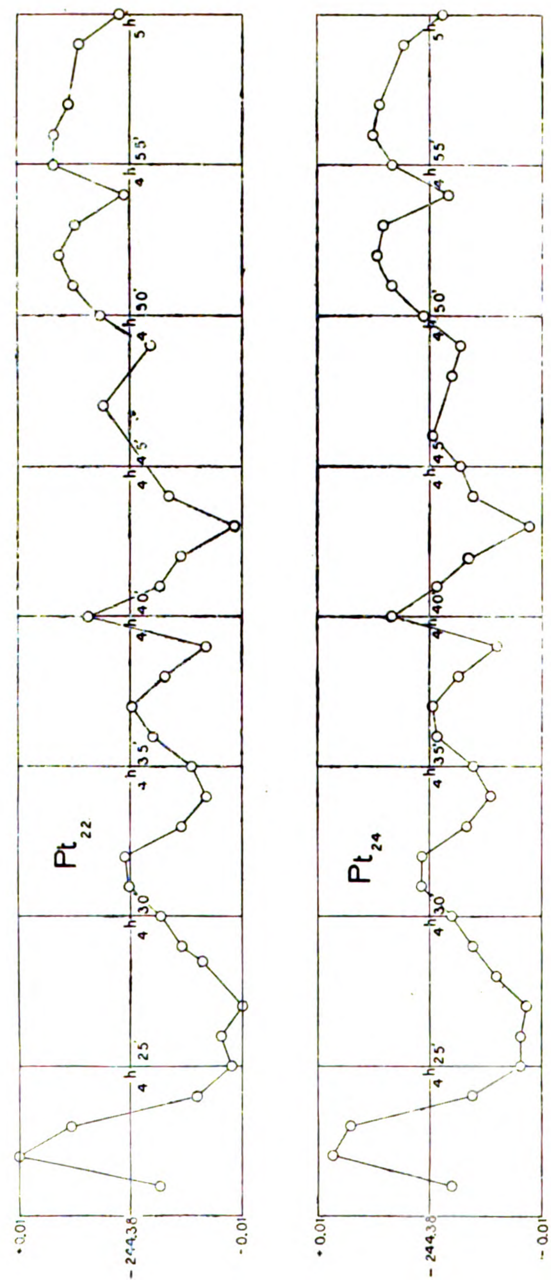


Fig. 2.

possibly not be attainable at all[1]). This fear turned out to be unfounded. Temperatures of — 250° and — 251° do not procure any special difficulties.

It was mentioned above, that the temperature in the experimental space could be kept constant to .01 of a degree during a long time. In confirmation of this we here reproduce two curves drawn according to the galvanometer-deflections of the two thermometers Pt_{22} and Pt_{21}, representing the deviations of the temperature as a function of the time. They refer to an observation at — 244.°38. It must be mentioned, however, that the management of the cryostat is far from easy or simple. A great deal of routine and experience is required to work the cryostat in such a way that the above constancy of the temperature is attained, and it is only after repeated vain efforts that even our highly trained technical staff has now learned to make the apparatus answer to the slightest hint.

Much is due to Mr. G. J. FLIM, chief mechanic of the cryogenic laboratory, under whose able guidance the apparatus was designed and constructed, and further to Messrs. L. and A. OUWERKERK both attendants 1st class, under whose supervision the apparatus now works so excellently.

[1]) Comp. Leiden Comm. No. 151*a* p. 4 note 1.

Physiology. — "*On the Influence of the Season on Laboratory Animals*". By Prof. H. ZWAARDEMAKER.

(Communicated at the meeting of January 29, 1921).

This technical subject appears to be of general application. In previous publications the present writer and his co-workers have superadded to J. LOEB's balancing of ions of the circulating fluids, expressed in the equation $\frac{Na + K}{Mg + Ca} = $ constant, the balancing formula $\frac{K + (UO)_2 + Th}{Ca + Sr + Ba} = $ constant. In the latter formula the radio-physiological antagonism between K and $(UO)_2 + Th$. need not be taken into account [1]).

Moreover, in earlier discourses the replacement of potassium by the other radio-active elements Rb, $(UO)_2$, U, Th, Io, Ra, Em, has been repeatedly discussed [2]).

Now the present writer wishes to point out that the dosages in which these elements are to be administered must be much smaller in summer than in winter. Of course, this difference is not brought about by the radio-active elements as such, but by the fact that in summer the organs are more sensitized by certain substances [3]).

These substances can be washed out, so that in the transition periods the functionating of a summer-organ during some hours' perfusion with an artificial but nonetheless efficient circulating fluid, suffices to transform a summer-organ into a winter-organ.

As regards sensitizing power, that of the washed-out substances is analogous to that of adrenalin.

The organs operated upon were the hearts of frogs and of eels.

A detailed publication will appear elsewhere.

---

1) C. R. des Séances de la Soc. de Biologie 7 Juin 1919.

2) Journal of Physiology Vol. 53 p. 273 1920.

3) Proceedings of this Acad. 25 Sept. 1920.

Physics. — "*On the principles of the theory of quanta.* By PAUL S. EPSTEIN. (Communicated by Prof. P. EHRENFEST).

(Communicated at the meeting of January, 29, 1921.)

1. *Introduction.* The quantum-theory in the form, which in 1911 PLANCK [1]) has given it, depends on the application of statistical mechanics in the so-called "phase-space" of the canonical position- and impulse-coordinates $q_1\, q_2 \ldots . q_f;\ p_1\, p_2 \ldots . p_f$, and consists in dividing this space into elementary regions of probability. The method obtains a considerable simplification for the soluble mechanical systems, since for them each impulse-coordinate $p_i = p_i\,(q_i)$. Instead of the $2f$-dimensional phase-space (f being the number of degrees of freedom of the system) it is then sufficient to consider the f "phase-planes" (p_i, q_i), which, as the author showed a few years ago [2]), gives great advantages in the treatment of these systems. In each of these planes the successive conditions of the system are represented by a curve. For the class of the "conditioned-periodic motions", the only ones for which so far quantum-conditions have been established, the curves in question are as a rule closed. The only exception is formed by the "cyclic coordinates" which bear the character of a plane angle; a cyclic coordinate varies from 0 to 2π and the corresponding impulse is constant; hence the representive curve becomes a segment of a straight line parallel to the axis of abscissae. [3])

PLANCK's hypothesis, as extended by SOMMERFELD and the author, consists in the assumption of the existence among the states of the system of certain preferential or "stationary" motions, which are represented by discrete curves in the diagram, the area of the phase-plane between two successive stationary curves being equal to the universal constant h

$$\iint dp\, dq = h. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If the area of the narrowest of these curves (or for cyclic coor-

[1]) M. PLANCK. Verhandelingen van het Solvay-congres.

[2]) P. S. EPSTEIN. Ann. d. Phys. **50**, p. 489; **51**, p. 168, 1916.

[3]) This case was discussed for the first time by P. EHRENFEST. Verh. d. D. phys. Ges. **15**, p. 451. 1913.

dinates the one nearest the axis of abscissae) is equal to h_0, that of the $(n+1)\,st$ stationary orbit will be

$$\int pdq = h_0 + nh \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

h_0 has therefore to be determined, in order that all the stationary curves be fixed.

For this purpose PLANCK [1]) lays down the principle, that the narrowest orbit must coincide with the *natural boundary of the phase-plane*; i.e. if on any grounds, connected with the nature of the system, the integral (1), which is essentially positive, cannot fall below a definite value, the latter has to be taken as h_0. In most cases a lower limit of that kind does not exist and the integral may be taken equal to zero, whence

$$\int pdq = nh \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2')$$

In his treatment of the relativistic Kepler-motion SOMMERFELD [2]) found the case to be different; he there gave a lower limit $p_0 = \varkappa e^2/c$ [3]) for the *constant azimuthal impulse*; this would give $h_0 = 2\pi\, p_0$. It would therefore, as pointed out by PLANCK, be necessary to take (2) as the fundamental relation, whereas experiment (the Balmer-series) can only be reconciled with supposition (2′). SOMMERFELD [4]) tried to remove this contradiction by pointing out, that when the motion of the nucleus is taken into account the numerical value of the limiting impulse is smaller than $\varkappa e^2/c$. In what follows we hope to prove that *the limitation of the phase-plane by the value $p = p_0$ is only an apparent one, even if the motion of the nucleus is left out of account*, and that p can very well fall below this value: at the same time the character of the motion is then essentially changed.

The admissibility of stationary orbits of azimuthal impulse $p = 0$ which on SOMMERFELD's theory seemed to be excluded is thereby proved in principle. As long as we are dealing with attractive forces (nucleus and electron) these orbits are hardly of practical importance, as they must lead to a collision of electron and nucleus. But the case changes, when the forces are repulsive (nucleus and α-particle); the orbits are then hyperbolic. If the quantization of such orbits is admitted, interesting physical conclusions follow which appear to

---

[1]) M. PLANCK. Ann. d. Phys. **50**, p. 385. 1916.

[2]) A. SOMMERFELD. Ann. d. Phys. **51**, p. 57. 1916.

[3]) Here e is the charge of an electron, $\varkappa e$ that of the atomic nucleus and c the velocity of light.

[4]) A. SOMMERFELD. Münchener Ber., p. 137, 1916.

give an explanation of certain recent experimental results of RUTHERFORD's [1]). The question raised by the author [2]) before as to the quantization of non-periodic motions is therefore put once more and discussed in a different manner (§ 4, 7).

§ 2. *The apparent boundary of the phase-plane $p = p_0$.*

The relativistic Kepler-motion is given by the following equation between the polar coordinates r, φ (cf. l.c. p. 819).

$$\frac{1}{r} = \frac{B}{p^2 - p_0^2}\left[1 - \varepsilon \cos \frac{\sqrt{p^2 - p_0^2}}{p^2}(\varphi - \varphi_0)\right] \quad . \quad . \quad . \quad . \quad (3)$$

with the abbreviations

$$\left.\begin{aligned} B &= \pm p_0 c\left(m + \frac{a}{c^2}\right) \quad ; \quad \varepsilon = \frac{\sqrt{B^2 + A(p^2 - p_0^2)}}{B} \\ A &= a\left(\frac{a}{c^2} + 2m\right) \quad ; \end{aligned}\right\} \quad . \quad . \quad . \quad (4)$$

a represents the energy of the system, c the velocity of light, m the mass of the moving particle. The positive sign of B refers to the case of attraction, the negative sign to repulsion; φ_0 is the azimuth of the radius vector with respect to the aphelion.

For $p > p_0$ with negative energies ($A < 0$) and attracting forces ($B > 0$) the orbit is an ellips with perihelion-motion. The procession of the perihelion increases in speed, the smaller the difference $p^2 - p_0^2$, and in the limiting case $p = p_0$ the orbit converges on the nucleus in a manner similar to an Archimedian spiral [3]):

$$\frac{1}{r} = \frac{B}{p^2}(\varphi - \varphi_0)^2 - \frac{A}{B} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

But nothing prevents us from now taking $p < p_0$; the expression (3) then assumes the form:

$$\frac{1}{r} = \frac{B}{p_0^2 - p^2}\left\{\varepsilon \cosh \frac{\sqrt{p_0^2 - p^2}}{p^2}(\varphi - \varphi_0) - 1\right\} \quad . \quad . \quad . \quad . \quad (6)$$

The right-hand side of this expression for a very large positive or negative value of φ becomes exponentially infinite independently of the value of the excentricity ε. The two extremities of the orbit thus approach logarithmic spirals. It further follows from (4) that

[1]) E. RUTHERFORD. Phil. Mag. **37**, p. 537, 1919.

[2]) P. S. EPSTEIN. Ann. d. Phys. **50**, p. 815, 1916. This paper will be quoted here as l.c.

[3]) Comp. A. SOMMERFELD Ann. d. Physik. **51**, p. 50 1916.

for $A \lesseqgtr 0$, $\varepsilon \gtreqless 1$. Thus with negative energy r always remains finite, the particle moves out from the centre and again returns to it.

When the energy disappears or becomes positive the orbit divides into two branches which run from the centre to infinity or vice versa. In the limiting case $p=0$ r is only finite for $\varphi=\varphi_0$, i.e. the motion is rectilinear.

Thus it is seen, that in reality there is not a limit $p=p_0$ at all: with small positive values of $p-p_0$ the orbit encircles the centre many times, while r diminishes, but remains at a finite distance from it which passes through a minimum and then increases again. For $p=p_0$ the curve runs into the centre as an Archimedian spiral. The approach to the centre is even more rapid when $p<p_0$, the spiral becoming logarithmic. It must not be supposed that the particle in its motion on the spiral will permanently remain near the centre: for although the spiral encircles the point an infinite number of times, its total length is finite and the time to describe it from a finite distance, as a simple calculation shows, is also finite and practically very small. Therefore the collision will occur very soon.

§ 3. *Quantization of the spiral orbits.* In the last section we have shown, that in the relativistic Kepler-motion, even with negative energy, besides the ellips-like orbits other forms are possible which are of finite length and are only once described.

The question now arises, whether these motions can be submitted to quantum-conditions and in what manner this would have to be done. Our answer to the first question is implicitly contained in the above discussion: the disappearance of the limiting value h_0 in assumption (2) we have explained by the fact that orbits have to be taken into account for which p is less than the azimuthal quantum p_0. It follows that these orbits join on continuously to the others and must be equivalent to them from the point of view of the quantum theory. Since for $p>p_0$ the stationary motions are given by the relation $p=nh/2\pi$ $(n=1, 2, \ldots .)$, it follows that for $p<p_0$ the only possible stationary condition is $p=0$. This conclusion is strengthened by the circumstance that when the movement of the nucleus is taken into account (as proposed by SOMMERFELD) similar spiral-shaped orbits have to be considered in order to explain the possibility of $p=0$: this can be easily shown to be the case.

We have therefore only to discuss the quantisation of the radial impulse: its dependence on the radius vector r and on the constants of the problem is given by the equation (l. c. p. 823).

$$p_r = \sqrt{A + 2B\frac{1}{r} + (p_0{}^2 - p^2)\frac{1}{r^2}} \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

which is represented graphically in Fig. 1 for $p < p_0$. The curves nearest the axis of ordinates correspond to large negative values of the energy constant α. With increasing energy the curves bend out more and more and for $\alpha = 0$ they divide into two branches which approach asymptotically to the axis of abscissae. For α positive the asymptotes are straight lines parallel to this axis.

For small values of r (7) reduces to

$$p_r = \sqrt{p_0{}^2 - p^2}\,\frac{1}{r}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7')$$

i.e. at a distance from the axis of abscissae the curves are hyperbolic. The area of such a curve is logarithmically infinite and the difference between the area of two curves is also always infinite, unless we apply artificial means such as the formation of the principal values of the integral. Since according to the quantum theory the areas of two successive stationary curves must differ by the finite quantity h, it follows that in this case the stationary energy stages must be infinitely dense, i.e. all values of the energy are "selected" in the sense of the theory. Whereas the selected values of p form a series of discrete numbers, those of α form a continuum. There are thus an infinite number of motions which starting from the zero reach as far as we like. All these orbits lead to a collision with the nucleus and for this reason they are not very important physically.

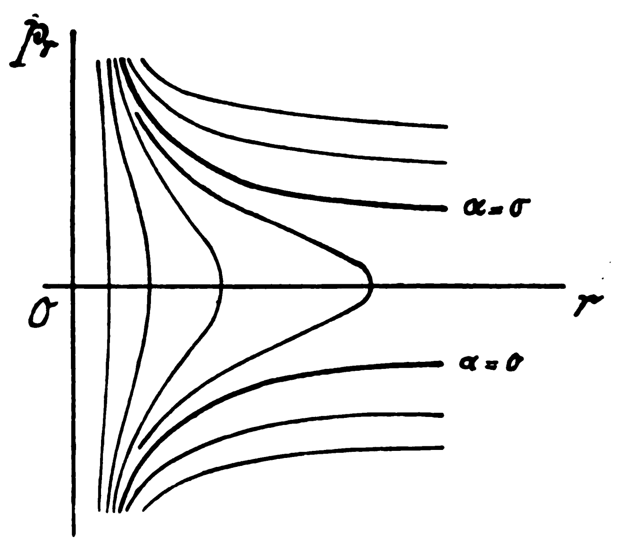


Fig. 1.

But for our purpose it is important that these orbits are possible in principle irrespectively of how long an electron can move along it.

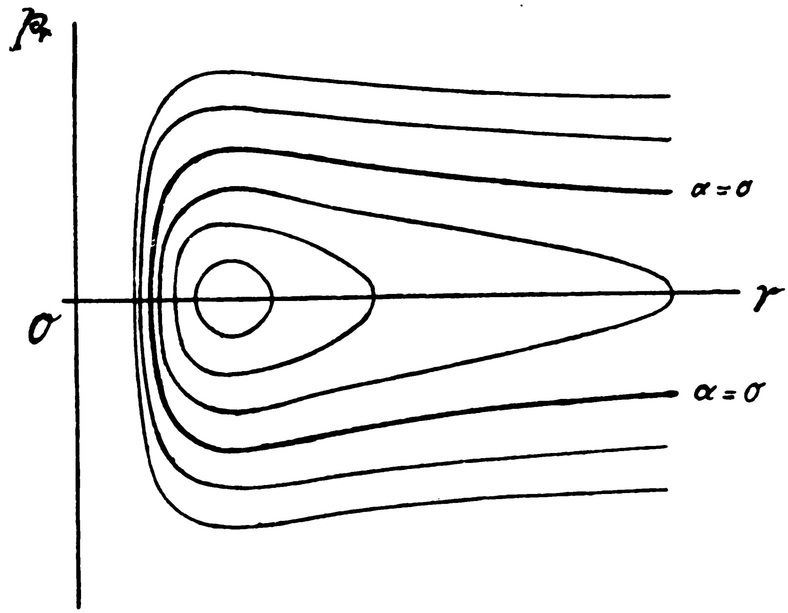


Fig. 2.

§ 4. *Quantization of the hyperbolic curves.* The problem becomes of greater importance physically, if repulsive forces are considered, so that the orbits are hyperbolic. The question, how these orbits have to be quantizized was discussed by me several years ago (l.c.). The method adopted then, which was explicitly stated to be provisional, I do not wish to adhere to in all its particulars. But the fundamental idea of submitting such orbits to quantum-conditions still appears to me a sound one. Quite a long time ago I have in the Munich colloquium developed certain views on this subject which appear to me still to deserve attention. For simplicity we shall here disregard the relativity correction ($c = \infty$): the radial impulse according to (7) and (4) then assumes the form:

$$p_r = \sqrt{2 m \alpha + 2 \varkappa m e^2 \frac{1}{r} - p^2 \frac{1}{r^2}} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

For $\alpha < 0$ the motion is elliptical, for $\alpha = 0$ parabolic, for $\alpha > 0$ hyperbolic. The aspect of the curves in the phase-plane (p, r) is seen in Fig. 2. The part of the plane, where $\alpha < 0$ is bounded by the heavily drawn curve $\alpha = 0$, both ends of which approach the axis of abscissae asymptotically. Inside this region the curves are elliptic and it is easy to fulfil the condition that the area between

two successive curves be equal to h. In the region outside the curve $a=0$ each one of the curves possesses two asymptotes parallel to the axis of abscissae. The strip between two curves whose energy-constants differ by a finite amount has an infinite area. Just as in the case of the spiral orbits we may conclude that the energy stages of the stationary motions must be infinitely dense. *Every positive value of the energy-constant is therefore a "selected" value in the sense of the quantum theory.* That hyperbolic orbits with all values of the energy are present, was already enunciated by BOHR on the ground of experimental results (by WAGNER and others). From our point of view this does not prove that the hyperbolic motion is beyond the controll of the quantum theory; on the contrary this fact is a natural inference of a consistent application of this theory.

This view naturally implies that the azimuthal impulse must also be subjected to quantic conditions. What these are cannot immediately be deduced from the case of the elliptic motion. Two possibilities seem to present themselves: we must extend the integral $\int p d\varphi$ either over the range of change of the coördinate φ, i. e. over the angle enclosed between the asymptotes, or, as in the case of the elliptic motion, from 0 to 2π. The former assumption would according to 2′ give

$$p=\frac{nh}{2\varphi}; \quad \ldots \ldots \ldots \quad (9a)$$

the latter

$$p=\frac{nh}{2\pi}; \quad \ldots \ldots \ldots \quad (9b)$$

In § 7 we shall meet with an argument in favour of the second assumption, but a decision between the two can ultimately only be brought by experiment.

§ 5. *Collision between an α-particle and an atomic nucleus.* We shall now investigate the case of repulsive forces in detail and thereby take into account the motion of the nucleus, neglecting the relativistic correction which is of no importance for our purpose. In the usual manner by means of the principle of the centre of mass we eliminate the co-ordinates of the one body and so reduce the problem to that of a system of two degrees of freedom. We shall choose as the variables the relative polar co-ordinates of the two particles, i. e. their distance and the angle φ under which the α-particle appears for an observer moving with the atomic nucleus,

and shall call p the impulse corresponding canonically to φ, m and M the masses, e, E the charges of the α-particle and nucleus respectively, and finally v the initial velocity of the α-particle, the atom being originally supposed at rest. The equation to the orbit then assumes the simple form

$$\frac{1}{r} = \frac{\mu e E}{p^2} \left[\varepsilon \cos (\varphi - \varphi_0) - 1 \right], \quad \ldots \ldots \quad (10)$$

where

$$\frac{1}{\mu} = \frac{1}{m} + \frac{1}{M} \quad ; \quad \varepsilon = \sqrt{1 + \left(\frac{pv}{eE}\right)^2} \quad \ldots \quad (11)$$

Hence the angle $\overline{\varphi}$ between the axis and the asymptote of the relative hyperbolic orbit of the α-particle is given by

$$\cos \overline{\varphi} = \frac{1}{\varepsilon}$$

or

$$tg\, \overline{\varphi} = \frac{pv}{eE}; \quad \ldots \ldots \ldots \quad (12)$$

We can now change to the absolute motion by considering, that the centre of gravity of the two bodies must move uniformly; originally this point moves in the direction of the α particle with the velocity $mv/(M+m)$ and this motion has therefore to be superposed on the relative motion. A simple calculation gives the following result[1]): after a sufficient time both bodies have assumed a uniform rectilinear motion. The direction of the final motion of the α-particle encloses an angle Φ with the initial direction (through this angle the α-particle is deflected by the collision)

$$tg\, \Phi = \frac{2\, tg\, \overline{\varphi}}{(m - M) + (M + m)\, tg^2\, \overline{\varphi}} \quad \ldots \ldots \quad (13)$$

the velocity V obtaining the value

$$V = \frac{v}{M + m} \sqrt{m^2 + M^2 - 2\, m\, M \cos 2\, \overline{\varphi}}. \quad \ldots \quad (14)$$

The angle between the direction in which the atom is propelled and the original direction of the α-particle is exactly equal to the angle φ of equation (12). The velocity of the atom is

$$u = 2v \frac{\mu}{M} \cos \overline{\varphi}, \quad \ldots \ldots \ldots \quad (15)$$

According to the view set forth in § 4 certain special motions of

---

[1]) Comp. C. Darwin. Phil. Mag. **27**, p. 499, 1914.

the system are to be allowed, namely those for which the azimuthal impulse p has a value satisfying the conditions ($9a$, b). In these the letter n represents a positive whole number, but $n=0$ which would be excluded according to SOMMERFELD must also be admitted on the point of view explained in § 2. In the latter case the assumptions ($9a$) and ($9b$) both give $p=0$; hence

$$\overline{\varphi}_0 = 0 \quad ; \quad u_0 = 2\frac{\mu}{M}v; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

in other words: *the nuclei or "recoilrays"* as RUTHERFORD has called them, *have for $n=0$ the direction of the primary α-rays.*

§ 6. *Recoil-rays of hydrogen.* Whereas the values (16) which obtain for $n=0$, hold generally for all kinds of atoms, the results are less general for $n=1$, 2 etc.; we shall only discuss the special case of the collision with a hydrogen atom. On the assumption ($9a$) we have according to (12)

$$\overline{\varphi}\, tg\, \overline{\varphi} = \frac{hv}{2\,e\,E}n; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

on assumption ($9b$)

$$tg\, \overline{\varphi} = \frac{hv}{2\pi\, e\, E}n; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

In these expressions we may substitute $h = 6.55 \times 10^{-27}$ erg. sec. $E = 4.77.10^{-10}$ e. s. units $e = 2E$; for v we shall take the velocity of the α-rays of $R\,a\,C$, for which RUTHERFORD gives the value $1.92.10^9$ cm/sec. we then obtain

$$\overline{\varphi}\, tg\, \overline{\varphi} = 13.8\, n, \quad \text{or} \quad tg\, \overline{\varphi} = 4.40\, n \quad . \quad . \quad . \quad . \quad (19)$$

The first of these equations gives

$n=1$, $\overline{\varphi}_1 = 84°$, $u_1 = 0{,}10\, u_0$, $R_1 = 0{,}001\, R_0$,
$n=2$, $\overline{\varphi}_2 = 86°{,}50'$, $u_2 = 0{,}055\, u_0$, $R_2 = 0{,}0002\, R_0$.

The velocities $u_1\, u_2$ are computed from (15), the corresponding ranges R_1, R_2 from the empirical equation $R : R_0 = u^3 : u^3$.

Similarly the second hypothesis gives

$n=1$, $\overline{\varphi}_1 = 77°$, $u_1 = 0{,}22$, $R_1 = 0{,}011\, R_0$,
$n=2$, $\overline{\varphi}_2 = 83°{,}30'$, $u_2 = 0{,}11$, $R_2 = 0{,}001\, R_0$.

On the view that the particles can only move on the special orbits allowed by the quantum theory, we obtain the following result: a portion of the recoil rays are emitted in the direction of the primary rays ($n=0$); besides there are only particles which start at considerable angles to that direction, the smallest angle in

the one case (9*b*) being 77° in the second (9*a*) even 84°. The corresponding ranges are exceedingly much smaller than for the *H*-atoms emitted in the direction of the primary α-particles.

These results agree with the result of RUTHERFORD's experiments [1]), who found all the *H*-atoms to be propelled in the direction of the primary rays. The range of this secondary radiation was 28 cms., which gives $R_1 = 0.028$ cm. or $R_1 = 0.31$ cm. according as we use (9*a*) or (9*b*). These values are too small for experimental verification, and were bound to escape detection.

§ 7. *Transition to the stationary orbits.* Up to the present time the quantum theory has only been applied to systems whose members permanently move round each other at a finite distance, i.e. systems which in the LAPLACE-sense are stable. My attempt of 1916 (l.c.) to apply the theory to the single passage of a particle through the sphere of action of a nucleus has not met with much sympathy among physicists. It therefore seems necessary to submit the difference between the two cases to a careful conceptual analysis.

The hypothesis of the theory as established by BOHR consists of two parts: 1. There are certain preferential or stationary orbits in which the system moves without radiation. 2. If the initial state is not a stationary one, the system passes into a stationary state with the emission of energy in the form of radiation. It is quite possible, that the real process is only formally represented by this division, but it has been confirmed in several cases and it forms for the present the only basis on which we can erect our further structures.

As regards the existence of stationary orbits, there does not seem to be any reason, why the quantum conditions should be solely applicable to finite orbits. Our views on this point have been expounded in §§ 3 and 4; but we shall try to strengthen them from a fresh point of view. The difference between motions which are finite and those which reach to infinity is expressed analytically by the fact, that for the former each cartesian co-ordinate may be represented as a FOURIER-series according to angular variables, whereas this is impossible for the latter. BOHR has established a relation between the terms of this FOURIER-series and the transitions which on the quantum-theory are possible from one stationary orbit to another.

In the case of the relativistic Kepler-motion the Cartesian coordinates are $x = r \cos \varphi$, $y = r \sin \varphi$. For shortness putting

[1]) E. RUTHERFORD, l.c.

$$\frac{\sqrt{p^2-p_0^2}}{p}(\varphi-\varphi_0)=\psi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

it follows from (3) that

$$x=\frac{p^2-p_0^2}{B}\,\frac{\cos\varphi}{1-\varepsilon\cos\psi} \quad ; \quad y=\frac{p^2-p_0^2}{B}\,\frac{\sin\varphi}{1-\varepsilon\cos\psi} \quad . \quad . \quad (21)$$

For a motion of elliptic type ($\varepsilon<1$) x and y are periodic in φ and ψ with a period 2π; φ and ψ are therefore angular coordinates of the problem and a FOURIER-expansion is possible [1]). Passing to the case $\varepsilon>1$ the angle ψ becomes limited and varies between the limits $\pm \arccos\left(\frac{1}{\varepsilon}\right)=\pm\bar{\psi}$. Only between these limits x and y have the meaning of the functions given in (21); hence they may now be represented by a FOURIER-integral

$$x=\frac{2}{\pi}\,\frac{p^2-p_0^2}{B}\cos\varphi\int_0^\infty \cos s\psi\, ds\int_{-\bar{\varphi}}^{+\bar{\varphi}}\frac{\cos\lambda}{1-\varepsilon\cos\lambda}\,d\lambda.$$

The case is different for φ: this angle also varies between two extreme values $\frac{\pm p\bar{\psi}}{\sqrt{p^2-p_0^2}}=\pm\bar{\varphi}$; but in contrast with ψ it possesses a physical periodicity: on changing φ by the amount 2π the same point of space is reached, so that x and y remain periodic with respect to φ. We may continue the dependence on φ in the ranges $\bar{\varphi}<\varphi<\pi$ and $-\pi<\varphi<-\bar{\varphi}$, where no motion takes place, just as we like. It would be simplest to assume the continuance of the law expressed by $\cos\varphi$ and $\sin\varphi$ over the whole range from 0 to 2π, in which case we should get

$$x=\frac{p^2-p_0^2}{\pi B}\int_0^\infty[\cos(s\psi+\varphi)+\cos(s\psi-\varphi)]\,ds\int_{-\bar{\varphi}}^{+\bar{\varphi}}\frac{\cos s\lambda}{1-\varepsilon\cos\lambda}\,d\lambda.$$

It seems to me that in this result lies a confirmation of the reasoning of § 4. The coefficient of ψ is the number s which may assume any value, whereas the coefficient of φ is the whole number 1. Extending BOHR's principles to this case we might conclude that the radial quantum which is subordinated to ψ underlies no limitations, whereas the azimuthal quantic number can only change by

[1]) These coordinates are not linear functions of the time. If we wish them to satisfy the latter condition, they have to be defined differently. But the conclusions to be drawn remain valid with this change in the definition.

1 each time. In this way it is made probable, that the azimuthal impulse possesses discrete special values and the analogy with the case of the elliptic motion imparts special probability to the hypothesis (9*b*).

Although the existence of stationary orbits is thus rendered probable, it does not follow that a particle which to begin with is not moving in a stationary orbit will have time and opportunity to pass into one. The giving off of energy requires time which is always available in the case of stable motion (in LAPLACE's sense). But for hyperbolic motion the case is different: the energy is not limited by any conditions, but the rotational impulse tends towards definite values which can only be reached by the process of radiation of electromagnetic moment of momentum. For this radiation the time available is only the one motion past the nucleus, and it is thus quite possible that the impulse lost by radiation is not sufficient and that the particle returns to infinity without having reached a stationary condition. On the basis of MAXWELL's theory this would even be the usual case. Calculation gives for the radiated impulse (for $p \gg p_0$)

$$2\frac{e^2v^2}{c^3}\frac{\varkappa e^2}{pv}\left\{\left(\frac{\varkappa e^2}{pv}+\frac{1}{3}\frac{pv}{\varkappa e^2}\right) bg\, tg\, \frac{pv}{\varkappa e^2}-1\right\},$$

that is an amount of the order 10^{-31} erg sec., whereas the steps of the constant p are about 10^{-27} erg sec., or about 1000 times larger. Under these circumstances no fraction of the particles worth mentioning could attain stationary orbits.

On the other hand we have the experimental fact, mentioned in the previous section, that the H-atoms are preferably emitted in the direction of the incident α-particles and it seems difficult to interpret this otherwise than on the quantum-theory. One of the possible explanations of RUTHERFORD's results seems therefore to be that the radiation is really stronger than would follow from MAXWELL's theory, sufficiently so to carry a considerable portion of the systems into the stationary condition. When we consider that even in the radiation of the hydrogen spectrum, where the distances from the nucleus are greater than 2×10^{-8} cms, a considerable deviation exists from MAXWELL's theory, the supposition in RUTHERFORD's case of a very much larger deviation does not appear to us too hazardous. For the distance from the nucleus is here of the order 3.5×10^{-13} and thus the acceleration about 1.5×10^{6} times larger than in the emission of the hydrogen lines. Moreover EINSTEIN[1]) has postulated a complete breach with MAXWELL's theory for elementary processes

[1]) A. EINSTEIN. Kleiner-Festschrift, Zürich 1918.

of this kind and we further know from the existence of a limit of the Röntgen-radiation on the side of the small wavelengths, that they cannot take place in accordance with the theory. According to RUTHERFORD's experiments the relative number of the emitted recoil-rays is strongly dependent on the rapidity or range R of the primary α-particles, as shown by the following table:

$R = 7{,}0$	5,3	4,5	3,7	3,0	etc.
$N = 100$	77	51	25	0.	

This might be interpreted as indicating that the radiation of rotational impulse decreases rapidly with the speed v, so that with falling v there are less and less particles which are able to reach the stationary orbit. If this view is correct, we would have in RUTHERFORD's table a new way along which to penetrate into the riddle of the quantum theory.

Side by side with particles which have completed the transition into the stationary orbit, others are to be expected, even with the highest velocities, which owing to a higher initial impulse have not succeeded in doing so. The directed radiation must therefore be surrounded with a scattered radiation. According to a kind personal communication of sir ERNEST RUTHERFORD's something of that kind is found experimentally: a new experimental method has shown that the recoil-rays are in reality less homogeneous than appeared originally and that side by side with the rapid H-particles observed at first, there are others of smaller speed [1]).

As suggested above it is probable that the large deviations from MAXWELL's theory, as required for a sufficiently strong radiation, are limited to the range of very high accelerations. This makes it doubtful, whether a similar approach to the stationary orbit $n = 1$ is to be expected as to the orbit $n = 0$; for the range near it correspondends to a much greater distance from the nucleus.

On a different occasion we hope to discuss the question, how the stationary orbits are distributed for nuclei other than of hydrogen. We shall only mention here, that for heavy atoms the equations (17) and (18) owing to the high value of the nucleus charge E make the steps of the discrete angular distribution so small, that the result cannot differ appreciably from a distribution in accordance with classical statistics.

---

[1]) Cf. E. RUTHERFORD, Phil. Mag. **41** (6), p. 307, 1921.

Mathematics. — "*Properties of Congruences of Rays*". By H. J. VAN VEEN. (Communicated by Prof. J. CARDINAAL).

(Communicated at the meeting of February 26, 1921).

§ 1. The rays of the space Σ_3 can be represented by a quadratic hypersurface O^2_4 in a five dimensional point-space Σ_5. By the aid of this representation I have derived the properties of a few complexes of rays (Nieuw Archief voor Wiskunde R 2, dl. XI, p. 232; id. dl. XII, p. 19; Handel. 17e Nederl. Natuur- en Geneesk. Congres 1919, p. 171). By the method followed there also the characteristic numbers for an arbitrary complex of rays can be determined. In what follows I shall make use of the representation mentioned to derive some of the principal properties of *congruences of rays.*

§ 2. In the first place I consider an *arbitrary congruence* of the *field-degree* p and the *sheaf-degree* q. This congruence is represented in Σ_5 by a surface V_2 of O^2_4, which has p points in common with any α-plane (representation of a field of rays) and q points with any β-plane (representation of a sheaf of rays).

§ 3. Let P be an arbitrary point of V_2. A hyperplane through P cuts V_2 in a curve that has one tangent in P; the tangents in P to V_2 form, therefore, a plane pencil. 2 of the straight lines through P on O^2_4 (images of plane pencils of Σ_3) lie in a linear space R_3 through P; for a ray of Σ_3 lies in 2 plane pencils of a bilinear congruence to which it belongs. The straight lines of O^2_4 through P form accordingly a threedimensional quadratic cone. The plane pencil and the cone lie in the hyperplane touching O^2_4 at P and have therefore two straight lines in common. Consequently:

Any ray of a congruence is intersected by 2 *consecutive rays,* or

On any ray of the congruence there lie 2 *focal points and through any ray of the congruence there pass* 2 *focal planes.*

The identity of the surface of the focal points with that of the focal planes (*the focal surface*) can easily be demonstrated now in Σ_3 (See e.g. STURM, Liniengeometrie II).

§ 4. The hyperplane R_4 touching O^2_4 at a point P, cuts V_2 in a curve V_1. This I project out of P on a linear space R_3 in R_4.

Of the α- (β-) planes through P two cut a straight line of R_3; they intersect R_3 therefore in the a- and b-lines of a quadratic surface O^2. On O^2 lies the projection k of the curve V_1; k has the a- and b-lines as p- and q-fold secants. The projections of k out of a point O of O^2 on a plane τ of R^3 gives a curve k', which has a p-fold and a q-fold point in the passages A_1 and B_1 of the a- and b-lines through O. The straight line A_1B_1 has only the points A_1 and B_1 in common with k', hence k' is of the order $p+q$. If r is the number of double points of k, hence also of k', the class of k' is:

$$(p+q)(p+q-1)-p(p-1)-q(q-1)-2r=2(pq-r)$$

so that out of A there can be drawn

$$2(pq-r)-2q=2q(p-1)-2r$$

and out of B

$$2(pq-r)-2p=2p(q-1)-2r$$

tangents to k', touching this curve elsewhere.

These numbers are at the same time the numbers of b- and a-lines touching k and also the numbers of the β- and the α-planes through P that have two coinciding points in common with V_2.

Now r is the number of bisecants of V_2 through P or the *axis-degree* (*rank*) of the congruence in consideration, hence:

The focal surface of a congruence of the field-degree p, the sheaf-degree q and the axis-degree r is of the order $2p(q-1)-2r$ and of the class $2q(p-1)-2r$.

§ **5**. I shall now consider the *complete congruence of intersection* of two complexes.

If these are of the order m and n, they have as images the multiplicities which O^2_4 has in common with two hypersurfaces V_4^m and V_4^n. These two hypersurfaces cut each other in a multiplicity V_3^{mn}. The bisecants of V_3^{mn} passing through a point P and cutting a plane π lie in the linear space $R_3 \equiv (P, \pi)$; they are also the bisecants through P of a curve in R_3 which is the complete intersection of a surface of the m^{th} order and a surface of the n^{th} order. The number of these bisecants is $\frac{1}{2}mn(m-1)(n-1)$, hence the bisecants of V_3^{mn} through P form a cone of 3 dimensions and of the order $\frac{1}{2}mn(m-1)(n-1)$. By choosing P on O^2_4 it appears that through this point there pass $mn(m-1)(n-1)$ bisecants of V_3^{mn} that lie on O^2_4, accordingly:

The congruence of intersection of a complex of the m^{th} order and a complex of the n^{th} order has the axis-degree $mn(m-n)(n-1)$.

§ 6. In order to find the order and the class of the focal surface of the congruence of intersection of two complexes, I pass two α-planes a_1 and a_2 through a point P of O^2_4. The β-plane through a ray a_1 of the plane pencil P, α_1 cuts α_2 in a straight line a_2. A plane γ through a_1 which cuts α_2 in a straight line a^1_2 and has two coinciding points in common with V_2^{mn}, touches the curve which the space $R_3 \equiv (a_1 \alpha_2)$ cuts out of V_2^{mn}; as this curve is the curve of intersection of a surface of the m^{th} order and a surface of the n^{th} order, it has the rank $mn(m+n-2)$; accordingly this is the number of the planes through a_1 intersecting α_2 in a straight line a_2^1 and at the same time touching V_2^{mn}. Between the rays a_2 and a_2^1 belonging to the same ray a_1, there exists a $[mn(m+n-2), mn(m+n-2)]$ correspondence, hence there pass through P $2mn(m+n-2)$ β-planes (and as many α-planes) touching V_2^{mn}. or:

The focal surface of the congruence of intersection of a complex of the m^{th} *and a complex of the* n^{th} *order, has the order and the class* $2mn(m+n-2)$.

Of course this result can also be derived from § 4 and § 5.

Anatomy. — "OMBREDANNE's *Theory of the "lames vasculaires" and the anatomy of the canalis cruralis*". By G. C. HERINGA. (Communicated by Prof. J. BOEKE).

(Communicated at the meeting of December 18, 1920.)

When adopting the reasonable and general view that the muscular fasciae are to be considered as compressions of loose connective tissue, originated under mechanic influences exerted by the surrounding individual muscles, a conception has been propounded, which brings before our mind the "muscle-compartments" in a clear and comprehensible way, and which is also of great practical value. Still, it would seem that the scientific value of this view may justly be contested. It would seem also that in order to obtain a clear notion of the matter these formations of connective tissue themselves should receive more of our attention than the spaces invested by the fasciae and filled up with muscles.

I would call upon the reader to consider a fascia as a thin layer of undifferentiated connective tissue, bounded on either side by a lamina of fibrillary connective tissue. Further we conceive a blood-vessel running in the middle layer and we imagine, in accordance with the publica opinio, that the two plane faces have, as it were, been attached through a polishing process to this interstitium of connective tissue by the mechanic action of the surrounding muscles, or by other tissues. This hypothesis approaches real facts, for a similar position we observe of the vasa plantaria med. und lat. in the septa intermuscularia pedis; a similar location we observe of the vena jugularis ext. in the fascia superficialis colli, of the aa. meningeae and the sinus durae matris in the hard cerebral membrane, of the vena saphena magna in the fascia lata, of the vasa epigastrica in the fascia transversalis [1]), and finally in a similar way numerous nerves — suffice it to mention only the n. cutaneus femoris lateralis, the branches of the n. femoralis, of the nervi superficialis colli — are running in the fasciae, prior to their ultimate intrusion into the skin.

The instances here enumerated, could easily be increased. They lend support to the roughly phrased conception that, generally

[1]) TESTUT et JACOB, Traité d'Anat. topogr., p. 45.

speaking, an inner layer of vessel-conducting connective tissue is the essential factor for the fasciae and *not* the polished fibrous tissue, which for the knife and for the eye is their characteristic aspect.

This is analogized by the mesentery, in which the functional essence is *not* constituted by the peritoneal epithelium lining it, but in reality by the vessels contained in its connective tissuelike substrate [1]). The thought underlying the view set forth here, is by no means new. It is OMBREDANNE's. This author has worked it out in his thesis (Paris 1900), particularly for the abdomen and the pelvis, but he also surmises its validity for the whole body:

"Il existe", OMBREDANNE says, "entre le peritoine de l'abdomen et du bassin une lame vasculaire, contenant les vaisseaux dans son épaisseur, émettant une lame secondaire, quand l'artère émet une série de branches dans un autre plan, que son plan de ramfication principal émettant une gaine vasculaire periviscerale quand l'artère émet un bouquet de branches allant envelloper un organe, présentant des renforcements du côté, d'où viennent les pressions, et capable de se souder au niveau de ses plicatures".

We see then that according to his description the fascia pelvis is such a "lame vasculaire" differentiated in the subperitoneal connective tissue, which "lame" is in the first instance attached to the large vessels lying against the pelvic wall, but from which "lame" determined by the outgoing ramifications, a number of lateral septa are emanating, which, in a frontal arrangement, divide the subperitoneal space into a number of partitions [2]) or leave the pelvis along with the outgoing vessels (vv glutaea, pudenda) and thus may assume the character of intermuscular septa. There is, according to OMBREDANNE, a close relationship between vessels and connective tissue, which invariably reveals itself in a condensation of the latter, in continuous connection with the vascular adventitia [3]). Fundamentally it is quite immaterial whether here, as in the case alluded to in the outset, pressure or distension on the part of the environing tissue influences the morphosis of the whole, or whether it does not. In the region of the pelvic fasciae examined by OMBREDANNE this manifests itself e.g. by the fact that a rather smooth plane of con-

[1]) This parallel between fascia and mesentery, which also OMBREDANNE has drawn, might still be extended, if, as is most likely, every fascia has u superficial lining of endothelium.

[2]) A clear and concise compendium of it may be found in TESTUT-JACOB. Tome II, p. 391-393.

[3]) In a section through the umbilical cord the more precise arrangement of the connective tissue elements round the vessels is distinctly noticeable.

nective tissue, the so-called muscular fascia of the M. levator ani, lines the muscular bottom of the pelvis; that, however, the frontal extensions, which do not undergo any pressure, gradually fade away in the subperitoneal connective tissue.

My starting-point then was the fundamental idea of OMBREDANNE's reasoning, which runs as follows: ".... que les vaisseaux ne sont nulle part libres entre deux feuillets fibreux, ou entre un plan fibreux, et un plan périostique; si l'on arrive à les isoler, à les disséquer, comme on dit, c'est à l'aide du tranchant d'un scalpel"... "Nous avons dit, que nous ne croyons pas les vaisseaux inclus libres entre deux feuillets dans une gaine, mois plongés dans l'epaisseur d'une lame: la formation de ces lames est la conséquence même de la formation du tissu conjonctif. Ce tissu est essentiellement un tissu de remplissage, de soutien; mais il n'existe que là, où il soutient quelque chose".... "Lorsque une artère s'épanouit en un éventail de branches, disposées dans un même plan, il constitue à cette artère et à ses branches une lame vasculaire, qui pourra s'infléchir, dont partiront des lames secondaires."

It was an investigation on a totally different basis which by chance, as it were, led me in the direction of OMBREDANNE's conception of which I had some cognizance through TESTUT and JACOB's work. But for the very reason that the results of this investigation compelled me to follow the general principle of OMBREDANNE's conception, I feel inclined to preface the publication of those results with a few theoretical considerations.

In the year 1919 FRANSEN of Leyden wrote a thesis in which the significance of the fasciae for the bloodstream in the venae was emphatically brought forward. As shown by him: "over the whole length of the lower extremities there exist formations of which the principal components are fasciae connected with muscles; formations, which guard the large venae and the arteriae from compression, and through their relation to the environment can perform the function of a suction-apparatus, by which the venous circulation of the blood and the continuous supply of arterial blood is guaranteed and promoted. Theorically as well as practically (with regard to the pathogenesis and the therapy of the varices) FRANSEN's clear exposition is no doubt of great value. It seems to me that he has hit the nail on the head in his roughly worded conclusion quoted above. Still, I presume to raise an objection to the details of his exposition, however cautiously they may have been laid before us. Among the instances prefixed by FRANSEN to the conclusion just quoted, we distinguish two groups of different nature. On the one

hand he describes the vena femoralis as lying "in a prismatic space", bounded by fasciae, and in which a negative pressure is supposed to arise with contraction of the M. sartorius through tension of the fascia lata. On the other hand in the same demonstration the circumstance that the vena poplitea is kept open on leaving HUNTER's canal, is correlated with an *arcus tendineus* stretched by adductor fibres and grown into one with the vascular sheath. Then again, according to the writer the same principle holds for the fossa poplitea and for the fossa *Scarpae*. On the other hand again, the arcus tendineus of the M. Soleus is supposed to act in the same way for the vasa tibialia, passing below it, as above the M. adductor magnus for the vv. poplitea.

It is obvious that two different principles have been mixed up here by the writer. Moreover the question arises whether these two mechanisms to keep the venae open actually coincide.

Now, I think, some objections may at once be raised against FRANSEN's interpretation of the effect of a negative pressure in the fossa Scarpae and the same applies of course to the fossa poplitea.

In the first place for a negative pressure-action to arise, as FRANSEN conceives it, it is necessary for the "space" containing the vessels, to be closed hermetically. It is both groundless and improbable to suppose that such a closure should be brought about by the fasciae.

In the second place that "space" round the vessels is quite filled with more or less closely woven connective, as is easily made out by the study of transverse sections. And though the intercellular, interfibullar groundsubstance of that connective tissue may be humid, it would require special experimentation to see whether any negative pressure could be propagated through such a colloidal medium.

In the third place, granting these two conditions to be fulfilled and the influence upon the *vena* as supposed by FRANSEN to really exist, two other conditions have to be satisfied for an effect on the *bloodstream in* the vena:

a. change of place or form of the vena as a whole on one side should be excluded. A "prismatic space" could not satisfy this condition

b. Since we cannot conceive of a sucking pump without valves, there must needs be at least one valve proximad from HYRTL's suction-apparatus. Now it is remarkable that on page 26 of his thesis FRANSEN says, in agreement with DELBET: "No valves occur (or if any, the are insufficient) in the vena fomoralis beyond the outlet of the vena saphena magna."

So it seems to me that FRANSEN's view is open to controversy on

several points, even though, for the rest, the anatomic relations were such as he supposes them to be. In fact, the last-named are decidedly more complicated. It seems to me that, on arguments to be brought up presently, we should really no longer consider the "fossa" Scarpae as a space enclosed by walls independent of its contents. There is, on the other hand, every reason to follow in OMBREDANNE's track, and assume it to be a connective tissue enclosed by muscles, carrying the vasa femoralia, besides the lymphatic vessels, and the lymphatic glands, of which connective tissue the superficial layers (i.e. those lying against the muscles) have been compressed into fascia lata, deep and superficial layer. In other words I am inclined to consider the fossa Scarpae, wall and contents taken together, as a "lame vasculaire" which has adapted its shape to the available space.

Still, although I cannot endorse FRANSEN's exposition, I think nonetheless that his pronouncement regarding the functional significance of the fascia lata for the bloodstream in the vena femoralis, is correct in so far as the fascia plays a part in keeping the vena open. But this, I believe, to be the consequence of the direct connection of the fascia lata with the connective tissue sheath of the vessels themselves.

Now about the relations in HUNTER's canal. Well, it seems to me that they may readily be looked upon from the same point of view that we just now suggested for the fossa Scarpae. This could be realised beforehand since HUNTER's canal is directly continuous with the fossa Scarpae. FRANSEN points out, with reason, that HUNTER's canal is invested by three aponeurotic fasciae. Well then, here also the vessels do not lie loose in that space, but wall and contents again form one whole. Here again, just as with the fossa Scarpae, the connective tissue layer bordering on the muscles, assumed the character of a fascia, in this case complicated because muscle-fibers attached themselves to this fascia and thus bestowed on it an aponeurotic character. It follows then in my opinion that FRANSEN's arcus tendineus is to be conceived as an aponeurotic fibre-bundle fascie, closely related to the vessels.

As I alluded to above, the investigation which led me in the direction of OMBREDANNE's theory was of quite a different nature, so that in order to discuss this question I must call the reader's attention to a totally different matter. I was induced to undertake my investigation by a question, put to me by Dr. LA CHAPELLE, at the time assistant at the surgical clinic of Leyden.

I will repeat the question here in its original form. The starting-

point was the following observation made by Dr. LA CHAPELLA: "After reposition of the stump of a tied up hernial sac in non-incarcerated hernia femoralis, a more or less sharp border may be felt when passing the finger under Poupart's ligament, the border constituting an arch over the os pubis. The question to be answered runs as follows: Has this border, which will incarcerate the hernial sac, even before Poupart's ligament does so, been preformed anatomically, and if so, what is it?

We might alter the question and say that we have to establish the identity of a clinically tactily ligament, which being concentric with Gimbernat's ligament, but located on a deeper level, narrows the entrance to the crural canal. Now, seeing that, as we read in surgical manuals, the incarcerating factor proper of the hernia femoralis is not known precisely, because after the removal of Poupart-Gimbernat often still further cleaving of deeper fibers, entwining the neck of the hernial sac, is required, we deemed it worth our while to look into this matter from a practical as well as from a theoretical point of view. This inquiry was begun by myself in conjunction with Dr. LA CHAPELLA. Only a considerable time later could I conclude it, thanks to Prof. VAN DEN BROEK's and Prof. BARGE's kindness in granting me the loan of their material at Utrecht. I also feel indebted to Dr. VAN RIJSSEL, at the time prosector for pathological anatomy at Utrecht, for yielding me an opportunity to verify the results obtained in the anatomical laboratory at an obduction corpse. I thus examined three corpses in all, two male bodies and one female, while of another female corpse sagittal frozen sections through the pelvis were examined.

When removing from the triangle of Scarpa the skin and the superficial layer of the fascia lata, and cautiously cleansing the large deep lymphatic vessels, it will be seen that the latter disappear into the medial upper angle of the regio, where by the aid of Gimbernat's ligament, the ligament of Poupart touches upon the pecten ossis pubis and fastens itself to the fascia pectinea. This convergency of the lymphatic vessels, besides the location *in situ* of one or more of the lymphatic glands (Rosemüller's glands) induce us to suspect Cloquet's septum to lie in that corner of the inextricable fibrous tissue. This septum Cloqueti is described as a subdivision of the fascia transversalis, which after fastening itself to Poupart's ligament extends to the os pubis and thus obturates the entrance to the crural canal, as a vertical septum pierced only by lymphatic vessels. But, if we closely consider what our preparation reveals, there seems to be something wrong, viz.

the entrance to the crural canal seems not to fill up all the space between Poupart and the os pubis; as none of the lymphatic vessels on its way to the pelvis passes directly beneath Poupart, but they all dive down to a deeper level. And when palpating cautiously along the frontal border of the complex of lymphatic vessels, we feel that the frontal border of the entrance to the crural canal is formed *not* by Poupart's ligament but by another tightly stretched connective tissue band, which runs parallel to Poupart, but is about 1 c.m. lower and lies somewhat deeper, and is approximately concentric with the curving fibres of Gimbernat's ligament; it also appears that this frontal border as well as Gimbernat's ligament converged with the fascia pectinea.

Through this palpation we were able to corroborate to some extent LA CHAPELLE's finding and now it will not be difficult to lay bare the bundle of fibres under consideration. If namely the lymphatic vessels are prepared away, we can see this bundle as well as when we felt it just now. (Fig. 1). We see then that the entrance to the crural canal is bounded at the front by this strand, which is separated from Poupart by a loose-meshed connective tissue.

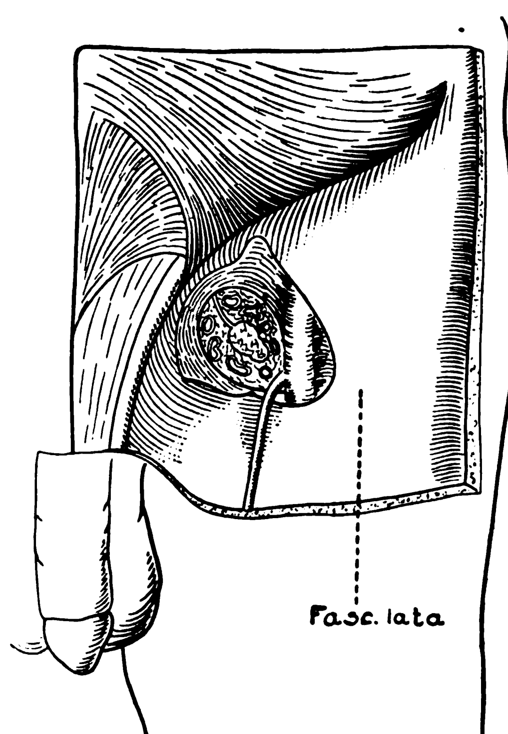


Fig. 1.

Furthermore, if we expose the deeper layer entirely by cutting Poupart through and folding the flap back, we see that what seemed to be a strand of connective tissue, is in reality the sharp, free lower border of a fascia, which descends from above behind Poupart.

This being established it was not difficult to expose this fascial leaf upwards, and to ascertain its continuity with the fascia transversalis abdominis. When this fascia is laid bare by preparation,

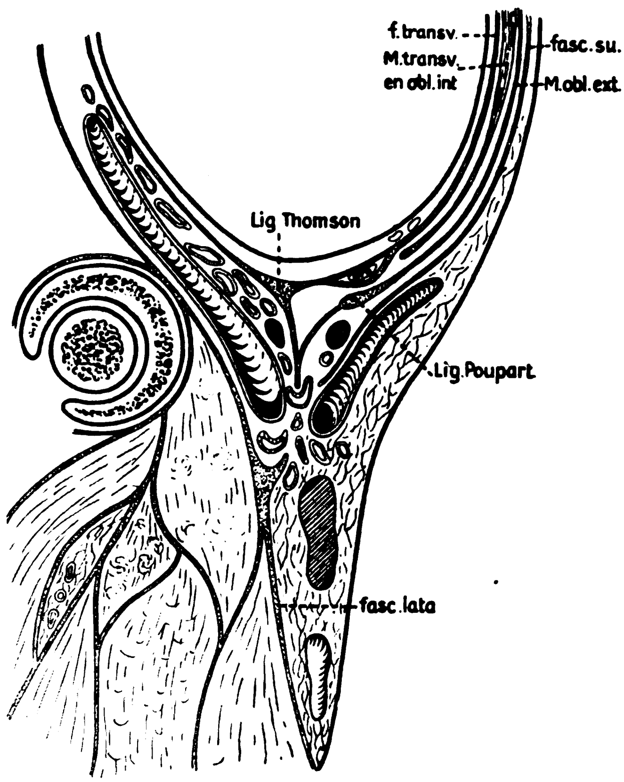


Fig. 2.
Sagittal section of a female pelvis about the vv. femoralia.
Semi-schematic.

after removal of skin and muscles, it will be seen to descend farther, after having passed behind Poupart's ligament, under exchange of a more or less distinct supply of fibers, till it meets at a right angle the vasa iliaca externa, which runs subperitonically on the bottom of the pelvis. These vessels it envelops in the well-known manner in which the fascia pelvis encloses the organs which pierce the bottom of the pelvis; it envelops the vessels with a fibrous formation, which partly runs proximally with the vessels

partly accompanies them distally and *passes with them to the crural regio beneath Poupart's ligament.* The fascia layer, of which, as described above, we could spy the free border after removal of the fascia lata (superficial layer), is indeed nothing else but the lower end of the fascia transversalis, which *could not but pass on to* the thigh, because it had been inseparably attached to the vessels by the formation of the vessel-sheath.

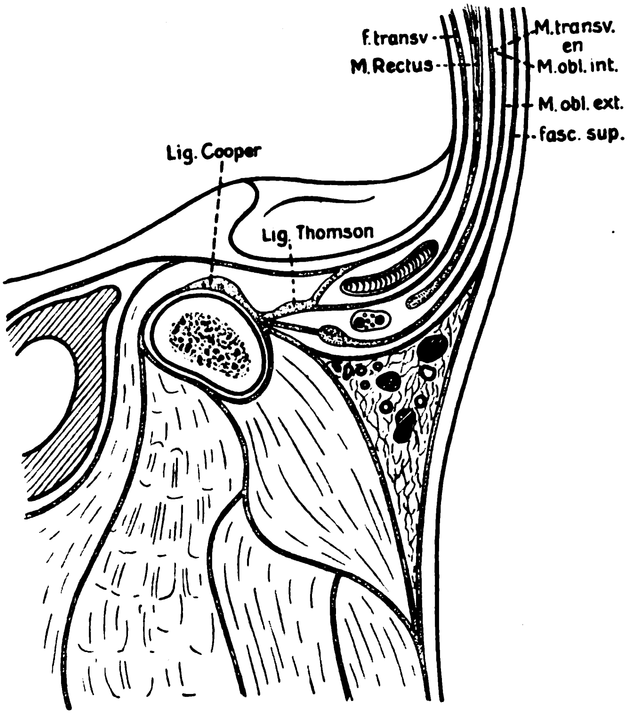


Fig. 3.

Paramedian sagittal section through a female pelvis. Showing semi-schematically the insertion of THOMSON's ligament on the os pubis. It is remarkable that, in consequence of an exchange of fibers of the m. rectus and the f. transvers, a close connection is brought about between THOMSON's lig. and the insertion of m. rectus.

In harmony with this the free border of this extension of fascia, just now described, and radiating out medially into the fascia pectina, concurs in forming the vessel-sheath where it could no more be separated from the fibers of the fascia lata.

When I had proceeded so far in my investigation, I felt justified in establishing that the fascia transversalis, instead of inserting itself to the os pubis and of forming CLOQUET's septum, terminates medial to the femoral vessels with a free border, and that, therefore, the lymphatic vessels, instead of piercing the fascia (in loco Sept. of CLOQUET), pass undermeath it to the

subperitoneal areas of the pelvis. That more laterally the fascia transversalis attaches itself the ruscular sheath of the femoral vessels, did not surprise me as I was not ignorant of TESTUT-JACOB's description of the "insertion inférieure" of the fascia transversalis: "Puis, continuant son traject descendant, ce fascia rencontre les vaisseaux femoraux: il se fixe sur leur pourtour, en contractant avec eux, avec la veine particulièrement, des adhérences intimes" (p. 43).

However my view was considerably altered when a close study of the sagittal sections taught me the following facts:

1. That the free border alluded to was an artificial product originated during the removal of the fascia lata for that there is a continual relationship between the fascia transversalis and the fascia lata.

Fig. 3 shows how some of the fibers of the fascia lata attach themselves to Poupart, resp. blends with the fascia abdominis superficialis; how others, the larger number of its fibers, running along *behind* Poupart get connection with that lower end of the fascia transversalis, which, as described above, reaches the crural region together with the vessels. Thus formulated this remark may sound strange to the reader; on closer inspection, however, our representation is not so strange, not even new. Also in the literature mention is made everywhere of the junction of the fibers of these two fasciae in the ligamentum inguinale, to which, from above, the fascia transversalis, from below the superficial layer of the fascia lata, attaches itself. And besides, already DELBET (POIRIER. Traité, T. 5, pag. 89) reports a concurrence of the f. transversalis with the vessels on the thigh.

2. We saw just now that the fascia lata and transversalis were connected when passing before the vessels, on the tacit understanding that with f. lata we meant only the superficial leaf. Now the sagittal sections afford us another datum of fundamental significance, viz. that while passing behind the vessels the fascia transversa is in precisely the same way connected with the *deep* layer of the fascia lata, i.e. with the layer that lines the bottom of the fossa scarpae. This is no novelty either, since, as we know, COOPER's ligament lining the pecten ossis, is generally acknowledged to be an interlacement of fascia transversa and deep fascia-lata-fibers.

When combining the facts mentioned under 1 and 2, we arrive at the conclusion that, as said above, the fascia transversa, on meeting the vasa iliaca, not merely attaches itself to them, but forms round it a closed sheath. For further illustration we add the following particular:

Just as Poupart's and Cooper's Ligament are formed by transversal fiber bundles interwoven where fascia-extensions, coming from different directions, meet, we also find in a number of cases a fully developed frontal fiber-bundle there where passing behind Poupart f. transversalis and fascia lata become merged. This is the very bundle felt by La Chapelle in vivo. It naturally runs parallel to the ligamentum inguinale in a somewhat deeper level. This ligament, which I observed most distinctly in two of the four corpses examined, and which in the frontal section of the female pelvis was highly conspicuous as a bundle of 2 mm in diameter (see figs 2 and 3), is no doubt the same as Thomson's ligament (Bandelette iliopubienne) described especially by French writers.

3. While then the fascia transversalis forms distally a closed vascular sheath about the vasa femoralia, proximally te same thing occurs, as pointed out above. This bears more particularly on Ombredanne's region. According to a number of authors the fascia transversalis reaches no further down than to the upper limit of the pelvis, which means to say down to the os pubis. It goes without saying that this view is incompatible with the relation described by me. Besides relying on direct observation of my sagittal sections which in my judgment, proves irrefutably the connection between fascia transversalis and fascia pelvis, I also call to witness other authors making special mention of this connection. First of all Spalteholtz in his well-known atlas (p. 613); secondly Paul Delbet, whom we have referred to in connection with the crural extension of the fascia transversalis. There we read: „En bas il (F. tr.) descend seulement jusqu' au pubis (Charpey, Pierre Delbet); le plus souvent, si j'en crois mes recherches, confirmées par les récents travaux d'Omrredanne, il se continue jusqu' au plancher pelvien. Il recouvre alors sur la ligne mediane le pubis, qu'il sépare de la vessie, latéralement le ligament de Gimbernat; plus en dehors l'orifice des vaisseaux fémoraux. Une partie de ce tissu cellulaire suit les vaisseaux dans la cuisse, l'autre descend directement derrière l'orifice en formant le septum crural. Par sa face antérieure il adhère intimement aux arcade de Douglas, en bas il contracte avec la face postérieure du pubis des adhérences laches", etc. It will be remembered that in the exordium of this article we have stated that Ombredanne has emphatically asserted that the vessels in the subperitoneal pelvic connective tissue are not merely lined by the fascia pelvis, but that this fascia provides them with a complete sheath, that as Ombredanne puts it, then run *into* this fascia. Now when correlating this description, which as already

mentioned, was endorsed by many French authors, with my observations concerning the fascia transversalis, which yielded data quite in keeping with OMBREDANNE's conception, we conceive the following image of the entire complex of fasciae in these regions and its relation to the large vessels:

From their subperitoneal course along the floor of the pelvis, up to the leg the vessels run through one continuous tubular sheath, formed by a series of interconnected fascia-formations, in which, however, as OMBREDANNE rightly points out, the vessels are not free but are enclosed by, embedded in, massive connective tissue. It seems, therefore, admissible to assume that these fasciae are, indeed, nothing but compressions, (due to some mechanic cause), of the vascular connective tissue, where its surface was exposed to friction and pressure from environing elements. [1])

Still there is one more item I should like to discuss in this connection. The importance which OMBREDANNE ascribes to the vessels in his treatise, and which reveals itself also in his terminology (lames vasculaires) has caused him to be accused of one-sidedness. That OMBREDANNE's conception requires, indeed, to be worked out a little more, cannot be demonstrated better than in the very region of Scarpa's triangle. For, if we once more consider the boundary of the "fascial tube", the frontwall will be seen to be formed by the superficial layer, the back-wall by the deep layer of the fascia lata, both of which are continued in the fascia pelvis with the co-operation of the fascia transversalis. It is clear, therefore, that what is known in topographical anatomy as "Fossa Scarpae" is nothing else but a considerable distension of the "lame vasculaire"; which distension may be explained by the presence of the large number of lymphatic glands *in situ*, which will apparently occur also in that "lame". For the very reason that the lymphatic vessels require so much space the "fascial tube" cannot enfold the vessels closely, on its passage under THOMSON's ligament; this is why this ligament of THOMSON, instead of uniting medially to the vessels with the deep layer of the fascia lata, attaches itself only much farther medially to the f. pectinea; and finally this is why we observe on the medial side of the vessels a mass of connective tissue, furrowed with lymphatic vessels, bridged over at the front by THOMSON's ligament, at the place where (in my judgment wrongly)

---

[1]) The term „fascial tube" should not bring before our minds individual, independent formations. I could not find a better expression. From what has been said and will still be said, it will be clear to the reader, I hope, that it is just the passive capacity of assimilation of the connective tissue that I wished to accentuate.

observers presume to have detected the Septum of Cloquet. In reality what has been described as such, is nothing but a mass of connective tissue bundles filling up the space left between the lymphatic vessels. I am firmly convinced that, together with the vessels, also the lymphatic vessels run from the leg as far as the pelvis and perhaps further, in a permanent connective tissue substrate, surrounded by connected fasciae, which according as the local relations vary, will be outlined more or less sharply.

In the foregoing I do not at all presume to have brought forward new facts. Such facts as were disclosed in my preparations have already been discussed by others. How could it be otherwise, considering that the same field has already been worked up thoroughly numberless times by numberless anatomists? If then, in spite of this I have been bold enough to take up again the anatomy of the crural canal, it is because I believe that the bringing together of some details, about which some authors still disagree, may serve two purposes:

In the first place this inquiry may be conducive to increase the appreciation of the connective tissue in the strict sense of the word, also in macroscopic anatomy, without derogating from the fasciae. We only wish to lay stress on the fact, that as OMBREDANNE argues, the connective tissue is essentially a supporting tissue: „. . . mais il n'existe que là vù il soutient quelque chose". Moreover, apart from the appreciation due to FRANSEN's valuable work described in his thesis, my paper may tend to forestall the view that the fossae of the topographical anatomy are to be considered essentially as spaces, in which a vacuum could readily be induced.

In the second place what has been reported in this paper, may be of some value for applied anatomy, also in another respect. For instance our conception of the canalis cruralis is somewhat modified by it. Whereas hitherto it has been described as a region, more or less independent, and enclosed by independent muscular fasciae, it would be more proper, I think, to look upon this path, along which the hernia proceeds, as a subdivision of a large continuous complex of connective tissue. It is generally imagined that an entrance into this forbidden space is made by forcing Cloquet's septum, which is supposed to stand at the beginning of the tunnel. I, on the contrary, would contend that, since there is no real entrance of the tunnel anyhow not in that sense, a spot must be found somewhere else, when the intrusive peritoneum can press itself into the fascial tube. Indeed, I believe to have found a "weak spot" in the fascial wall, which may be deemed answerable for such a dereliction of duty. When we consider that the fascia transversa, running across

the os pubis, leaves the pelvis at the place, where the vasa femoralia is located (see Fig. 4), while it covers the posterior part of

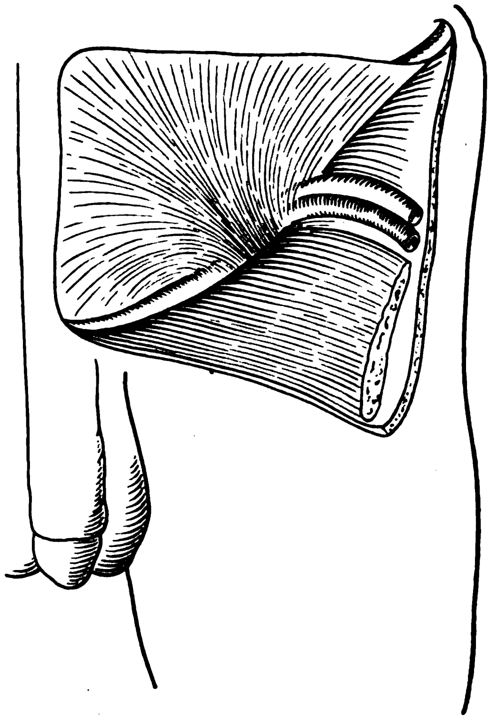


Fig. 4.
Front-view of the f. transversalis after detaching and folding back all layers of the anterior abdominal wall lying before it.

this bone, i.e. the interior of the pelvic wall, medially behind the corpus ossis pubis, we may naturally expect that in one spot or other, situated between these two places, the fascia transversa must cross the os pubis. Now, this crossing-point could easily be found in our anatomical preparation, because at the same place a transparent spot became at once conspicuous. Moreover this spot was less resistant to the pressure of the palpating finger, so that it is not out of the bounds of probabilities to state that there, that is exactly on the inside of THOMSON's ligament, the fascia yields to the peritoneum and allows it to force its way into the canal. It should be noticed that, if this supposition is correct, the hernial sac must enter the canal from the medial side, on the medial side of the vessels, consequently precisely at the spot where the septum femorale is usually localized, so that my view does not clash with daily experience. However, further investigations may throw more light upon the matter.

Mathematics. — "*Explanation of some Interference Curves of Uniaxial and Biaxial Crystals by Superposition of elliptic Pencils*". By J. W. N. LE HEUX. (Communicated by Prof. HK. DE VRIES).

(Communicated at the meeting of February 26, 1921).

In LISSAJOUS' "Etude optique des mouvements vibratoires"[1]), the name of "Unisson" is given to the curve, resulting from the composition of two vibrations, which only differ in amplitude and in phase.

When the amplitudes are supposed to be equal and not diminishing with continued movement, when the directions are at right angles and the difference of phase increases from 0° to 90° — the unisson may be considered as a pencil of ellipses, whose envelope is a square[2]).

The null-ellipse of this pencil is a diagonal d_1 of the square, the end-ellipse is the circle, inscribed in the square. Let the other diagonal be d_2.

Two equal unissons, U_1 and U_2, partially covering each other, produce certain "watered curves" (moiré), which may be divided into two sets:

1°. those similiar to hyperbolas, when the exact covering of U_1 and U_2 may be obtained by moving the centre of the pencil along d_1 (fig. 1) and

2°. those, similiar to lemniscates, when the exact covering may be obtained by moving the centre along d_2 (fig. 2).

The "watered curves"[3]), above mentioned, bear a strong resemblance to the interference curves of some crystals — it will be examined, whether these interference curves may be explained by superposition of two pencils of ellipses.

Therefore, the image of the hyperbolas will be compared to the interference curves of a uniaxial crystal in convergent light, the crystal-plate being cut parallel to the optic axis and the image of the lemniscates to the interference curves of a biaxial crystal, the plate being cut perpendicular to the first diameter.

---

1) Annales de Chimie et de Physique, 3ième série. t. LI. Octobre 1857.

2) Proc. Kon. Acad. v. Wet. pp. 857—870 March 1914.
Mathésis, 3ième série t. X pp. 209—212, 1910.

3) On stereoscopic curves, see Comptes Rendus t. 130 p. 1616.
Also: Harmonic Vibrations and Vibration Figures by J. GOOLD, C. E. BENHAM, R. KERR and Prof. L. R. WILBERFORCE, Newton, London.

The "isophase" surface of BERTIN, by means of which interference curves usually are explained, is the locus of points with a constant difference of retardations:

$$\vartheta = dV\left(\frac{1}{V_1} - \frac{1}{V_2}\right).$$

In this formula d means the length of the way in the crystal, supposing, with BERTIN[1]), that the bifurcation of the ray is neglected, V the velocity of light in the medium, V_1 the velocity of the ordinary and V_2 that of the extra-ordinary ray in the crystal.

V being constant, we may write $\frac{\vartheta}{V} = \frac{d}{V_1} - \frac{d}{V_2} =$ constant.

The centre of the ellipsoid of polarisation is supposed to be in the centre of light in the lower side of the plate.

Let P be a point of the image of interference curves in the upper side of the plate, where $d = d_1$.

Suppose $\frac{d_1}{V_1} = m$, then $d_1 = m V_1$, so P lies on a surface $\varrho = m V_1$ ($m =$ constant), homothetic with the blade $\varrho = V_1$ of the surface of the wave.

Suppose $\frac{d_1}{V_2} = n$, then $d_1 = n V_2$ and P lies also on the surface $\varrho = n V_2$ ($n =$ constant), homothethic with the blade $\varrho = V_2$ of the surface of the wave.

The surfaces $\varrho = m V_1$ and $\varrho = n V_2$, each being cut by the upper side of the crystal plate in a pencil of curves C_1 and C_2, when m and n are variable, it is evident, that each curve of the image is the locus of the points of intersection of those curves of the pencils, which correspond to $m - n =$ constant.

The forms, into which the wave is found to diverge, are a sphere and an ellipsoid for uniaxial crystals; so the sections with the upper side of a plate, cut parallel to the optic axis, are a circle and an ellipse, having the same tangent in the extremities of the minor axis of the ellipse (the section in the upper side is an approximate form of that in the lower side of the plate).

For the wave in a biaxial crystal, we find two surfaces, which are in fact one continuous surface. A plate, cut perpendicular to the first diameter, gives two ellipses, one of which is wholly surrounded by the other.

Thus, it may be said generally, that interference curves may be considered as "watered figures" of two concentric pencils of ellipses E_1 and E_2.

[1]) Ann. de Chim. et de Phys. (3). 63 pp. 57—92, 1861 en Sér 2. T 63. 1861.

W. N. LE HEUX: "Explanation of some Interference-Curves of Uniaxial and Biaxial Crystals by Superposition of Elliptic Pencils".

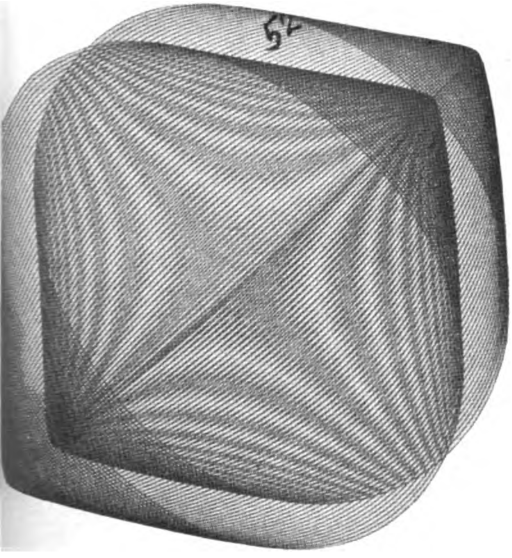


Fig 1.

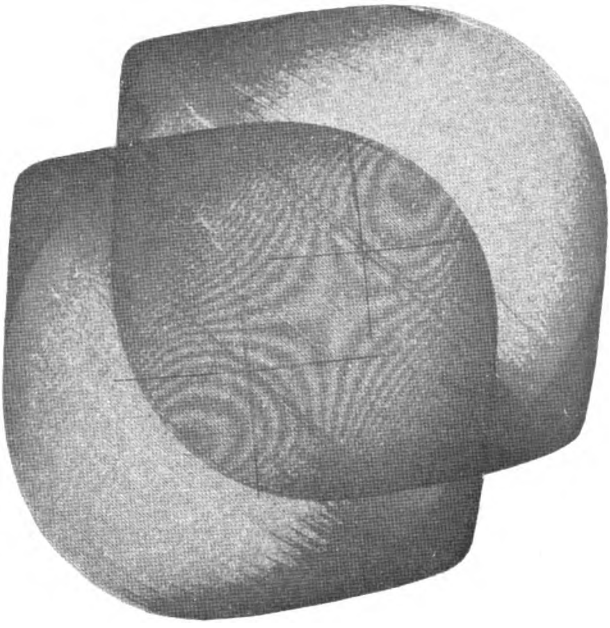


Fig. 2.

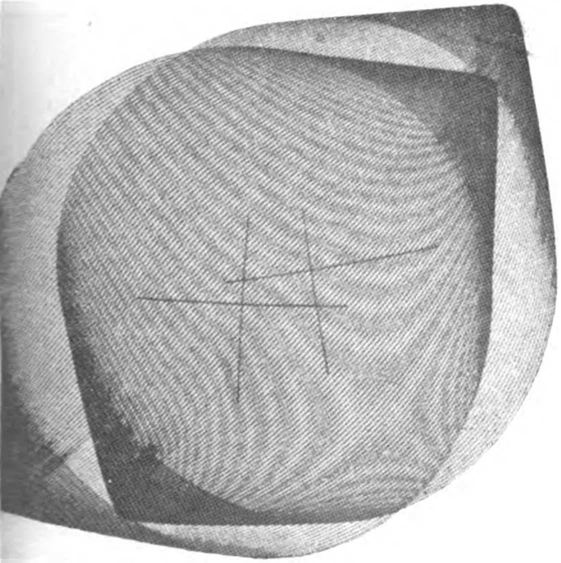


Fig. 3.

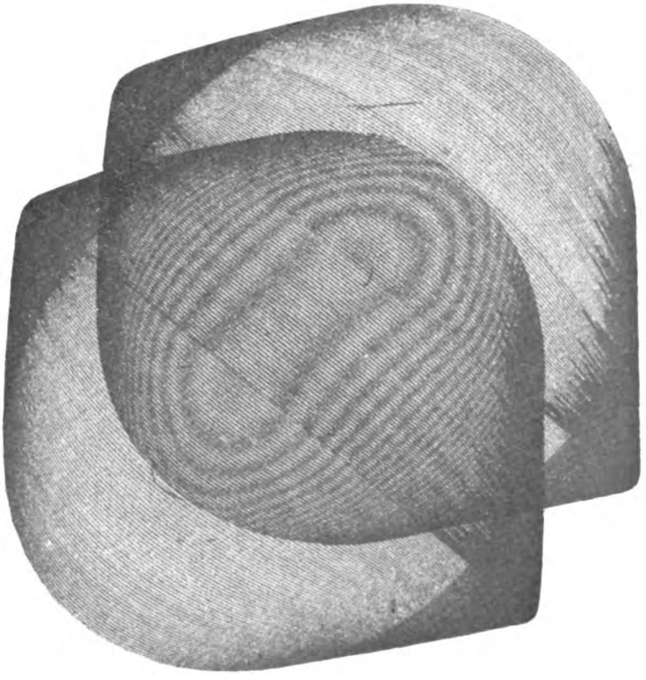


Fig. 4.

Now, it is possible by means of a simple algebraïc transformation, to derive these concentric pencils E_1, and E_2, from the excentric pencils U_1, and U_2.

The curves of the new pencils however are composed of four vibrations, which differ only in phase and whose directions two by two are parallel to those of the compounding vibrations of the unissons.

To support these conclusions by experiment, I have made use of an instrument [1]) planned by myself, with four pendulems, two of which describe a LISSAJOUS-curve on a plane, which describes such a curve itself. In our case, those two curves are unissons. The resulting movement is a spiral, which may be considered as a pencil of concentric circles. By altering the difference of phase, the circles are converted into ellipses.

It is shown now by experiment, that always hyperbolas are obtained by superposition of two pencils, which have for null-curves a circle and an ellipse, having the same tangent in the extremities of the minor axis, and lemniscates, when the null-curves are two ellipses, one of which is surrounded by the other.

Because the pencils U_1 and U_2, by inversion, may be derived from the pencils E_1, and E_2, it is evident, that the "watered curves" of the unissons are approximate images of interference curves of uniaxial and biaxial crystals.

This theory is supported by further experiments. The centre of the hyperbolas is displaced by a small rotation of one of the unissons. The image resembles the interference-curves of a uniaxial crystal, cut non-parallel to the optic axis (fig. 3).

With a small number of curves, the image of the lemniscates has both poles surrounded by the inner curve, just as is taught in crystal-optics. (fig. 4). Two concentric pencils of circles (each compounded of four vibrations), show the phaenomenon of Newton's rings.

Probably it is also possible to obtain AIRY's spirals in this manner, and the way, in which a certain image appears by superposition of two pencils of ellipses may throw more light upon the phaenomena of refraction and polarisation of light in crystals.

[1]) Other instruments are described in: Harmonic Vibrations and Vibration Figures by GOOLD etc.

Also: A. C. BANFIELD. The Photo-Ratiograph, Illustrated London News, Sept. 25th 1920. pg. 470.

Comptes Rendus t. 130. pg. 1616.

H. J. OOSTING. Hand. XIV Ned. Nat. en Geneesk. Congres March 1913. Ann. d. Phys. u. Chem. N. F. 33. p. 415 1888; Maandbl. voor Natuurwetenschappen 1898; Zt. f. d. Physik. u. Chem. Unterr. 8 p. 190 1895 and 11, p. 221 1898.

Mathematics. — "*On analytic functions defined by certain* LAMBERT *series.*" By J. C. KLUYVER.

(Communicated at the meeting of March 26, 1921).

The definition of the analytic function was based by WEIERSTRASS on his theory of power-series. From a given analytic expression we deduce an element of the analytic function, that is a power-series converging within a determinate circle, and by the continuation of this element an analytic function is defined existing within the region that is covered by the set of the circles of convergence. One and the same analytic expression in distinct regions may define several functions. So, for instance, TANNERY's series

$$\sum_{n=0}^{n=\infty} \frac{z^{2^n}}{1-z^{2^{n+1}}}$$

for $|z|<1$ will represent the analytic function $\varphi_1(z)=\sum_{k=1}^{k=\infty} z^k = \frac{z}{1-z}$, whereas for $|z|>1$ the expression defines the analytic function $\varphi_2(z)=-\sum_{k=1}^{k=\infty} z^{-k} = -\frac{1}{z-1}$. Both functions, each of them defined in a separate region, can be continued over the whole plane, but manifestly they remain everywhere essentially distinct.

In fact, from the general theory it follows that the concept of an analytic function is not co-extensive with the concept of functionality as expressed by an analytic expression and it is precisely this fundamental idea that, as BOREL repeatedly pointed out, sometimes leads to conclusions which are not always in every respect satisfactory [1]).

BOREL supposes that a given analytic expression $F(z)$ defines a function $\varphi_1(z)$ inside a certain closed curve C and moreover a second function $\varphi_2(z)$ in the region outside C, the singularities of these functions being everywhere-dense on the curve, so that C for both functions constitutes a socalled natural limit. He then shows that the series of polynomials representing $\varphi_1(z)$ under certain conditions remains convergent, absolutely and uniformly, when the variable z along certain radii crosses the boundary C. Otherwise said, it occurs

[1]) Leçons sur les Fonctions monogènes uniformes d'une variable complexe. Chap. III.

that the value of an analytic expression, coinciding at first with that of the function $\varphi_1(z)$, can be made to change continuously into the value of the function $\varphi_2(z)$ and this possibility more or less seems to be incompatible with the theory, according to which the functions $\varphi_1(z)$ and $\varphi_2(z)$ are wholly unconnected.

In the present paper I propose to treat two simple examples in which the transformation of $\varphi_1(z)$ into a series of polynomials is not necessary, and that, as I believe, yet give an insight into the tendency of BOREL's remarks.

Let the given analytic expression be the series of LAMBERT

$$F(z) = \sum_{n=1}^{n=\infty} \frac{1}{n^s} \frac{z^n}{1-z^n},$$

where the exponent s may supposed to be real.

Clearly, whatever be the value of s, we can expand $F(z)$ into an integral series, and as for $|z| < 1$ we have

$$\left| \frac{1}{n^s} \cdot \frac{z^n}{1-z^n} \right| < \frac{1}{n^s} |z|^n \cdot \frac{1}{1-|z|},$$

$F(z)$ defines an analytic function $\varphi_1(z)$ inside the circle C of radius unity. However, if $s > 1$, we may write

$$F(z) = -\sum_{n=1}^{n=\infty} \left(\frac{1}{n^s} + \frac{1}{n^s} \cdot \frac{\frac{1}{z^n}}{1-\frac{1}{z^n}} \right) = -\zeta(s) - \sum_{n=1}^{n=\infty} \frac{1}{n^s} \cdot \frac{\frac{1}{z^n}}{1-\frac{1}{z^n}},$$

and from $F(z)$ we derive also an integral series in $\frac{1}{z}$, that is a second analytic function $\varphi_2(z)$ existing in the region outside C.

The functions $\varphi_1(z)$ and $\varphi_2(z)$ represented in distinct regions by the same analytic expression satisfy the relation

$$\varphi_1(z) + \varphi_2\left(\frac{1}{z}\right) = -\zeta(s), \qquad (|z| < 1)$$

but the main question is, whether either of them is, or is not an analytic continuation of the other. The decision can be based on a transformation of $F(z)$. Corresponding to the rational numbers $\frac{p}{q}$ of the interval (0,1) we can arrange the so-called rational points $a_n = e^{2\pi i \frac{p}{q}}$ on the circle C as a sequence (a_n) and denoting by q the denominator of the rational fraction that corresponds to a_n, it will be seen that we have

$$F(z) = -z\,\zeta(s+1) \sum_{n=}^{n=\infty} \frac{1}{q^{s+1}} \cdot \frac{1}{z-a_n}.$$

This series of fractions represents $\varphi_1(z)$, if $|z| < 1$, $s > 0$, on the other hand it is equal to $\varphi_2(z)$ as soon as $|z| > 1$ and at the same time $s > 1$. We now can apply a theorem due to Goursat [1]) and conclude that the points a_n without exception are singular points of the functions $\varphi_1(z)$ and $\varphi_2(z)$. Hence, as these points form a set dense on C, the continuation of either of the functions across the circle is excluded. [2])

By application of EULER's summation-formula we can calculate the values taken by the functions $\varphi_1(z)$ and $\varphi_2(z)$, when z along the radius approaches one of the singular points. In this way I find in the first place, when z has a positive value $x < 1$, the following asymptotic expression for $\varphi_1(z)$

$$\varphi_1(x) = \frac{1}{\log\frac{1}{x}} \cdot \zeta(s+1) + \left(\log\frac{1}{x}\right)^{s-1} \Gamma(1-s)\,\zeta(1-s) - \tfrac{1}{2}\,\zeta(s) +$$

$$+ \frac{B_1}{2!}\log\frac{1}{x} \cdot \zeta(s-1) - \frac{B_2}{4!}\left(\log\frac{1}{x}\right)^3 \zeta(s-3) + \frac{B_3}{6!}\left(\log\frac{1}{x}\right)^5 \zeta(s-5) - \ldots.$$

holding for all non-integer values of s.

The result is less simple, when z tends along the radius to the point $e^{2\pi i \frac{p}{q}} = e^{i\beta}$. Putting $z = \varrho e^{i\beta}$, I get for $\varrho < 1$ and supposing again s to be a non-integer [3])

---

[1]) Bulletin des Sciences Math., t. XI, p. 109. Sur les fonctions à espaces lacunaires.

[2]) This results also from one of the propositions concerning the series $\sum_{n=1}^{n=\infty} b_n \frac{z^n}{1-z^n}$, enunciated in a previous communication (Verslagen en Mededeelingen. XXVIII. p. 269) according to which the continuation of the function across the circle is impossible, as soon as $b_n > 0$ and $\lim_{n=\infty} b_n = 0$.

[3]) For integer values of s the result is obtained by making s tend to the integer limit. So for instance, if s tends to zero, we will find

$$\varphi_1(x) = \frac{C - \log\log\frac{1}{x}}{\log\frac{1}{x}} + \tfrac{1}{4} - \frac{B_1{}^2}{2\,.\,2!}\log\frac{1}{x} - \frac{B_2{}^2}{4\,.\,4!}\left(\log\frac{1}{x}\right)^3 - \ldots\ldots,$$

and

$$\lim_{\varrho \to 1} \left\{ \varphi_1(\varrho e^{i\beta}) - \frac{C - 2\log q - \log\log\frac{1}{\varrho}}{q\log\frac{1}{\varrho}} \right\} = \tfrac{1}{4} - \frac{i}{2q}\sum_{h=1}^{h=q-1} h \cot\frac{h\beta}{2}.$$

The former of these formulae was obtained by SCHLÖMILCH, the latter I deduced in a previous paper: On LAMBERT's series.

$$\begin{aligned}
\varphi_1(\varrho e^{i\beta}) = {} & \frac{1}{q^{s+1}\log\frac{1}{\varrho}}\cdot\zeta(s+1) + q^{s-1}\left(\log\frac{1}{\varrho}\right)^{s-1}\Gamma(1-s)\,\zeta(1-s) - \tfrac{1}{2}\zeta(s) + \\
& + \frac{i}{2q^s}\sum_{h=1}^{h=q-1}\zeta\left(s,\frac{h}{q}\right)\cot\frac{h\beta}{2} + \\
& + \log\frac{1}{\varrho}\cdot\left\{ + \frac{B_1}{2!\,q^{s-1}}\,\zeta(s-1) - \frac{1}{1!\,2^2 q^{s-1}}\sum_{h=1}^{h=q-1}\zeta\left(s-1,\frac{h}{q}\right)(D\cot\nu)_{\nu=\frac{h\beta}{2}}\right\} + \\
& + \left(\log\frac{1}{\varrho}\right)^2\cdot\left\{ - \frac{i}{2!\,2^3 q^{s-2}}\sum_{h=1}^{h=q-1}\zeta\left(s-2,\frac{h}{q}\right)(D^2\cot\nu)_{\nu=\frac{h\beta}{2}}\right\} + \\
& + \left(\log\frac{1}{\varrho}\right)^3\cdot\left\{ - \frac{B_2}{4!\,q^{s-3}}\,\zeta(s-3) + \frac{1}{3!\,2^4 q^{s-3}}\sum_{h=1}^{h=q-1}\zeta\left(s-3,\frac{h}{q}\right)(D^3\cot\nu)_{\nu=\frac{h\beta}{2}}\right\} + \\
& + \left(\log\frac{1}{\varrho}\right)^4\cdot\left\{ + \frac{i}{4!\,2^5 q^{s-4}}\sum_{h=1}^{h=q-1}\zeta\left(s-4,\frac{h}{q}\right)(D^4\cot\nu)_{\nu=\frac{h\beta}{2}}\right\} + \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

In this formula $\zeta(p,a)$ stands for the function that, if $p > 1$ and $0 < a < 1$, is represented by the series $\sum_{n=0}^{n=\infty}\frac{1}{(a+n)^p}$.

It may be noticed that in both equations the absolute value of the error committed by stopping at any particular stage in the series is always less than a finite multiple of that of the last written term.

In particular we may deduce, supposing $s > 1$,

$$\lim_{x\to 1}\left\{\varphi_1(x) - \frac{1}{\log\frac{1}{x}}\,\zeta(s+1)\right\} = -\tfrac{1}{2}\zeta(s),$$

$$\lim_{\varrho\to 1}\left\{\varphi_1(\varrho e^{i\beta}) - \frac{1}{q^{s+1}\log\frac{1}{\varrho}}\,\zeta(s+1)\right\} = -\tfrac{1}{2}\zeta(s) + \frac{i}{2q^s}\sum_{h=1}^{h=q-1}\zeta\left(s,\frac{h}{q}\right)\cot\frac{h\beta}{2}.$$

$$\left(\beta = 2\pi\frac{p}{q}\right)$$

Hence, as z approaches along the radius a rational point $e^{2\pi i\frac{p}{q}}$ it is only the real part of the value of the function that increases indefinitely and at all points $e^{2\pi\frac{ip}{q}}$ which correspond to the same denominator q the real parts are ultimately equal.

The function $\varphi_2(z)$ behaves in quite similar manner because of the relation

$$\varphi_1\left(\frac{1}{z}\right) + \varphi_2(z) = -\zeta(s)\ ,\ (|z| > 1)$$

by means of which $\varphi_2(z)$, as soon as z along the radius tends to $e^{2\pi i \frac{p}{q}}$ from the outside of the circle, is expressed in $\varphi_1\left(\frac{1}{z}\right)$.

The rational points on C thus having been recognized as singularities of $\varphi_1(z)$ and $\varphi_2(z)$, we now must turn our attention to other points on the curve, and as such I will consider the points $e^{2\mu i\xi}$, where ξ is a root of an irreducible algebraic equation of degree $\mu > 1$ with integer coefficients. Evidently these points $e^{2\mu i\xi}$ which I will call the algebraic points of order μ on C, determine a new enumerable set, everywhere-dense on the circle.

Let $z = \varrho\, e^{2\mu i\xi}$, then it is readily seen that for all values of ϱ

$$\left|\frac{1}{z^n} - 1\right| > 1 \qquad , \text{ if } \cos 2\pi n\xi < 0,$$

$$\left|\frac{1}{z^n} - 1\right| > |\sin 2\pi n\xi| \ , \text{ if } \cos 2\pi n\xi > 0.$$

Now in the latter case $n\xi$ is an irrational number increasing with the index n, hence there exists an integer k, such that $|n\xi - k| < \frac{1}{2}$. But, as $\cos 2\mu(n\xi - k) = \cos 2\mu n\xi > 0$, we must have $|n\xi - k| < \frac{1}{4}$ and $\sin 2\mu\,|n\xi - k|$ being the sine of an acute angle is greater than the angle itself multiplied by $\frac{2}{\pi}$.

Therefore, if $\cos 2\pi n\xi > 0$, we may write

$$|\sin 2\pi n\xi| = \sin 2\pi\,|n\xi - k|$$

and

$$\left|\frac{1}{z^n} - 1\right| > 4n\left|\xi - \frac{k}{n}\right|.$$

Now according to LIOUVILLE's known theorem about algebraic numbers, we have

$$\left|\xi - \frac{k}{n}\right| > \frac{1}{Mn^\mu},$$

where M is a finite number independent of n and only depending on the coefficients of the equation of which ξ is a root.

In this way we conclude that

$$\left|\frac{1}{z^n} - 1\right| > \frac{4}{Mn^{\mu-1}}$$

and consequently that we have for all values of $\varrho = |z|$

$$\left|\frac{1}{n^s} \cdot \frac{z^n}{1 - z^n}\right| < \frac{1}{n^s} \qquad , \text{ if } \cos 2\pi n\xi < 0,$$

$$\left|\frac{1}{n^s} \cdot \frac{z^n}{1 - z^n}\right| < \frac{M}{4} \cdot \frac{1}{n^{s-\mu+1}} \quad , \text{ if } \cos 2\pi n\xi > 0.$$

Therefore the series of LAMBERT $F(z)$ converges absolutely on the radius of the point $e^{2\pi i\xi}$, as soon as $s > \mu$, the convergence being then independent of ϱ and uniform on any segment of the radius. Supposing z to move continuously along that radius, the value of the analytic expression $F(z)$ which for $s < 1$ is equal to that of the function $\varphi_1(z)$ changes also continuously into the value of the function $\varphi_2(z)$ as soon as ϱ becomes greater than unity. Besides, if s is taken sufficiently above the number μ, for instance, if we take $s > 2\mu - 1$, the series obtained by differentiating term-by-term the series $F(z)$ with regard to ϱ in exactly the same way will give the value of $\frac{d\varphi_1(z)}{dz}$ or that of $\frac{d\varphi_2(z)}{dz}$ according to the value of ϱ. In this order of thought we may ascribe to the functions $\varphi_1(z)$ and $\varphi_2(z)$ a common definite value at the point $e^{2\pi i\xi}$, though of course that point is not an ordinary point. Making $\varphi_1(e^{2\pi i\xi})$ and $\varphi_2(e^{2\pi i\xi})$ both equal to the finite limit $\lim\limits_{\rho \to 1} F(\varrho e^{2\pi i\xi})$, we obtain

$$\varphi_1(e^{2\pi i\xi}) = \varphi_2(e^{2\pi i\xi}) = -\tfrac{1}{2}\zeta(s) + \frac{i}{2}\sum_{n=1}^{n=\infty}\frac{\cot \pi n\xi}{n^s}$$

and the series $\sum\limits_{n=1}^{n=\infty}\frac{\cot \pi n\xi}{n^s}$ will certainly be convergent, if only $s > \mu$.

Hence, we have established a certain connexion between the functions $\varphi_1(z)$ and $\varphi_2(z)$ which according to WEIERSTRASS's theory we must regard as essential distinct and in no wise connected. In fact, we have shown that in this very special case in which the classical continuation by means of power-series is impossible, a new kind of continuation, as complete as could be desired, is furnished by the series of LAMBERT along the radii of an enumerable infinite set.

The question arises, whether cases exist in which the continuation by means of a series of LAMBERT is effected along the radii of a set having the power c of the continuum. The answer is in the affirmative, we only want to choose a LAMBERT series the coefficients of which are decreasing more rapidly. For instance I will consider the series

$$G(z) = \sum_{n=1}^{n=\infty}\frac{1}{n!}\cdot\frac{z^n}{1-z^n}.$$

Again in this new series the coefficients are positive and zero is their common limit, hence according to the proposition mentioned in the footnote on p. 1228, the rational points on the circle C are singularities of the analytic functions $\psi_1(z)$ and $\psi_2(z)$ defined by $G(z)$ inside and outside C.

Again some insight in the behaviour of these functions in the neighbourhood of the singularities is obtained by the application of EULER's summation-formula. Giving in the first place z the positive value $x < 1$, I find

$$\psi_1(x) = \frac{1}{\log\frac{1}{x}}\{\mathrm{Li}(e) - C\} - \tfrac{1}{2}(e-1) + \frac{B_1}{2!}.e\log\frac{1}{x} - \frac{B_2}{4!}.5e\left(\log\frac{1}{x}\right)^3 +$$

$$+ \frac{B_3}{6!}.52\,e\left(\log\frac{1}{x}\right)^5 \ldots\ldots$$

and the absolute value of the error committed by stopping at any particular stage in the series always will be less than that of the last written term.

Putting then $z = \varrho e^{2\pi i\frac{p}{q}} = \varrho e^{i\beta}$ and making ϱ tend to unity, we will find

$$\lim_{\varrho \to 1}\left\{\psi_1(\varrho e^{i\beta}) - \frac{1}{\log\frac{1}{\varrho}}\sum_{k=1}^{k=\infty}\frac{1}{(kq)!\,kq}\right\} = -\tfrac{1}{2}(e-1) - \sum_{h=1}^{h=q-1}\left(\frac{h}{q} - \tfrac{1}{2}\right)e^{\cos h\beta + i\sin h\beta}$$

The function $\psi_2(z)$ behaves in the neighbourhood of a singular point in a similar manner because of the relation

$$\psi_2(z) + \psi_1\left(\frac{1}{z}\right) = -(e-1). \qquad (|z| > 1)$$

Now, let ξ be a transcendental number of the interval (0,1) the expansion of which in a continued fraction gives

$$\frac{1}{a_1+}\,\frac{1}{a_2+}\,\frac{1}{a_3+}\ldots\ldots\frac{1}{a_k+}\ldots\ldots,$$

where all integers a_k are less than a given finite number l.

Evidently these numbers ξ, and therefore also the points $e^{2\pi i\xi}$ form a set of power c, the set of points $e^{2\pi i\xi}$ however being not dense on the circle. By the known properties of continued fractions we have, k being an arbitrary integer, $\frac{T_n}{N_n}$ the n-th convergent

$$\left|\xi - \frac{k}{n}\right| > \left|\xi - \frac{T_n}{N_n}\right| > \left|\frac{T_{n+2}}{N_{n+2}} - \frac{T_n}{N_n}\right| =$$

$$= \frac{a_{n+2}}{N_n(a_{n+2}N_{n+1} + N_n)} > \frac{1}{2N_n N_{n+1}} > \frac{1}{2(a_{n+1}+1)N_n^2} > \frac{1}{2(l+1)N_n^2},$$

and as N_n is manifestly always less than $(l+1)^n$, we may write

$$\left|\xi - \frac{k}{n}\right| > \frac{1}{2(l+1)^{2n+1}}.$$

Determining then the integer k by the condition $|n\xi - k| < \frac{1}{2}$ and putting $z = \varrho e^{2\pi i \xi}$, we get by the same reasoning as before

$$\left|\frac{1}{z^n} - 1\right| > 1 \qquad \text{, if } \cos 2\pi n\xi < 0,$$

$$\left|\frac{1}{z^n} - 1\right| > 4n\left|\xi - \frac{k}{n}\right| > \frac{2n}{(l+1)^{2n+1}} \qquad \text{, if } \cos 2\pi n\xi > 0,$$

and consequently

$$\left|\frac{1}{n!}\cdot\frac{z^n}{1-z^n}\right| < \frac{1}{n!} \qquad \text{, if } \cos 2\pi n\xi < 0,$$

$$\left|\frac{1}{n!}\cdot\frac{z^n}{1-z^n}\right| < \frac{(l+1)^{2n+1}}{2n\,.\,n!} \qquad \text{, if } \cos 2\pi n\xi > 0.$$

Hence the series $G(z)$ will converge absolutely on the radius of the point $e^{2\pi i \xi}$ and the convergence will be uniform on any segment of that radius.

Thus then, we have shown that in this case the functions $\psi_1(z)$ and $\psi_2(z)$ are connected at all points of an aggregate of power c and that along the radii of these points' the series of LAMBERT $G(z)$ procures a faultless continuation, whereas the analytic continuation necessarily fails [1]).

The elementary examples I discussed show as well as the examples of BOREL that sometimes we are led to regard as a single function a group of distinct analytic functions existing in separate regions. And from the fact that in these cases a non-analytic continuation can be effectuated, the question arises whether a certain extension should not be given to the concept of functionality. BOREL made a step in this direction by developing the theory of a class of non-analytic, monogenic functions existing in a so-called domain of CAUCHY [2]).

---

[1]) As we have $\frac{1}{n!} < \frac{1}{n^s}$ for all values of s, if only n is sufficiently large, we are certain that the series $G(z)$ also furnishes the continuation along the radii of algebraic points of order whatever.

[2]) Leçons sur les fonctions monogènes uniformes d'une variable complexe. Chap. V.

Geology. — *"On the Composition and the Xenoliths of the Lavadome of the Galunggung"*. (West-Java). By Prof. H. A. BROUWER. (Communicated by Prof. G. A. F. MOLENGRAAFF).

(Communicated at the meeting of November 27, 1920).

During an eruption of the Galunggung, which commenced on the 18th of July 1918 and produced only an inconsiderable fall of ashes, a lavadome was formed in the crater, which formed on the 20th of July an islet in the craterlake [1]), and which gradually grew so large that the entire lake disappeared. On the 11th of August only the N. W. part of the Warirang-crater was covered with water [2]).

On a visit to the Galunggung-crater Dr. W. VAN BEMMELEN, on my request, searched for xenoliths in the rocks of the dome, in order to ascertain whether any crystallization had taken place in the magma under the crater similar to that in the dome of the Ruang (Sangi Islands) [3]). The collection transmitted to me through the "Headoffice of the Mining Department" comprises numerous samples of lava from the dome with fine-crystalline to coarse-grained homoeogeneous xenoliths, which will be described lower down.

The lava of the dome.

All the rocks examined are brownish-red, porous hypersthene-augiteandesites with phenocrysts of zonary plagioclases, among which frequently narrow basic and more acid zones occur alternately, so that the marginal zone, even in the case of markedly zonary structure, is often only little more acid than the central part; also the succeeding zones differ but little as to basicity.

Carlsbad twins occur; in sections of the symmetrical zone we determined that, on an average, the composition of the plagioclases is like that of bytownite Ab_{25} An_{75}. Inclusions, among which some of ore and of a glassy substance are generally few in number; a zonary arrangement, the inclusions being limited to certain zones, is sometimes met with. The hypersthene- and augite phenocrysts

1) R. D. M. VERBEEK et R. FENNEMA. Geologische Beschrijving van Java en Madoera. II.

2) B. G. ESCHER. De uitbarsting van den Goenoeng Galoenggoeng. De Taak 12 Oct. 1918 pp. 126—127 en Mededeeling namens B. G. ESCHER door G. A. F. MOLENGRAAFF. Versl. Geol. Sectie. Geol. Mijnb. Gen. II. Oct. 1919.

3) H. A. BROUWER. Crystallisation and Resorption in the magma of the volcano Ruang. (Sangi Island). Proc. Kon. Akad. v. Wetensch. Vol. XXIII, p. 561.

are often accumulated; sometimes they are found together with plagioclase- and larger ore crystals. Where the pyroxenes are contiguous to groundmass they are commonly encircled by a narrow zone of ore, which is lacking where they are in contact with plagioclases. Apparently this is a chemical exchange between the phenocrysts of pyroxene and the still liquid part of the enclosing magma. In some samples of the dome resorbed brown amphiboles were found; however, they exhibited no idiomorphous crystalform and we venture to assume that they are not crystals formed in the dome but fragments transported by the rising magma. Some of them may be fragments of the same crystallization products, from which the amphibole-rich homoeogeneous xenoliths originate.

The groundmass of the dome-rock is rich in glass and contains lath-shaped plagioclase and grains or skeleton-shaped individuals of ore, while pyroxene has not (or only to a small degree) crystallized in this ground-mass.

The xenoliths of the domerock.

Among them we distinguish the following types:

1. medium-grained, occasionally porphyric xenoliths, consisting of plagioclase and amphibole with a small quantity of a more or less devitrified glass.

2. medium-, to coarse-grained xenoliths, made up of plagioclase, amphibole with little pyroxene and sometimes a little olivine. Glass occurs also in these xenoliths.

3. medium-, to coarse-grained, sometimes porphyric xenoliths with plagioclase, (little olivine), amphibole and much augite and hypersthene. The olivine was seen only in some xenoliths, ore sometimes occurs in a small quantity outside the resorption-rims of the amphibole. The relative quantity of amphibole, augite, and hypersthene is variable. All xenoliths, in which the number of pyroxenes is not very small, have been included here. Glass with microlites was observed only in some xenolith and in a very small quantity.

4. porphyric xenoliths with phenocrysts of plagioclase in a fine-grained groundmass in two generations with plagioclase, pyroxene and ore. Larger pyroxene-crystals do not occur in the xenoliths, but are accumulated in a small marginal zone against the enclosing rock.

5. fine-grained xenoliths, made up of plagioclase, augite, hypersthene and little ore. Much glass with microlites is found in some samples between the other minerals.

6. xenoliths of older andesites, some of them bearing amphibole, others devoid of amphibole.

The xenoliths mentioned sub 1 (Plate fig. 1. and 2) are first of all characterized by plagioclases, which are distinguished from those of the phenocrysts of the enclosing rocks by the almost total absence of the frequent alternation of more basic and more acid zones. The crystals, often zonary and with a broad basic central part are chiefly composed of basic bytownite, the marginal zone is more acid.

The amphibole is strongly pleochroitic, from a brownish red to light-yellow, and invariably shows a resorption-rim, often only narrow in many xenoliths and sometimes entirely absent where the crystals are contiguous to plagioclase, while it is often well developed where the amphibole is in contact with the glass-rich mass with microlites. These narrow resorption rims are composed of a black mass of ore. It is obvious that in part the amphibole has crystallized later than the plagioclases, which form idiomorphous crystals and then guard against resorption that portion of the amphibole with which they are in contact. In the xenoliths with more strongly resorbed amphibole, there occur entirely resorbed crystals, which can only be recognized as original amphibole by their crystalform. In the crystals that are partly unaltered, the resorption-rim consists only of a mass of ore or a marginal zone of ore, separated from the intact part of the crystal by an irregularly shaped pyroxene-rich zone, which is sometimes lacking and which sometimes occurs mixed with unmodified amphibole. Out of the resorption rims larger crystals of ore do not occur in the xenoliths.

Some of the xenoliths with strongly resorbed amphiboles present a porphyric structure, the groundmass, which contains large, more or less idiomorphous amphibole crystals, consisting of plagioclase. Between these smaller plagioclase crystals, as in the non-porphyric xenoliths with larger plagioclases, more or less devitrified glass is found. Plagioclase-microlites and ore-skeletons are easy to distinguish in this mass; only a small number of pyroxene microlites are distinguishable; the devitrification is sometimes complete.

The limit between the xenoliths and the enclosing lava is always such that the lava has adapted itself to the shapes of the xenolith. The crystal faces of the plagioclases and amphiboles have reached full development at the margin of the xenoliths so that the boundary line with the lava proceeds irregularly. Also the enclasped glassy mass shows that the minerals had not been perfectly crystallized, when the xenoliths were taken up in the enclosing magma, so they were still molten to a certain extent and may therefore be considered as an almost perfectly crystallized crust on the magma, which was effused from a larger depth and has produced the dome. The

groundmass of the lava of the dome and the devitrified glass of the xenoliths have then crystallized almost simultaneously.

The xenoliths mentioned sub 2 do not differ much from those just described; they are all characterized by a small amount of pyroxene, while also a small amount of olivine was found in some. The amphiboles show a narrow resorption rim, which contains much ore, mixed with pyroxene (principally augite).

Veins of the same composition as the resorption rim are found in the more central part of the crystals, which are larger than those of the other minerals, which present an irregular outline and may enclose all the other components entirely or partly, also olivine, if it occurs in the xenoliths. The olivine presents rounded shapes without distinct crystal faces and has been altered to some degree into a brownish red or black substance, rich in iron. The glass-bearing mass is rich in laths of plagioclase and also contains many pyroxene-microlites. The glass is very dark and the devitrified portion is apparently rich in iron; ore hardly shows itself in separate grains.

The xenoliths with more pyroxene, mentioned sub 3 (Pl. fig. 3) are distinguished from the preceding chiefly by the decrease of the amount of amphibole and the increase of pyroxene. Various structures occur. Pyroxene (augite as well as hypersthene) occurs occasionally with a few larger amphibole-, and plagioclase-crystals in a finer crystalline mixture, consisting mainly of plagioclase with little pyroxene. The pyroxene (above all the augite) then exhibits similar skeleton-like structures to those of the amphibole, and then incloses, just like the last-named mineral, numerous plagioclase crystals, which also penetrate into the augites with idiomorphous crystal form, so that these are one of the last crystallisation products of the xenolith. In connection with this the smaller plagioclases are entirely free from enclosed dark minerals, the larger nearly so. It is these xenoliths that contain olivine with rounded shapes and mostly enclosed by the amphibole. Devitrified glass occurs in small quantity.

In other xenoliths also bearing occasionally olivine, for the rest little different from the others, the pyroxenes are chiefly restricted to the fine-crystalline bulk of the xenolith, while only few larger crystals occur with some of partly resorbed amphibole and of plagioclase. A portion of the xenoliths of this group displays the normal, medium grained structure without larger crystals; here we find evidence of the posterior crystallization of the amphiboles, because they occur in skeleton shaped crystals, which occasionally enclose the pyroxene. The different stages of resorption of the amphibole, mentioned already for the xenoliths described sub 1,

were also met with here. No olivine was found in the samples of the mediumgrained xenoliths examined.

The xenoliths mentioned sub 4, found only in some specimens, are free from amphibole, just as those mentioned sub 3. They are characterized by a remarkable structure, large plagioclase-phenocrysts lying in a groundmass, consisting of smaller crystals of plagioclase, few ore crystals and very few of augite and hypersthene. The latter in their turn are surrounded by a fine-crystalline mixture of the same minerals, whose constituents — with the exception of the small plagioclases — occur in a large number as inclusions in the plagioclases of the first and te second generation (Pl. fig. 4). Zonary structure does not occur with these plagioclases or only in a small measure and without the alternation of basic and more acid zones, which distinguish them from those of the enclosing lava. The fine-crystalline groundmass is almost entirely absent in a narrow marginal zone of the xenolith where this is contiguous to the enclosing lava. Here we see a mixture of plagioclase, like those occurring everywhere in the xenolith as small-sized phenocrysts, together with the augite-, hypersthene-, and ore-crystals, which are seen only in small number in the central parts of the xenoliths as small phenocrysts. It appears then that pyroxene and ore are accumulated in the marginal zone. The structures described heretofore point to the fact that the crystallization of the xenoliths *was still to take place for the most part*, when they had already been taken up in the enclosing lava. In an early stage the plagioclases, the pyroxenes, and the ore-crystals of the second generation have crystallized. The latter two have accumulated in the marginal zone of the xenolith. That the crystallization of the plagioclases was the first to be finished here, is proved by the idiomorphous shape of the crystals relative to the pyroxenes in the marginal zone and the enclosure of plagioclase by pyroxenes, which occurs frequently here. In the central parts we see that the crystallization of the ore and of the pyroxene of the fine-crystalline groundmass had already begun during the crystallization of the plagioclases, some of which have grown into larger phenocrysts. Then followed the ultimate crystallization of the fine-crystalline groundmass, in which occurs the plagioclase in rounded shapes, which points to a crystallization about simultaneous with that of the pyroxene. The plagioclases of the marginal zones are poor in or destitute of inclusions and seem, therefore, to have crystallized before those of the central part of the xenolith, or the crystallization of the fine-crystalline groundmass has taken place in the marginal zone later

and in a smaller degree, so that the quantity of it is small there. The boundary between the xenolith and the enclosing lava, of which the latter has adapted itself to the shapes of the crystals in the xenolith, proves moreover that the xenolith has not been enclosed at greater depth in a solid state, but that it has been taken up in the enclosing lava as a partly crystallized mass.

The xenoliths mentioned sub 5 consist of plagioclase, augite and hypersthene with a few ore-crystals, the interspaces being filled up with various quantities of a partly devitrified, dark substance, against which feldspar-microlites stand out sharply outlined. The plagioclases and the pyroxenes present mostly idiomorphous outlines (especially the plagioclases); in contradistinction to that of the xenoliths thus far described, the structure of the plagioclase is zonary with frequent alternation of more acid and basic zones, like those of the phenocrysts in the enclosing lava of the dome mentioned above.

The xenoliths of older andesites mentioned sub 6, display differences as regards mineralogical composition and structure. In some of them a few plagioclase phenocrysts occur in a groundmass, consisting of plagioclase and pyroxene with ore.

In others amphibole was observed, occasionally as phenocryst in similar rocks to those mentioned above, sometimes in aureoles round orecrystals, occurring porphyrically with plagioclase in a dark glass-bearing groundmass. Frequently the microscopic aspect changes considerably, e. g. as to the amount of ore and as regards the colour of the groundmass, while the rocks give an impression of being modified through contactmetamorphism, in which process recrystallizations have taken place. The porphyric plagioclases have been strongly eroded by the clear mixture of which the present groundmass consists; we then suppose the groundmass to have been entirely recrystallized and the phenocrysts only in their marginal zone. The aureoles of amphibole round ore-crystals in a partly devitrified groundmass find an explanation in the assumption that what has taken place here is just the reverse of what happened with the resorption of amphibole. The enclosed andesite fragment, heated anew, has been for some time submitted to temperature- and pressure-relations, which do not affect amphibole and this mineral has crystallized instead of components that otherwise build up resorption rims.

The various crystallizations in the Galunggung magma.

The boundaries of the homoeogeneous xenoliths relative to the enclosing lava proves that the xenoliths had not crystallized completely when they were taken up in the lava. The residual magma

has crystallized as a glass-bearing mass with microlites, just as the groundmass of the enclosing rocks.

That in many xenoliths the amphiboles exhibit a resorption-rim where they border on the glass-bearing mass alluded to, points out that the amphibole remained stable down to the moment of the eruption of the lava dome. After this the pressure in the lava and the xenoliths decreased rapidly, which made the amphibole instable and augite could crystallize during the time when the temperature of the residual magma fell, and a complete solidification had not yet been effected. This interval may have been longer or shorter for different portions of the dome, hence the resorption in amphiboles of different xenoliths was varying. Already before we accounted for the mineralogical differences between xenoliths and dome-lava of the RUANG [1]) we have assumed that during the eruption of the volcano the outpouring magma enclosed fragments of its completely or partly solidified dioritic crust. The same applies to dome and xenoliths of the Galunggung. The occurrence of pyroxene-poor and pyroxene-rich amphibole-bearing xenoliths may be accounted for by the assumption that they originate from zones at various depths in this crust. It depends on the difference of pressure and temperature of these zones whether amphibole only or first pyroxene and later, on further cooling, amphibole has crystallized. [2])

In that case the pyroxene-bearing xenoliths originate from deeper zones according as they are richer in pyroxene, whereas at a greater depth with a higher temperature much less crystalline components and only pyroxenes occur in the outpouring magma, which does not contain amphibole as phenocrysts.

The amphibole-free xenoliths with different structures described above, may have crystallized already before the eruption at a great depth, so above temperatures, at which the amphibole is stable, while in that case complete crystallization has taken place after the eruption had commenced, when the amphibole was not stable either, in connection with the suddenly modified pressure and temperature relations. Maybe some of these xenoliths have crystallized at a pressure lower than that of the stage of stability of the amphibole and may therefore belong to parts of the magma that have cooled down more slowly, that could crystallize more completely along the walls of the vent and were only then carried along by the outpouring magma.

[1]) H. A. BROUWER, Crystallizations etc. loc. cit. p. 665.

[2]) F. BECKE. Gesteine des Columbretes. Anhang. Tscherm. Min. u. Petrogr. Mitt. XVI. 1897. blz. 327 e.v.

Geology. — *"On the Alkalirocks of the Serra do Gericino to the northwest of Rio de Janeiro and the Resemblance between the Eruptive Rocks of Brazil and those of South-Africa."* By Prof. H. A. BROUWER. (Communicated by Prof. G. A. F. MOLENGRAAFF).

(Communicated at the meeting of January 29, 1921).

On the boundary between the State of Rio de Janeiro and the Districto Federal lies near the station of Maxambomba of the E. F. Central do Brazil, the Serra do Gericino [1]), extending in W.S.W.-N.N.E. direction over a length of about twenty k.m. and a breadth of about eight k.m. It is chiefly composed of nephelinesyenites like the Tingua eruptive province which lies about 30 km. farther to the North and of which the alkali-rocks have been described by GRAEFF [2]) and DERBY [3]). As I could not personally visit the Serra do Gericino during my stay in Brazil, several samples for further investigation were sent me by GONZAGA DE CAMPOS, Director of the "Serviço geologico e mineralogico do Brazil".

Geological Survey.

The alkali-rocks constitute an eruptive centre amidst the old gneisses of the mountain-range Serra do Mar, which runs parallel to the Brazilian Coast. (Fig. 1). We only know that the gneisses are intruded by alkali-rocks which consequently must be younger than the gneisses. No data were obtained concerning the exact geological age.

Coarse-grained, as well as dyke-, and effusive rocks occur, just as in other Brazilian occurrences of alkali-rocks. Although nearly the whole region and especially the highest parts consist of coarse-grained rocks, the occurrence of effusive rocks allied to them, still

---

1) MATHIAS G. DE OLIVEIRA ROXO. Resumé of the preliminar note on the Gericinó eruptive centre. Empresa Brasil Editora 1920.

2) FR. GRAEFF, Mineralogisch-petrographische Untersuchung von Eleolithsyeniten von der Serra de Tingua. Neues Jahrb. f. Min., Geol. u. Pal. 1887. II, blz. 222 e.v.

3) O. E. DERBY. On nepheline rocks in Brazil. Quart. Journ. Geol. Soc. Vol. XLIII 1887, blz. 457; Vol. XLVII, 1891, blz. 251.

proves that the Serra do Gericino constitutes the strongly denuded remnant of a volcano or group of volcanoes, like the Serra de Tingua further northward. That remains of lava-flows have been preserved only locally, and only between the eruptive rocks, whereas they do not occur in the surrounding gneisses, points to the circumstance that these effusive rocks, which originally must have extended far beyond the present mountain, have long been protected from erosion through overhead stoping, the roof having locally sunk down. These effusive rocks occur near the station of Maxambomba of the E. F. Central do Brazil, and near the façenda D. Eugenia close to the west of this station.

Dyke-rocks were e.g. met with near the station of Maxambomba (tinguaite), near the façenda Mascarenhas and in the western part of the eruptive province, between Cava and Ypiranga (aegerine- and amphibole-Sölvsbergite).

The coarse-grained rocks, which chiefly compose the eruptive province, are generally characterised by table-shaped felspars; consequently they belong to the foyaites as far as they contain nepheline. Of the western part, known as Serra de Marapicu, samples of nepheline-free umptekite were examined, while among the foyaites, which seem to build up the greater part of the mountain-ridge between the Serra de Marapicu and the station of Maxambomba, also alkali-syenites (partly pulaskites) occur[1])

The granular rocks.

The following types may be distinguished:

1. Foyaites.
2. Alkali-syenites.
3. Umptekites.
4. Pulaskites.

Foyaites.

They seem to be the most common rocks of the Serra do Gericino.

Type 1 is a *pyroxene-amphibola foyaite*, collected near Cancella Azul along the road which crosses the rivulet Cachoeira. The composing minerals are light-coloured orthoclase and microperthite,

---

[1]) Most of the samples received, which had been provisionally determined as nepheline-syenite, belonged to the alkali-syenites. The typical foyaites originate from Cancella Azul along the road intersecting the rivulet Cachoeira. Therefore, because many of the rocks have apparently been mistaken for nepheline-syenites, whereas they are actually alkali-syenites, the data occurring on a map on which different types of nepheline-syenites have been separated cannot be relied on.

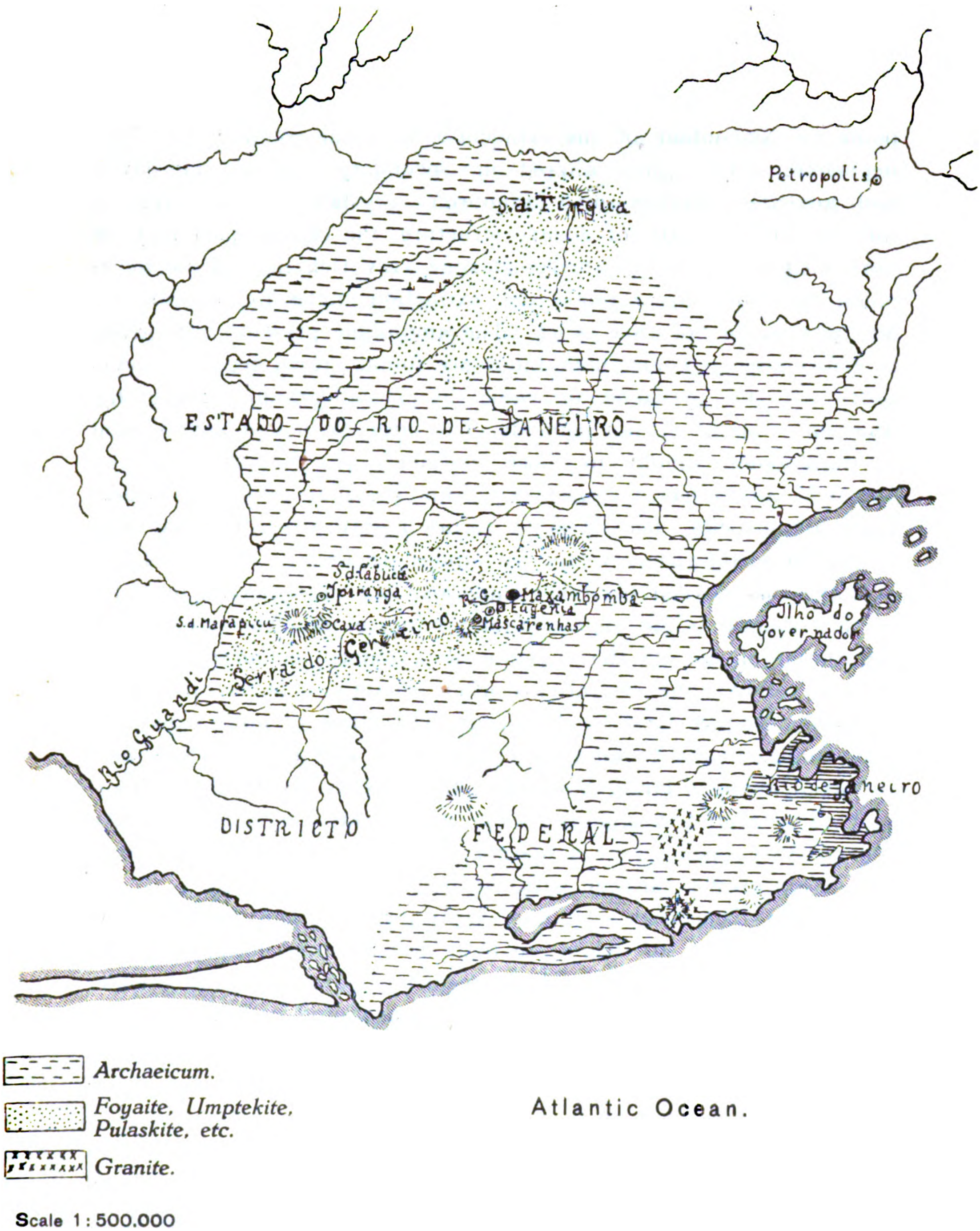


Fig. 1. Geological Sketchmap of the environs of the Serra do Gericinó to the Northwest of Rio de Janeiro. R. C. = Rio Cachoeira, (compiled from observations of Dr. MATHIAS DE OLIVEIRA ROXO).

nepheline, sodalite, aegirineaugite, greenish-brown amphibole with a slight quantum of analcite, lavenite, biotite, pyrite and magnetite, while muscovite and calcite occur as secondary minerals.

Large extinction angles in sections normal to the obtuse bissectrix point to Na-content of the orthoclase. Like the sodalite, the clear nepheline, only slightly altered into secondary minerals (muscovite and analcite), exhibits some idiomorphic crystals. The pyroxene is for the greater part of zonary structure, the central part may be very rich in augite-, the marginal zone very rich in aegirine molecules, but in most crystals the extinction angles for central part and margin do not vary much. The amphibole presents olive-green or bluish-green colours, both kinds are found grown together sometimes with the pyroxene, in which the crystallographic axes of the different minerals do not coincide. The large extinction angles point to amphiboles similar to those described by WRIGHT [1]) and by USSING [2]) and collected respectively from Brazilian and from Greenland alkali-rocks. The lavenite forms highly pleochroic and with strong birefringence crystals sometimes occurring with irregular crystal-form between the other minerals, like the analcite, in so far as this mineral is not an alteration-product of the felspathoids. The absorption-scheme of the lavenite is $\mathfrak{c}$ (canary-coloured) $> \mathfrak{b} = \mathfrak{a}$ (bright yellow); the crystals are often (sometimes polysynthetically) twinned; simple crystals also occur. The plane of optic axes is at right angles to the twinning plane cf. (100) and the cleavage-lines; the axial angle is large. Sometimes the crystals are partly idiomorphic. Pyrite and magnetite occur in separated crystals, but often the pyrite is enclosed by a margin of magnetite and both minerals also occur grown together with the other dark minerals.

Type 2. This rock is more finely grained than the preceding and is composed of white- to light flesh-coloured felspars (chiefly microperthite) with grey or black greenish-coloured liebeneritepseudomorphs after original felspathoids. It was found near the façenda D. Eugenia. Beyond strongly weathered ore no original dark minerals can be recognized in the rock.

Alkalisyenites.

A rock, also collected near the façenda D. Eugenia, consisting for the greater part of light-coloured microperthite which is rich in albite and strongly weathered ore, contains only little of a substance

---

[1]) F. E. WRIGHT. Die syenitisch-theralitischen Eruptivgesteine der Insel Cabo Frio, Brasilien, Tscherm. Min. u. Petr. Mitt. 1901, XX, blz. 249.

[2]) N. V. USSING. Geology of the country around Julianehaab, Greenland, Meddelelser om Grönland. Vol. XXXVIII, 1911.

consisting of small muscovite-flakes, which may also be alteration products of original felspathoids. These, however, were then present only in a very small quantity.

Umptekites.

The Serra de Marapicu i. e. the Western part of the Serra do Gericino seems to be chiefly built up of these rocks. The composing minerals are for the greater part light-coloured microperthite and amphibole, and small quantities of pyroxene, titanite, analcite magnetite and apatite.

The felspars are microperthites with a variable amount of acid plagioclase, which is sometimes absent altogether. Large extinction angles in sections normal to the obtuse bissectrix point to a Na-content of the orthoclase. The amphibole differs from that of type 1 of the foyaites, the extinction-angles remain smaller and frequently varying colours occur in one and the same crystal; a greenish variety in the marginal zone, a brownish in the central part, but both varieties form separate crystals. The amphiboles are very much like those which WRIGHT[1]) has described in an umptekite near Cabo Frio. In crystals of zonary structure we see in sections normal to the acute negative bissectrix of a small axial angle the following absorption : $\mathfrak{c}$ central part: reddish brown-green; margin: green) $\pm = \mathfrak{b}$ (central part: brown with a greenish tint; margin: brownish-green).

In sections parallel to the plane of symmetry the extinction-angle increases towards the green marginal zone up to $\pm$ 22°; we often see for the absorption parallel to the $\mathfrak{a}$-axis a homogeneous light yellow-brown colour, without any difference for central part and marginal zone. The only slight quantity of pyroxene consists of a green augite with extinction angles as high as 40° relative to the cleavage lines Apatite is present in numerous idiomorphic crystals.

Pulaskites.

This term comprises the alkali-rocks rich in mica, sometimes with a small amount of felspathoids.

Type 1. The felspars have a more reddish tint than those of the rocks described above. The sample was collected along the road from Maxambomba to Mascarenhas, close to the façenda D. Eugenia. Biotite and titanite are visible macroscopically in numerous crystals. The composing minirals are: microperthite and a small amount of plagioclase, sodalite and analcite, biotite, augite with a margin of aegirineaugite, apatite and ore. As secondary product occurs a chlo-

---

[1]) F. E. WRIGHT. l. c. p. 246.

ritic mineral of rather strong double refraction, which has been formed to the cost of the pyroxene.

The large extinction angles in sections normal to the obtuse bissectrix again point to a Na-content of the orthoclase of the microperthites. The analcite has been formed partly at the cost of the felspar, optic anomalies occur. The biotite is highly pleschroic, the colour ranging from brownish black to brownish-yellow. The pyroxene is almost colourless and is often encircled by a rim of green aegirine augite, but both also occur separately. Biotite, pyroxene, titanomagnetite, titanite and apatite often are grown together, in these intergrowths all or some of the minerals referred to occur.

Type 2. The felspars in this rock are partly green and the dark minerals chiefly occur only in small crystals. It was found in that part of the Serra do Gericino which is known as Serra de Cabuçu, along the road between Mascarenhas and Cabuçu.

The felspars consist of orthoclase or microperthite, which is poor in plagioclase. Not a trace of felspathoids is distinguishable. The markedly pleochroic biotite (from brownblack to light brownish-yellow) has often partly or completely been converted into green mica, while at the same time grains of a light yellow-green highly refracting, isotropous mineral having the properties of garnet, are formed. These grains are also found scattered in the felspars and the conversion may have taken place already before the complete crystallization of the rock.

The rock contains also titanomagnetite which has been entirely or partially converted into leucoxene.

The dyke-, and the effusive rocks.

We distinguish the following types of rocks:

1. Alkalisyeniteporphyries.
2. Nephdinesyeniteaplites.
3. Tinguaites.
4. Sölvsbergites.
5. Trachytes.

Alkalisyeniteporphyries.

If these rocks contained originally felspathoids, the latter have been completely converted into secondary minerals.

Type 1. A rock, collected where the road to the façenda D. Eugenia crosses the rivulet Cachoeira, contains white to bright reddish felsparphenocrysts in a grey fine-grained ground mass.

The felsparphenocrysts consist of orthoclase in which felspar with stronger double refraction is seen in small quantities. Pseudomorphs

also occur. They consist of muscovite flakes; whether the original mineral has partially belonged to felspathoids, which the form sometimes seems to suggest, could not be made out with certainty. In the groundmass the same felspars occur, the laths contain more of the strongest double-refracting felspar mentioned above, this felspar sometimes exhibits polysynthetic twins and occurs also in a few separate crystals. The groundmass contains also muscovite, calcite, rather much apatite and ore, which occurs also in some larger crystals, is strongly weathered and consists partly of pyrite.

Type 2. Near the facenda D. Eugenia a rock was collected with white to faintly reddish coloured felsparphenocrysts in a light-gray, finely crystalline groundmass. The rock is strongly sericitised, although the felsparphenocrysts have been altered very little. Initially it may have contained felspathoids. Ore, leucoxene and titanite occur.

Nephelinesyeniteaplites.

These rocks are known only as boulders near Mount Sapé in the Serra di Marapucu (western part of the Serra di Gericino). Macroscopically it presents itself as a medium-, to fine-grained light-grey rock, with numerous black points chiefly consisting of magnetite. The constituents are: clear albite, less clear orthoclase and microperthite, nepheline and analcite, magnetite and little pyrite, titanite apatite and green or brownish biotite.

The nepheline is often enclosed by the felspars. The albite reveals itself in a large quantity in polysynthetically twinned crystals. There is an abundauce of analcite; a Cl-reaction with a negative result points to the absence of sodalite.

Tinguaites.

Typical tinguaites were collected near Maxambomba, the rocks seem to form a dyke here and also a flow, the latter of a thickness of more than 100 meters. Only a single sample was examined, most likely several varieties and also typical effusive rocks occur here.

The sample contains in a grey finely-crystalline groundmass a few phenocrysts of light-coloured felspar, consisting of Na-bearing orthoclase or anorthoclase. They have been partially converted into natrolite. Microscopically the groundmass seems to consists of felspar laths aegirine, natrolite, analcite and a little nepheline. Some prisms with high refractive indices and strong birefringence, which show parallel extinction and are optically positive, point to zircon.

The colourless substance with low refractive index which exists in large quantity between the felspar laths, is probably chiefly composed of analcite, which is partially an alteration product of original nepheline.

Muscovite flakes also occur as alteration products of nepheline. Larger crystals of aegirine show distinctly a higher augite content in their central part the needles often show a sheaf-shaped or radial arrangement.

Sölvsbergites.

Under this name we have grouped the rocks in which probably felspathoids occurred, but in smaller quantity than in the tinguaites. The felspathoids cannot be recognised any more as such, the secondary minerals, however, are indicative of their having been present originally. Then the rocks approach the tinguaites.

Type 1 (*with pyroxene*). It is distinguished from the above-mentioned tinguaite by the dark grey colour of the finely crystalline groundmass, against which numerous white or light-red phenocrysts, which no doubt consist for the greater part of felspar, are sharply outlined, while also a few larger pyroxene crystals occur. It was collected from a dyke between the Serra de Cabuçu and the Serra de Marapicu.

Orthoclase is the predominant mineral of the phenocrysts; in small quantity polysynthetically twinned felspars occur with small extinction-angles. More or less regularly defined groups, consisting chiefly of acid plagioclase and cancrinite, sometimes mixed with analcite, possibly point to original felspathoids. Beside larger crystals of aegirine-augite with a high augite-content, which decreases in zonary crystals in a narrow marginal zone, also a few phenocrysts of brown amphibole and very little biotite occur together with larger ore-crystals. The groundmass consists of numerous pyroxene-needles sometimes of zonary structure and consisting of aegirine and aegirine-augite. Sometimes the central part of zonary crystals is of a violet colour with a great extinction-angle indicating the presence of titaniferous augite, which was also observed in some large crystals. Very few amphibole prisms occur. In the colourless mass between them felspar can be recognised, originally it probably consisted chiefly of felspar and felspathoids; at present there is an abundance of cancrinite and analcite as products of alteration. Inclosures of ore are numerous.

Type 2 (*with amphibole*). It was collected near type 1 also from a dyke. It is a dark grey fine-crystalline rock with some felspar-phenocrysts. Original felspathoids are not noticeable, but the cancrinite-content of the groundmass points to their former existence. True amphibolephenocrysts do not occur, though we do see accumulations of brown-green amphibole and ore which sometimes show a regular outline.

The groundmass is composed of a good many plagioclase-laths which are sometimes polysynthetically twinned, of markedly pleo-

chroic amphibole, the colour ranging from dark brownish green to light brownish yellow, without large extinction-angles; of ore, cancrinite, analcite, fluorite and little calcite.

Trachyte.

The grey, compact rock was collected from a lava flow more than 100 M. in thickness, near the façenda D. Eugenia.

A microscopic examination shows beside portions, in which crystalline constituents with weak birefringence are scarcely visible, other parts, in which distinctly felsparlaths without polysynthetic twins and with nearly parallel extinction, have been largely developed. There are also larger felsparcrystals; in sections normal to the acute bissectrix they present a rather small axial angle. Some of the larger felspars exhibit polysynthetic twins with small extinction-angles. Parts with a more or less regular form and consisting of muscovite flakes remind somewhat of liebeneritepseudomorphs after nepheline. However, sometimes quartz occurs in large quantity mixed with muscovite flakes. The quartz, which we take to be a secondary product, also occurs scattered in the rock. Finally pyrite must be mentioned as one of the composing minerals.

Resemblance between the Eruptive Rocks of Brazil and those of South Africa.

Rocks, rich in alkalies, some of which have been described above, are of frequent occurrence in Brazil as well as in South-Africa, and the various types in both regions show many points of resemblance, which will be discussed in detail lower down. This resemblance exists also with regard to other eruptive rocks. On a journey through Brazil in 1920 I was struck by the marked resemblance of some groups of sedimentary rocks with which I got acquainted in South Africa in 1910. Anyhow the differences are not greater than are known for adjacent regions of the African continent at a much shorter distance.

As the principal groups of eruptive rocks whose resemblance in composition and geological aspect will be discussed below, we mention:

1. Old granites, intrusive in rocks of probably archaean age.

2. Younger granites, intrusive in deposits of Devonian age and older than permo-carboniferous rocks.

3. Younger rocks, rich in alkali, (nephelinesyenites, alkalisyenites with accompanying abyssal- and effusive rocks).

4. Jurassic volcanic rocks and intrusive dolerites (the determination

of age is connected with the prolonged denudation before Upper-cretaceous time.

5. Kimberlites, alnoites etc. in pipes and dykes, younger than the dolerites mentioned sub 4 [1]).

Old Granites.

The archaean rocks classed together for Brazil under the term Brazilian complex, are granites, gneisses, quartzites, marbles and crystalline schists. They may be compared with the Malmesbury system of the Southern Cape Colony, the Swarzland system of the Transvaal and Rhodesia and the Fundamental complex with intrusive old granites of South-West Africa. Both the east coast of Brazil in the Serra do Mar and the opposite West Coast of South- and Central Africa consist for the major part of these rocks and they often impart to the landscape in both continents a similar topographic aspect. As to the petrographic features of these rocks no data are known sufficient for a minute comparison of the rocks near the opposite shores.

Young granites.

An instance of this type in South Africa are the granites of the "Bushveld Igneous Complex" in The Transvaal, occurring in combination with the gabbros, norites and ultrabasic rocks, the Erongo granite in Hereroland, and the Branaberg granite in the North Western part of Damaraland. The first-named are intrusive in the ? Devonian Waterberg Sandstone; the Eronga-granite has intruded the lowermost division of the ? Cambrian Nama System, hence they are younger than the old granites from which they also differ in petrographic composition, but their exact age is not known.

In Brazil the extensive granite areas and their contacts with the environing sediments have been studied very little. However, here also granites are known as intrusions in the algonkian or old-palaeozoic Minas Serie, as e. g. appears from the gold-bearing dyke of Passagem [2]) in Minas Geraes, ultra-acid granite apophysis, and intrusive in the so-called itabirite-formation of the Minas Series. In the neighbourhood a granite occurs and similar gold-bearing quartz-dykes are known in several places in the States of Minas Geraes and Goyaz. In the southern states mention is made of the occurrence

[1]) For the literature on parts of the coastal regions on either side of the Atlantic Ocean we refer to: J. C. BRANNER. Geology of Brazil. Bull. Americ. Geol. Soc. 1919. P. A. WAGNER. The Geology and Mineral Industry of South-West-Africa Geol. Surv. Memoir N⁰. 7, 1916.

[2]) E. HUSSAK. Der goldführende kiesige Quarzlagergang von Passagem in Minas Geraes. Zeitschr. f. Prakt. Geol. 1898. Oktober, blz. 345 e.v.

of granites, intrusive in rocks of probably old-palaeozoic. For instance by E. P. DE OLIVEIRA, and according to a communication to the present writer by GONZAGA DE CAMPOS also in the State of Sao Paulo granites have distinctly metamorphosed old-palaeozoic rocks. As with the old granites, still too little is known of the petrographic features of the Brazilian young granites to compare them with those of South-Africa.

Alkali-rocks.

First of all we refer to places, where alkalirocks occur at or near the opposite coasts, as in Brazil in a number of places in the Serra do Mar[1]) (Itatiaya, Serra do Gericino, Serra de Tingue, Cabo Trio) and in Africa near the coast of Lüderitzland, and near Cape Cross to the North of Swakopmund.[2]) It is most likely that similar rocks occur out of these better known regions in a number of other localities near the coasts. We know e.g. already pyroxene foyaite from Angola and much farther northward different alkali-rocks, from the Los Islands (9°13′ N.East).

Abyssal-rocks and the related dyke- and effusive-rocks are associated with each other. They are in South-West Africa syenites, nepheline-syenites, essexites, and theralites with phonolites, tinguaites, bostonites, camptonites, monchiquites alnoites. Similar rocks are known to occur in the Brazilian coastal region, we cite only the well-examined foyaites, essexites, phonolites, and basic dyke-rocks, besides tinguaites and bostonites in and near the State of Rio de Janeiro. The association with the related effusive rocks points in both regions to the circumstance that the alkalirocks are in part intrusive into their own effusive rocks and that they have crystallized at a small depth below the earth's surface Erosion caused the volcanoes to disappear, which formerly existed near the two opposed coasts of the Atlantic Ocean, as they now arise near the East-African Lake-region, where also alkali-rocks are of frequent occurrence. Farther removed from the two coasts alkalirocks exist in various localities. We confine ourselves to mentioning only two largest eruptive provinces, hitherto examined on both continents, viz that of Poços de Caldas[3]) in the South of the State of Minas Geraes, and the that of the Pilandsberg[4]) in the district of Rustenburg (Transvaal). These two large provinces, the

---

[1]) O. E. DERBY. On Nepheline rocks in Brazil. l. c.

[2]) E. KAYSER. Bericht über geologische Studien während des Krieges in Süd-West-Afrika. Abh. der Giessener Hochschulegesellsch. II, 1920, blz. 18.

[3]) O. E. DERBY. loc. cit.

[4]) H. A. BROUWER. Geology of the alkali rocks in the Transvaal. Journ. of Geology, 1917, XXV, p. 741 sqq.

first with a diameter of about 25 to 30 k.m., the second of about 30 k.m., are both remnants of volcanic centres of large extent. In both provinces the effusive-rocks include phonolites, leucite-rocks, volcanic breccias and tuffs; among the abyssal-rocks foyaites and syenites are known. In both provinces aegirine or aegirineaugite is a common dark constituent and tinguaites occur as independent rocks or as marginal zone of nephelinesyenites.

Volcanic rocks and intrusive dolerites.

The volcanic rocks of the Stormberg-series, whose lavas are widely spread over the whole of South-Africa, point to a volcanic episode in the mesozoic history of this country. At the same time and shortly after this the instrusion of the so-called Karroo-dolorites took place, which occur chiefly as dykes and intrusive sheets. Near the westcoast the Kaoko-formation, composed of horizontal sandstones and augiteporphyrite, extends over a wide area between 18° and 21° S. Lat.

In Brazil similar rocks have a great extent. Dykes and intrusive sheets of diabase occur in various places in the states of Minas Geraes and Sao Paulo in rocks of permian and of triassic age. Just as in South-Africa a thick series of volcanic rock occurs in the upper series of the Sta Catherina System, which is the equivalent of the South-African Karroo-System. These rocks are considered to be of Jurassic age and cover large surfaces in the States of Rio Grande do Sul, Santa Catharina, Parana, Sao Paulo, and Matto Groso, even parts of The Argentine, Urugay and Paraguay.

Rocks like those in the above-named Kaoko-formation in South-Africa occur also in Brazil near the opposite coast in the Southern States of Santa Catherina, and Rio Grande do Sul. In both regions these formations overlie for the greater part archaean rocks.

Kimberlites, Alnoites, etc.

The frequent occurrence of these rocks in South-Africa as far as in the Congo State is well-known, in connection with the occurrence of the diamond in some of these rocks, especially in some diamond-pipes which are generally filled up by a volcanic breccia of serpentinised ultrabasic material.

Suchlike rocks have been known long since in Brazil. They have been described by HUSSAK [1]) as picriteporphyrite. He points out a certain resemblance between the diamond-bearing deposit of Agua Suja in West-Minas Geraes and the Kimberlites of South-Africa, while later on Kimberlite was recognized in dykes in the State of

---

[1]) E. HUSSAK. Uber das Vorkomen von Palladium und Platin in Brasilien. Zeitschr. f. prakt. Geol. XIV, 1906, blz. 284 e.v.

Rio de Janeiro together with picrite porphyrites, alnoites and limburgites, besides similar rocks in dykes and pipes in the Western part of the State Minas Geraes [1]).

Just as the Kimberlite rocks near the West-Coast of South-Africa, the known Brazilian rocks also belong nearly all to the basaltic varieties, which are poor in mica.

Horizontal movement of the Atlantic Coasts.

The resemblance between some groups of sedimentary rocks on either side of the Atlantic Ocean is also striking. We merely mention the South-African Karroo System and the Brazilian Santa Catharina System. The Orleans conglomerate in Sta. Catharina and Rio Grande de Sul agrees with the Dwyka conglomerate of South-Africa and in either continent the higher divisions are built up of the above-named thick series of volcanic rocks, such as those of the Drakensberg in Cape Colony and those of the Serra Geral in Rio Grande de Sul.

When we reconstruct the volcanoes of alkali rocks which existed in earlier periods along the present coasts, and imagine the two continents to be brought close together, we obtain a configuration similar to the aspect of the East-African Lake region, where at the present day the volcanoe Kenia and Kilima Ndsjaro built up of alkali-rich rocks, arise. This picture illustrates WEGENER's [2]) interpretation of the origin of the Atlantic Ocean [3]). More should be known, than has been recorded in the foregoing, about the resem-

---

[1]) E. RIMANN. Uber Kimberlite und Alnoite in Brasilien. Tscherm. Min. u. Petr. Mitt. 1915. Id. A Kimberlita no Brazil. Annaes da Escola de Minas de Ouro Preto. N⁰ 15, 1917, blz. 27 e.v.

[2]) A. WEGENER. Die Entstehung der Kontinente und Ozeane, Die Wissenschaft. Bd. 66, 1920.

[3]) Still other fissures of the African continent may be reconstructed of similar character to, but of higher geological age than, those of the present East African fractures. We refer to the system of dykes of alkali-rocks with a uniform northwestern to northern trend, occurring on either side of the old volcanic centre of the Pilands Berg in the Transvaal and can be traced over a distance of more than 100 K.M., cutting through all older formations. In the part of the earth's crust, which has disappeared here through erosion the fault-system may have exhibited here an aspect similar to that of parts of the present East-African fracture-system; it seems however that the horizontal movements on either side of these faults soon ceased and that they did not produce any considerable gaps. Then the fissures will disappear at greater depths and many similar faults may have existed in an earlier stage of erosion on the African Continent as intruded or gaping, fissures, of which no trace is visible now. (Cf. fig. 2 and p. 765 in H. A. BROUWER, Geology of the Alkali rocks etc. l. c.)

blance of the eruptive rocks and the petrographic provinces near the opposed shores, to lend support to the above interpretation. Still, in any case the resemblance of the rare eruptive rocks, is striking. According to WEGENER the present coastlines of Africa and South-America represent the borders of a fissure, which is supposed by that writer to have gradually widened to the present Atlantic Ocean through horizontal movements of the two present continents.

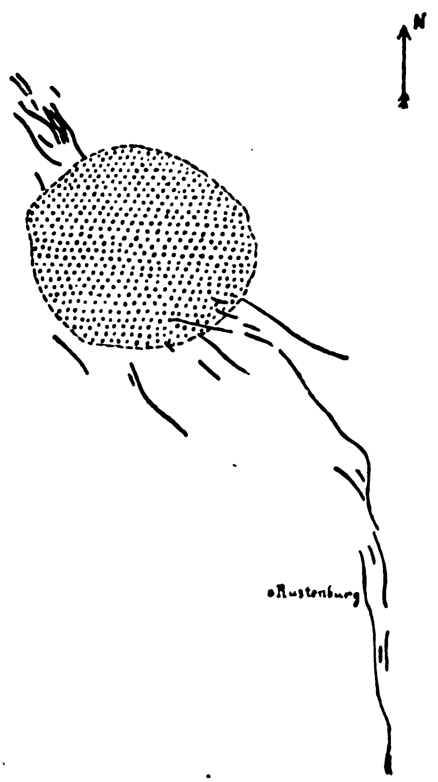


Fig. 2. An older African fault-system.

The old Pilandsberg vulcano (Transvaal).

Dikes of (nepheline) syenitic rocks.

Scale ± 1 : 1100.000.

This hypothesis is at variance with the view that the Atlantic Ocean should have arisen through the subsidence of a continental region, while Africa and America are supposed not to have moved in a horizontal direction.

The vertical movements executed on the surface of the earth are evidenced e.g. by upheaved shore-terraces and reefcaps, drowned river valleys etc. In connection with this the genesis of sea-basins is explained by vertical downward movements, because the horizontal movements are not established in a similar manner and consequently escape our direct observation. But with rising rows of islands the horizontal component of the rate of movement is sometimes much greater than the vertical one. The latter is distinguishable by upheaved coralreefs and shore-deposits whereas the former must be derived from far less distinguishable phenomena such as the form of the reefcaps and the character of the fault-movements. [1]) The mesozoic rows of islands of the Tethys have executed chiefly horizontal and far less significant vertical movements,

[1]) H. A. BROUWER. Uber die horizontale Bewegung der Inselreihen in den Molukken. Nachr. Ges. der Wiss. zu Göttingen. 1920, Math. phys. Kl. Id. Breuken en Verschuivingen nabij de oppervlakte van bewegende geantiklinalen. Versl. Kon. Akad. v. Wet. Amsterdam, XXVIII, 1920, p. 1151.

at present their masses overlie each other in the overthrust sheets of the mountain chains. These movements are explained by the fact that the old continental flocks of Eurasia and Indo-Africa have moved towards each other, in which process continental areas have executed horizontal movements. Similar movements may have co-operated to originate the Atlantic Ocean. Whether horizontal or vertical movements have prevailed may to some degree be made out by comparing the geological composition and structure of the opposed coastal regions. The points of similarity enumerated by WEGENER and contested by SOERGEL [1]) have still retained their significance in some measure and the concordance of the eruptive rocks discussed by us does not clash with the prevalence of horizontal movements.

[1]) W. SOERGEL. Die atlantische „Spalte". Zeitschr. der Deutschen Geol, Gesellsch. 1916, Monatsber. Bd. 68, S. 200 folg.

Physiology. — "*A direct proof of the impermeability of the blood-corpuscles of man and of the rabbit to glucose*". By S. VAN CREVELD and R. BRINKMAN. (Communicated by Prof. H. J. HAMBURGER).

(Communicated at the meeting of December 18, 1920).

1. *Introduction.*

The question dealing with division of glucose between the red bloodcorpuscles and the bloodplasma, which has been discussed so often already in the literature, has come to the front again through recent research.

In 1919 one of us together with Miss E. VAN DAM published a series of researches [1]) which clearly demonstrated that the permeability of the bloodcorpuscles for glucose is intimately related to the process of coagulation, and that the bloodcorpuscles of the frog and of man are found to be impermeable to gluclose only when the earliest incipiency of coagulation has been prevented. In the case of the frog the physiological impermeability could be shown by direct chemical analysis.

Such a direct chemical proof could at that time not be given for the human bloodcorpuscles. In the osmotic experiments these blood-corpuscles were invariably found to be impermeable in cases where the blood had not yet coagulated, and it was held that all the authors who had found the bloodcorpuscles to be permeable to sugar had used blood of which the commencement of coagulation had not been prevented.

Shortly after this publication there appeared an article by W. FALTA and M. RICHTER-QUITTNER [2]) on the distribution of sugar, chlorides and residual-N between plasma and bloodcorpuscles in the circulating blood. Also these investigators came to the conclusion that in man the sugar in the blood occurred only in the plasma. The method used by them could be considered as a direct chemical one. They determined the amount of sugar in the blood as a whole and in the plasma, and from these two values calculated the volume of the bloodcorpuscles, taking for granted that all the sugar occurred

[1]) BRINKMAN and v. DAM. Arch. Intern. de Physiologie XV. 105. 1919.

[2]) FALTA and RICHTER—QUITTNER: Biochem. Zeitschr. 100. 140. 1919.

in the plasma. The volume of the bloodcorpuscles found in this way corresponded in a large number of cases to that determined in the haematocrite.

The results of the Austrian investigators have, during the past year, been contradicted from different quarters by others who had used the same method, but had come to opposite results [1]). This did not surprise us seeing that FALTA and RICHTER-QUITTNER had used hirudine to obtain the bloodplasma. Before them, however, several other investigators had already used hirudine blood and had found the bloodcorpuscles to be permeable.

The explanation of this we thought could be sought in the fact that hirudine does not prevent the first phases of coagulation. Only after this had been prevented in another way it was found by the osmotic experiments that also in the hirudine blood the bloodcorpuscles are impermeable to sugar [2]). If FALTA and RICHTER-QUITTNER in spite of using hirudine blood had obtained the same results, then, we thought, it was to be attributed not to the hirudine but to the separation of the plasma and bloodcorpuscles by direct and rapid centrifugalisation. Whether, by setting to work in this way, the bloodcorpuscles are indeed found to be impermeable to sugar is, however, still subject to grave doubt owing to the many failures of experiments done with hirudine blood by others.

The great theoretical and practical value of the question under discussion demands however direct chemical proof which can be regarded as being absolute. Also this we think cannot be said of the experiments of FALTA and RICHTER-QUITTNER.

According to our train of thought such direct proofs could be given only by examining plasma which was free from bloodcorpuscles and which had been drawn directly from the bloodvessels, or had been obtained outside the body from blood which had remained perfectly fluid without the addition of a single one of the substances which prevent coagulation, for these, after all, do not prevent the first phases of coagulation. The amount of sugar in the plasma ought, if the bloodcorpuscles were impermeable, to be able to be calculated approximately from the total amount of sugar in the blood, and the volume of the bloodcorpuscles [3]).

---

[1]) See f. i Biochem. Zeitschr. **107**. 246 and 248. 1920.

[2]) BRINKMAN and v. DAM l.c.

[3]) We say "approximately" because we want to take for granted for the time being that the blood corpuscles have a share in the socalled restreduction. This is however very small according to the investigation of R. EGE (Biochem. Zeitschr. **107**, 229, 1920) when determined by the Bang-method which we used.

We have succeeded in obtaining plasma in both of these abovementioned ways, first from the rabbit and afterwards from man, and in subsequently demonstrating in a direct chemical way the impermeability of the bloodcorpuscles towards glucose.

II. *Determination of the amount of sugar in the bloodplasma of the rabbit obtained from a vein isolated from the body.*

To obtain blood-plasma from a blood-vessel our primary idea was that we could make use of the property of the bloodcorpuscles of female animals (especially pregnant ones) of settling rapidly compared to those of male animals [1]). Accordingly we several times clamped the marginal vein of a she-rabbits ear which did not show apparent anastomoses, the rabbit being bound on a rabbit plank and the ear in question held vertically. We did not succeed, however, in obtaining sedimentation in this way, probably because, after all, there still existed small anastomoses on account of which the blood could still circulate in the clamped vein.

By another method, however, the desired result was obtained with the same animals.

ARTHUS [2]) has found that when blood is kept in a vein which is taken from the body and ligated at both ends, the blood remains fluid in this vein, and, what is of great importance with regard to the question under discussion, shows no glycolysis. This method for obtaining uncoagulable plasma has, practically speaking, up to the present been followed only with the jugular of the horse and is therefore known as the jugular-method.

We have applied it twice to obtain pure plasma from rabbits. Here we set to work in the following way: The jugular on one side was laid open over a length of at least 4 cms. and dissected free from the neighbouring tissues and the greatest length between two of its confluent veins was doubly ligated at both ends. This part which was ± $2^1/_2$ cms. long in both cases was then removed from the body and held vertically. Seeing that it would take too long to wait for the bloodcorpuscles to settle down when the vein was hung up we placed it in a small centrifugal tube in which the vein just reached to the bottom, and centrifugalized rapidly. After some minutes there could be seen through the wall of the

---

1) FÄHRAEUS. Biochem. Zeitschr. 89. 355. 1918.

2) ARTHUS. Arch. de Physiologie 1891—1892.

vein the distinct division between the dark mass of bloodcorpuscles and the pale yellow translucent plasma. The bloodvessel was then ligated in the lowest layer of plasma and a prick hole made in the top portion from which the oozing plasma was caught up on two pieces of Bang's paper. In both cases we obtained enough plasma to enable us to make a reliable double determination of the plasma sugar. Simultaneously with this blood was drawn from a vein in the ear of the same rabbit and in this the amount of blood sugar and the relative volume of blood corpuscles was determined. The result of both experiments was the following:

	a. Plasmasugar.	*b.* Total bloodsugar.	*c.* Volume of bloodcorpuscles.	*d.* Amount of plasma sugar calculated from *b* and *c*.
Exper. I.	0.266 %	0.194 %	27	0.2657 %
» II.	0.255 »	0.1935	27	0.265

We have therefore obtained with the jugular-method the result almost surprisingly accurate and accordanto that *in the case of the rabbit the blood sugar occurs almost exclusively in the plasma.* It can be remarked here still that both rabbits which were operated upon under a light ether anaesthesia showed a pronounced hyperglycaemia. This *hyperglycaemia* could therefore be reduced totally to a hyperglucoplasmia.

III. *To show the impermeability of the bloodcorpuscles of man towards sugar by the paraffin method.*

To investigate also in the case of man the impermeability of the blood corpuscles towards sugar along directly chemical lines we first used the vein method for obtaining the plasma. Upon the advise of Prof. HAMBURGER the umbilical cord was used as human vein. Through the condencension of Prof. NIJHOFF and the house obstetricians of the obstetrical clinic in Groningen we had for some weeks at our disposal perfectly fresh umbilical cords. We tried repeatedly to bring about in pieces of umbilical cord a division between plasma and bloodcorpuscles in the large vein which could not always be traced distinctly, because this vein could only with great difficulty be dissected free from the neighbouring tissue with which

it was intimately connected. Neither by centrifugalizing in suitable tubes nor by hanging up vertically pieces of ligated cord did we succeed in this however in more than a few cases. The strong contortions of the most umbilical cords and the consequent twistings were the chief reasons for this. Only once up to this did we succeed in bringing about in a cord with few contortions, the division and comparing the plasma sugar with the quantity of sugar in the blood of the cord as a whole and the relative volume of the blood corpuscles. The concentration of the sugar in the plasma was found to be markedly higher that that of the blood in the cord.

It appeared that some of the large veins which are constantly found on the surface of the foetal side of the placenta could be more easily isolated and then centrifugalized. In these the blood remained fluid for a markedly long time. Also in these cases we can up to this boast of only one reliable determination for comparing the plasma and the blood as a whole. This however also proved to be in favour of the plasma. We have not been able to make a sufficient number of determinations by this method to come to a conclusion through them whether human bloodcorpuscles are permeable or impermeable to sugar.

We succeeded in doing this in the meantime by another method viz the paraffin method; one way of keeping blood uncoagulated without adding one of the known substances is by collecting it in tubes which have been thoroughly cleaned and then waxed to make them perfectly smooth. By using thus small and narrow waxed tubes the blood collected in them can by rapid centrifugalization be divided into its corpuscular and plasmatic parts which takes place without the occurrence of coagulation. In larger tubes coagulation took place fairly regularly during the process of centrifugalization. The way in which plasma was obtained now was very simple.

From a carefully cleaned finger tip in which a deep prick was made with a needle, we allowed a few drops of blood to fall into two tubes which had been waxed shortly before the experiment. These were then rapidly centrifugalized for a period of from one to two minutes and the plasma then sucked off by means of a waxed pipette and dropped on BANG's paper. At the same time blood was collected for the determination of the total amount of bloodsugar and the relative volume of the bloodcorpuscles.

From a number of these experiments conducted with different persons at different times of the day the following results were obtained:

Total bloodsugar.	Sugar in the plasma.	Volume of the bloodcorpuscles.	Calculated sugar of the plasma.
0.117 %	0.178 %	38 %	0.188 %
0.103	0.161	39	0.169
0.110	0.165	38	0.180
0.135	0.223	43	0.237
0.137	0.192	38	0.221
0.111	0.188	41	0.190

In our opinion these results afford a direct chemical proof that the bloodcorpuscles of man are free from sugar.

This decision will have to be taken into account in clinical examinations so that besides the determination of the amount of sugar in the whole of the blood the volume of the bloodcorpuscles will have to be determined.

The contradiction which we find with the many authors on this subject [1]) we hold has its origin in the following facts which we will return to in extenso later:

1. Only in blood in which no signs of coagulation have appeared, we find the bloodcorpuscles free from glucose (hirudine and other substances which are supposed to make the blood uncoagulable, do not arrest the very first phases of the process of coagulation).

2. The existence of a glucose-colloid-compound must be taken into account.

3. Experiments which purpose the examination of the permeability of the bloodcorpuscles towards glucose with the aid of the introduction of fresh glucose must be judged with great care because the relative permeabilities of the α- and β-modifications of the α-glucose which result on solution of the latter are by no means equal [2]).

Groningen, December 1920. *Physiological Laboratory.*

[1]) See f. i. FALTA en RICHTER-QUITTNER. l.c.
R. EGE. Biochem. Zeitschr. **111**. 189. 1920.
BÖNNIGER. „ „ **103**. 306. 1920.
BRINKMAN en v. DAM. Biochem. Zeitschr. **105**. 93., **108**. 74. 1920.
HAGEDORN. „ „ **107**. 248. 1920.
FEIGL. „ „ **112**. 54. 1920.
M. B. WISHART. Journ. Biol. Chemistry. **44**. 563. 1920.
TACHAU. Zeitschr. Klin. Med. **79**. 421. 1914.
GRADWOHL and BLAIVAS. Journ. Lab. and Clin. Med. II. No. 6. 1917.

[2]) HAMBURGER. Proceedings of the Royal Acad. of Sciences XXI. N°. 4, XXVIII p. 318 and 327.

Physiology. — *The Significance of the concentration of calcium-ions for the movements of the stomach caused by stimulation of the N. Vagus*". By R. BRINKMAN and Miss E. VAN DAM. (Communicated by Prof. H. J. HAMBURGER).

(Communicated at the meeting of December 18, 1920).

The great significance of the calcium-ion as an antagonist of the Na- and K-ions has been set forth by numerous researches [1]) since the fundamental experiments by RINGER and LOEB. The physico-chemical explanation of the action of calcium-ions must be sought in the balancing effect that this ion has towards the monovalent Na- and K-ions, as is very clearly illustrated, for instance, by the researches of NEUSCHLOSZ [2]), published but latily, about the influence of salt-equilibration on the surface-tension of lecithine-soles in water. From the table below one can form an idea of this action. In this list it is stated how the strongly-increasing influence which definite (physiological) NaCl-concentrations exercise on the surface-tension

TABEL I.

Total concentration of the mixture.	1 NaCl.	1 NaCl/ 1 $CaCl_2$	1 NaCl/ $^1/_5$ $CaCl_2$	1 NaCl/ $^1/_{10}$ $CaCl_2$	1 NaCl/ $^1/_{20}$ $CaCl_2$	1 NaCl/ $^1/_{30}$ $CaCl_2$	1 NaCl/ $^1/_{50}$ $CaCl_2$	1 NaCl/ $^1/_{100}$ $CaCl_2$
1 mol.	90.3	89.4	83.8	80.3	76.3	77.5	82.7	88.8
$^1/_2$ mol.	90.7	90.3	84.4	80.8	76.6	78.4	83.6	90.6
$^1/_4$ mol.	92.9	90.8	85.6	81.7	76.8	79.4	84.4	91.5
$^1/_8$ mol.	94.5	91.7	86.1	82.1	76.4	80.0	85.2	92.4
$^1/_{16}$ mol.	92.9	89.7	84.4	81.2	76.0	79.8	84.9	91.0
$^1/_{32}$ mol.	87.6	89.0	84	80.5	75.9	79.6	83.3	90.3
$^1/_{64}$ mol.	83.6	88.3	83.6	80.1	75.7	78.5	82.5	89.4
$^1/_{128}$ mol.	80.1	87.5	82.7	78.5	75.9	77.5	81.9	88.1

[1]) Summary in HÖBER: Physikalische Chemie der Zelle und Gewebe, Kap. VIII (1914); v. TSCHERMAK: Allgemeine Physiologie, p. 120 (1916); BAYLISS: Principles of General Physiology, p. 215 (1915); HÖBER: Pflüger's Archiv. **166**, 531, 1917.

[2]) NEUSCHLOZS: Pflüger's Archiv. **181**, 17, 1920.

of lecithine-soles, is almost entirely neutralized by a definite concentration of $Ca^{\cdot\cdot}$-ions.

The surface-tension of a pure 1 % lecithine-sole amounted to **75.9.** [1])

Consequently it appears from this table that the influence of a definite concentration of an unbalanced NaCl-solution on the surface-tension of a lecithine-sole, may be neutralized almost entirely by the addition of Ca-ions, but it appears at the same time that only one definite $[Ca^{\cdot\cdot}]$ can do this and that this balancing effect can be produced neither by a too large $[Ca^{\cdot\cdot}]$ nor by a too small one. The degree of this balancing $[Ca^{\cdot\cdot}]$ depends on the ion-system present.

How we should explain this balancing is not known with certainty; it seems that LOEB [2]) and others have modified the theory of the electro-chemical ion-proteid-compound in favour of an ousting from the surface. In a biological respect examples have of late come to our knowledge from which it appears that also with the physiological ion-balancing the degree of $[Ca^{\cdot\cdot}]$ is decisive, and that very slight fluctuations of these $[Ca^{\cdot\cdot}]$ may have an important physiological consequence. [3])

It may be understood therefore that these $[Ca^{\cdot\cdot}]$ should be kept constant in the blood-plasm, as well as, e.g. the $[H^{\cdot}]$. The buffer-system by which this is principally effected has been indicated by RONA and TAKAHASHI [4]). According to these authors there is for the free calcium-ion-concentration in the blood the equation: $[Ca^{\cdot\cdot}] = K.\frac{[H^{\cdot}]}{[HCO']}$ (K being about 350), a relation we could entirely confirm by direct measurément of the $[Ca^{\cdot\cdot}]$. [5]) As the $[H^{\cdot}]$ practically varies very little in physiological and also in pathological cases, the $[Ca^{\cdot\cdot}]$ will consequently be controlled chiefly by the concentration of the bicarbonate-ions. An increase of the $[Ca^{\cdot\cdot}]$ will depend in the first place on a decrease of the $[HCO'_3]$, in other words of an acidosis.

The main object of this communication is what influence the $[Ca^{\cdot\cdot}]$ and its fluctuations have on the irritability of the N. vagus. As a

---

[1]) Measured with the stalagmometer of TRAUBE: Handbuch der Biochemische Arbeitsmethoden V, Bd. 2. 1912.

[2]) LOEB: Journal of General Physiology, Vol. I en II.

[3]) HAMBURGER en BRINKMAN: Biochemische Zeitschrift 88, 97, 1918; BRINKMAN: Biochem. Zeitschr. 95, 101, 1919.

[4]) RONA en TAKAHASHI: Biochemische Zeitschrift 49, 370, 1913.

[5]) BRINKMAN and miss VAN DAM, Verslagen Kon. Akad. v. Wetenschappen, meeting of 25 Oct. 1919.

test-organ we selected the perfused, surviving frog's stomach, on which we can easily study the influence of the N. vagus on the motility.

The general significance of the Ca-ions for the irritability of the cerebro-spinal and the autonomic, central and peripheric nervous system has been known for some time.

LOCKE[1]) demonstrated that the Ca-ion is necessary for conducting the stimulus from a nerve to a voluntary muscle. OVERTON[2]) proved that it was equally indispensable for preserving the synapsis between nerve-ending and ganglion-cell.

BUSQUET and PACHON[3]) showed that the irritability of the N. vagus, which soon disappears on perfusing the heart with a pure NaCl-solution (HOWELL)[4]), returns by adding small quantities of Ca. They further found, as did also SABBATANI[5]) by testing many calcium-salts of widely differing degree of dissociation, that we are definitely concerned with an ion-influence and that undissociated Ca-salts were of no importance for the balancing effect.

It is the concurring opinion of all investigators that the explanation of this Ca-ion effect must be sought again in the influence on the synapsis-colloids which is antagonistic to Na and K. From the above-mentioned experiments of NEUSCHLOSZ[6]) as well as from said physiological experiments [7]) it appeared also that *this* $[Ca^{\cdot\cdot}]$ *must have a very special constant value,* and that slight variations of the physiological $[Ca^{\cdot\cdot}]$ may be of great influence. A total absence of $Ca^{\cdot\cdot}$-ions will never occur in vivo, but especially these slight fluctuations of $[Ca^{\cdot\cdot}]$ are important under physiological conditions.

It is true that in the literature of the subject there are indications to be found that a too large quantity of Ca is as detrimental as a

---

1) LOCKE: Zentralblatt f. Physiologie **8**, 166, 1894. See farther
CUSHING: American Journal of Physiology, **6**, 77, 1902;
MINES: Journal of Physiology, **42**, 251, 1911.

2) OVERTON: Pflüger's Archiv. **105**, 261 and 280, 1904.

3) BUSQUET et PACHON: Journal de Physiologie et de Pathologie Gén. **11**, 807 and 851, 1909.
MINES l.c.; LOEWI: Archiv. f. Exp. Pathol. **70**, 343, 1912.
HAGGAN and ORMOND: Amerc. Journ. o. Physiol. **30**, 105, 1912.
CAZZOLA: Archivio di Fisiol. **11**, 88, 1913.

4) HOWELL: Americ. Journ. o. Physiol. **15**, 280, 1906.

5) SABBATANI: C. r. Soc. Biol. **54**, 716, 1902.

6) NEUSCHLOSZ: l.c.

7) HAMBURGER and BRINKMAN: l.c.

too small one[1]), but a careful study, showing the relation between the [$Ca^{\cdot\cdot}$] degree and the vagus-irritability, has not come to our knowledge. For this reason we have tried to find this relation as it was also done with the surviving frog's kidney[2]) and the haemolysis[3]).

The perfusions were done as follows:

The abdominal wall, thorax wall, and clavicula of the frog (♂) are carefully cut away, also the extremities are removed and the test object is nailed to a board. For a better survey the intestines may also be removed as far as the duodenum provided the mesogastrium is not injured. The canula is inserted into the a. coeliaca; the a. mesenterica is tied tight. In this way stomach + liver and gall-bladder (art. hepatica) are perfused. The pressure may be regulated by the level of the liquid-reservoir and the orifice of the canula.

Should it be desired to perfuse the whole of the intestines + portal circulation, than the a. mesenterica is not tied; liver (arterial and venous), wall of gall-bladder, stomach and intestines are then perfused. The proximal part of the v. abdominalis must be tied.

The n. vagus is stimulated by inserting electrodes in the tubae Eustachii; this is done most easily, by hammering 2 copper nails through the tubae into the board.

With this method of stimulation we always see (by very constant coils-distance) the vagus-effects on heart and stomach-intestines.

The duration of each experiment was about $1^1/_2$ hours.

We have now observed the influence of the Ca-ion-concentration in about 75 perfusions. Beforehand the irritability and the motility of the stomach-wall of the newly-killed not-perfused frog was determined, which existed as much as possible under physiological conditions.

Afterwards the perfusions took place with the following solutions:

1. NaCl 0,6 °/₀.
2. NaCl 0,6 °/₀, then NaCl 0,5 °/₀, $NaHCO_3$ 0,20 °/₀, $CaCl_2$.6 aq. 0,040 °/₀, KCl 0,020 °/₀ $P_H = \pm 8{,}6$[4]).
3. NaCl 0,6 °/₀ + KCl 0,02 °/₀.
4. NaCl 0,6 °/₀ + KCl 0,02 °/₀ + $CaCl_2$. 6 aq. 0,005 °/₀, 0,010 °/₀, 0,012 °/₀, 0,014 °/₀ etc., 0,020 °/₀, 0,025 °/₀ etc.
5. NaCl 0,5 °/₀, $NaHCO_3$ 0,28 °/₀, $CaCl_2$.6 aq. 0,040 °/₀, KCl 0,02 °/₀, P_H varying considerably: from 8,6 to 7,2.
6. NaCl 0,6 °/₀, $CaCl_2$.6 aq 0,040 °/₀, KCl 0,02, $P_H = 8{,}6$, $NaHCO_3$ 0,05 °/₀, 0,010 °/₀, 0,0015 °/₀ etc.

---

1) JOSEPH E. MELTZER: Americ. Journ. o. Physiol. **29**, 1, 1911.
BENDA: Zeitschr. f. Biol. **63**, 11, 1914.

2) HAMBURGER u. BRINKMAN, l.c.

3) BRINKMAN: l.c.

4) Colorimetrical according to SÖRENSEN.

1. *The influence of a pure NaCl-solution on the motility and the irritability of the vagus of the muscular-stomachwall.*

When observing the stomach of a newly-killed frog, one often notices spontaneous local contractions or peristaltic waves in both directions. Stimulation of the vagus, brought about in the way described above, causes strong peristaltic movements, especially in the pyloric part; at the same time one can observe a frequent lengthwise contraction. The stimulation has a rather long after-effect (5 minutes). It was constantly found that the minimum degree of effective stimulation was with a coils-distance of 7 to 8 c.m.

If the stomach is perfused with a 0.6 °/₀ NaCl solution $P_H = 8.6$), we see that the spontaneous peristalsis has disappeared after 5 to 10 minutes and that the stomach has become quite limp; the mechanical irritability has completely disappeared.

The vagus-irritability is as follows: before the perfusion a vagus-effect is observed with a coil-distance of 7 to 8 c.m.; after a 5 minutes perfusion a distance of 5 c.m.; after 10 minutes a distance of 4 to 3 cm.; after 15 to 20 minutes even the strongest stimulation of the vagus takes no effect.

This disappearance of the vagus-irritability is reversible. If, after half an hour's perfusion with the pure NaCl solution, the liquid is replaced by a well-equilibrated salt-solution (NaCl 0.5 °/₀, $NaHCO_3$ 0.28 °/₀, $CaCl_2$ 6 aq 0.040 °/₀, KCl 0.020 °/₀, $P_H = 8.6$) spontaneous contractions are again observed after five minutes; after 10 minutes vagus-effect occurs at 10 cm. coil-distance, after 25 minutes vagus-effect can be observed clearly at a coil-distance of 7 cm.

So it is clear that after half an hour's perfusion with a pure NaCl solution, the harmful action is still perfectly reversible.

2. *The influence of $NaCl + KCl$.*

Now we have tried to find out which ions of the physiological solution in this respect caused the balancing effect. It soon appeared that the addition of K-ions, to which one has to assign such an important effect in heart-perfusion, have no effect of importance here. A concentration of K-ions which can cause the return of the vagus-irritability cannot be found.

3. *The influence of $NaCl + KCl + CaCl_2 . 6\,aq$.*

The vagus-influence may be re-established by the addition of a certain calcium-concentration to the (in itself insufficient) system of

NaCl 0.6 °/₀ + KCl 0.02 °/₀ ($P_H = 8.6$). The following experiments give a brief survey of it:

a. unperfused stomach, vagus-effect at a distance of 8 cm., afterwards NaCl 0.6 °/₀ + KCl 0.02 °/₀; after 10 minutes the vagus is un-irritable, the stomach is limp. Then NaCl 0.6 °/₀, KCl 0.02 °/₀, $CaCl_2$. 6 aq 0.002 °/₀; vagus-effect still fails to appear, stomach remains limp, though somewhat less so than when it is perfused with a pure NaCl solution;

b. unperfused stomach, vagus-effect at 7.5 cm., then NaCl 0.6 °/₀, KCl 0.02 °/₀; after 15 minutes the stomach is limp, stimulation of vagus has no effect. Then NaCl 0.6 °/₀, KCl 0.02 °/₀, $CaCl_2$. 6 aq 0.004 °/₀. Whereas the effect of this Ca-concentration on the perfused heart is clearly visible, there is no effect whatever on the stomach, except a slight tonic contraction.

c. Nor could a return of the vagus-irritability be established in numerous perfusions, when to the NaCl 0.6 °/₀ + KCl 0.02 °/₀ was added respectively $CaCl_2$. 6 aq 0.006 °/₀, 0.008 °/₀, 0.010 °/₀, etc. But

d. the addition of 0 015 °/₀ $CaCl_2$. 6 aq to NaCl 0.6 °/₀ + KCl 0.02 °/₀ caused the vagus-irritability to return completely. We must, however, stress the fact, that, to obtain this result, one should take special precautions. As namely the liquid does not possess at all a buffer-system against H-ions, fluctuations of $[H^{\cdot}]$ occur very easily. It is necessary that the P_H of this perfusion-liquid should be 8.6 and remain constant during the experiment. The use of a rubber tube is very dangerous in this experiment, as it nearly always makes the liquid too acid.

These precautions being taken, one can always demonstrate that a concentration of 0.015 °/₀ $CaCl_2$. 6 aq (and also 0.016 °/₀) is able to balance the concentration of alkali-ions; this concentration corresponds to a free $[Ca^{\cdot\cdot}]$ of about 9 milligrammes per litre.

It is an interesting fact that exactly the same concentration of Ca-ions proved necessary for the preservation of the impermeability of the glomerulal membrane for physiological quantities of glucose.[1])

e. a concentration of $CaCl_2$. 6 aq of 0.020 °/₀ and higher concentrations are unable to preserve or recall the vagus-irritability; then tonic contractions of the stomach-wall too disappear again completely in this case.

4. *The influence of a concentration of hydrogen-ions.*

By choosing the total quantity of Ca of the perfusion-liquid in

[1]) HAMBURGER and BRINKMANN, l.c.

such a manner that a high free [$Ca^{\cdot\cdot}$] cannot arise, it is possible to investigate the influence of the [$H^{\cdot}$] separately.

It appeared already in the above-mentioned perfusions with NaCl 0,6 °/₀, KCl 0,02 °/₀, $CaCl_2$.6aq 0,015 °/₀, that the [$H^{\cdot}$] must be kept within rather narrow limits.

When a buffer-system exists ($NaHCO_3 + CO_2$), the [$H^{\cdot}$] may vary within the limits of this system, as appears from the following experiments:

a. Perfusion with NaCl 0,5°/₀, $NaHCO_3$ 0,28°/₀, KCl 0,020°/₀, $CaCl_2$.6aq 0,015°/₀, $P_H = 8,6$; there are strong spontaneous peristaltic movements; vagus-irritability at a coil-distance of 7.5 cm. Then the same liquid but now with CO_2 passed through until $P_H = 7,1$; the stomach becomes limp in 10 minutes and can no longer be influenced by vagus-irritation.

b. Perfusion with NaCl 0,5°/₀, $NaHCO_3$ 0,28°/₀, KCl 0,02°/₀, $CaCl_2$.6 aq 0,015°/₀, $P_H = 8,6$; irritability at a distance of 8 cm., spontaneous contractions. Then $P_H = 8,3$, constant irritability at 10 cm.'s distance; spontaneous contractions of stomach. Then $P_H = 7,7$; irritability at 14 cm., spontaneous rapid peristalsis. Then $P_H = 7,3$; irritability at 14 cm., stomach contracted spastically. Then $P_H = 7,1$; stomach not irritable, spontaneous movements have disappeared. Then $P_H = 8,6$; after 10 minutes' vagus-stimulation at 8 cm. spontaneous movements of stomach.

This last survey is an example of many similar experiments, from which it appears that the slight [$H^{\cdot}$] fluctuations do not let the vagus-irritability disappear but certainly influence it.

The actions of the $H^{\cdot}$ and the $Ca^{\cdot\cdot}$ cannot be separated here, because their quantities are directly dependent on each other and because in general the colloid-action of the $Ca^{\cdot\cdot}$-ions depends on the H-ion-concentration which is present. The balancing effect of $Ca^{\cdot\cdot}$-ions can be indicated only with one definite $H^{\cdot}$-ion-concentration.

The fact that an alteration of the $Ca^{\cdot\cdot}$-ion-concentration in itself induces a variation of the vagus-irritability, is shown by the last series of experiments which correspond for the most part to conditions as they occur physiologically and pathologically.

5. *The influence of the $NaHCO_3$-concentration.*

When the $NaHCO_3$-degree of a liquid is modified systematically, the H-ion-concentration remaining constant, one obtains likewise a

modification of the Ca-ion-concentration, because the $[HCO'_3]$ and $[Ca^{\cdot\cdot}]$ are inversely proportional to each other, the influence of this modification is great, as appears from the following examples:

a. Perfusion with NaCl 0,6 %, KCl 0,02 %, $CaCl_2$.6 aq. 0,04%, $NaHCO_3$ 0,05 %. $P_H = 8,6$. After 5 minutes the spontaneous contractions have disappeared and the stomach is contracted spastically: no effect of vagus-irritability is visible.

b. Perfusion with NaCl 0,6%, KCl 0,02 %, $CaCl_2$. 6 aq 0,04%, $NaHCO_3$ 0,10%, $P_H = 8,6$. After 10 minutes the stomach is contracting with intense spasms, especially the pyloric part of it. The vagus is extremely irritable, at 15 cm's coil-distance deep waves arise in the stomach-wall which last very long and are displaced very slightly; finally we have a very spastically contracted stomach (pylorospasmus).

c. Perfusion with NaCl 0,5 %, KCl 0,02 %, $CaCl_2$. 6 aq 0,04 %, $NaHCO_3$ 0,15 %, $P_H = 8,6$. After 10 minutes the stomach shows very slight peristalsis with intense spastic contractions in the pyloric part. Vagus-irritability at 12 cm, tonic contractions lasting a very long time.

d. Perfusion with NaCl 0,5 %, KCl 0,02 %, $CaCl_2$. 6 aq. 0,04 %, $NaHCO_3$ 0,20%, $P_H = 8,6$. With this liquid the spontaneous peristaltic movements appear again; the vagus is irritable at a coil-distance of 9 cm. and produces a series of peristaltic movements; the spastic contractions are still present in a slight degree.

e. Perfusion with NaCl 0,5 %, KCl 0,02 %, $CaCl_2$. 6 aq. 0,04 %, $NaHCO_3$ 0,28 %, $P_H = 8,6$. With this liquid the vagus is irritable at a coil-distance of 7 cm.; there are normal peristaltic movements and no spastic contractions. The conditions are completely like dose of the unperfused stomach.

From these experiments appears clearly the great influence which a change in bicarbonate-concentration has on the irritability of the n. vagus and on the spontaneous rhythmical movements of the stomach-wall. The latter effect is probably also due to the influence on the autonomous plexus of AUERBACH. It cannot be decided with certainty whether we have to think here especially of a direct influence of the HCO_3-ions [1]) or only of the influence of the latter on the $[Ca^{\cdot\cdot}]$, but, in connection with the experiments with pure NaCl + $CaCl_2$ solutions, the primary significance of the Ca-ions seems to us by far the more probable.

We attach some significance to the fact that a decrease of $[HCO_3']$,

---

[1]) RONA and NEUKIRCH: Pflüger's Archiv. **148**, 285, 1912.

so an acidose, can cause spastic concentration of the stomach and an increased irritability of the n. vagus (vagotony). Whether a decreased [Ca¨] can cause similar phenomena, has not yet been investigated by us.

Physiological Laboratory of the University of Groningen.

December 1920.

Paleontology. — "*On the Significance of the Large Cranial Capacity of Homo Neandertalensis*". By Prof. EUG. DUBOIS.

(Communicated at the meeting of November 27, 1920).

Before the discovery of the fossil man of La Chapelle-aux-Saints our knowledge of the most important character of *Homo neandertalensis*, the cranial capacity, rested only on estimation, especially from the capacity of the calvaria. SCHAAFFHAUSEN, HUXLEY and SCHWALBE started from the supposition that the capacity of the calvaria of the Neandertal Man, which is human as regards its size, was in the same ratio to that of the whole skull as in Man of the present type. It is not surprising, that their results are pretty well concordant [1]).

First SCHAAFFHAUSEN [2]) measured the capacity of the Neandertal calvaria with water, on a level with the orbital plate of the frontal bone, with the deepest notch in the squamous margin of the parietal, and with the superior semicircular ridges of the occipital. He found for it 1033 cm.^{3}, and estimating the capacity of the missing part at 215 cm.^{3} from other skulls, he found 1248 cm.^{3} for the total capacity of the skull. Later, anew measuring the calvaria with water, "mit ihrem oberen Rande horizontal gestellt", he found 930 cm.^{3} for its capacity, and now for the whole capacity, through comparison with the corresponding part and the whole of a "roh gebildeten Schädel" of 1305 cm.^{3} capacity and of a negro skull, only 1093, resp. 1099 cm. [2]). Accepting the first calvaria measurement by SCHAAFFHAUSEN, HUXLEY [3]) estimated the capacity of the entire skull at about 75 cubic inches (= 1229 cm.^{3}). SCHWALBE [4]) measured the capacity of the Neandertal calvaria with peas up to the transversal

---

1) M. BOULE, Sur la capacité cranienne des Hommes fossiles du type de Néanderthal. Comptes rendus. Académie des Sciences. Tome 148, p. 1352. Paris 1909.

2) SCHAAFFHAUSEN, Zur Kenntniss der ältesten Rassenschädel. Archiv für Anatomie, Physiologie und wissenschaftliche Medicin (Johannes Müller). Jahrgang 1858. Berlin, p. 455 and p. 464.

H. SCHAAFFHAUSEN, Der Neanderthaler Fund, p. 43. Bonn 1888.

3) T. H. HUXLEY, Evidence as to Man's Place in Nature, p. 156 - 157. London 1863.

4) G. SCHWALBE, Der Neanderthalschädel. Bonner Jahrbücher. Heft 106, p. 50—52. Bonn 1901. SCHWALBE erroneously rejects SCHAAFFHAUSEN's second determination, „weil sie durch Wasserfüllung ermittelt ist", which would, indeed, also be applicable to the first determination. In this procedure errors *could* be avoided. It is not clear what caused SCHAAFFHAUSEN to arrive at so much lower capacity

glabella-inion plane, and found, on comparison with the skull of a New-Irelander, 1233 cm.^{3} for the capacity of the entire Neandertal skull. His confidence in these results was so great that he stated: "An der Thatsache, dass die Capacität des Neanderthalschädels nicht mehr als 1230 cm.^{3} beträgt, ist jedenfalls nicht zu zweifeln". Yet it has turned out that his conclusion was erroneous.

SCHAAFFHAUSEN's measurements dit not refer to parts of the cranial cavity that could be clearly defined. For this reason I measured the capacity of the Neandertal calvaria, already in 1897, up to a definite plane imaginable in the encephalon, the transversal plane through the frontal pole of the hemispherical axis (which plane in most human skulls, as also in those of Neandertal and of Spy and in Pithecanthropus, corresponds to the boundary of the lowest and middle third part of the area of the inferior frontal convolution) and the middle of the upper rim of the right sulcus transversus of the occipital bone (corresponding to the lower margin of the cerebrum). First I then measured the capacity of the calvaria of the Spy-skulls at Liège, in the laboratory of my regretted friend JULIEN FRAIPONT; the following day at Bonn, in the Provincial-Museum, with the permission of the director, Professor J. KLEIN, that of the Neandertal-calvaria in perfectly the same way, with the same material (rape-seed). I found 920 cm^3. for the Neandertal-calvaria, almost the same capacity as SCHAAFFHAUSEN found in his second measurement. This concordance is probably owing to this that the upper rim of the right sulcus transversus coincides in its horizontal course with the edge of the fracture[1]). Thus I determined the capacity of the calvaria of Spy I at at least 900 cm^3., of Spy II at at least 1050 cm^3. The two latter values can be so only approximately on account of the incompleteness and partial reconstruction of the skull walls, especially of Spy I.

---

of the fossil skull in his later comparison; probably because he took other limits of the calvaria space in the modern skulls than in the fossil one.

J. RANKE (Der Mensch. Zweite Auflage. Band II, p 478. Leipzig 1894) estimated the capacity, from the horizontal circumference and the breadth index according to WELCKER's table, at 1532 cm^3. L. MANOUVRIER („Deuxième étude sur le Pithécanthropus" in Bulletin de la Société d'Anthropologie de Paris, 4e série, tome 6, p. 585. Paris 1895) estimated it at 1500 cm^3. by assuming a basio-bregmatic height of 125 mm. and a cubic index of 1.25. The latter estimation, in Broca-measure, corresponds to a minimum of 1410 cm.^{3} real capacity. RANKE supposes, certainly erroneously, that the height, independent of the particular shape of the skull, is in the same relation to the horizontal dimensions as in ordinary human skulls.

[1]) Thus noted down at the time of my investigation. The protuberantia occipitalis interna, which cannot be sharply defined, lies $\pm$ 8 mm. higher.

In order to compare as much as possible with homologous capacities of recent men I chose three skulls of Europeans (Dutchmen) of different sizes, and a skull of a Javanese, and determined the capacities of the upper or calvarial part, to the same level, and of the entire skulls, with water, by the halves which had been made impermeable and were shut off by a glass plate.

	D.1	D.2	D.3	J.	
Cranial capacity	1260	1434	1500	1550	cm^3
Calvarial capacity	884	1000	1070	1150	„
Ratio	1.42	1.43	1.40	1.35	mean 1.4.

Accordingly the calvarial capacities of the examined individuals of the Neandertal-Man fall entirely within the range of the calvarial capacity (which is as much as possible homologous) of large-brained recent races. The total capacity was, therefore, certainly not smaller.[1]) A simian flattened upper part of the skull must have gone together, as in the Apes, with a comparatively larger lower part of the skull than in the high-vaulted skull of recent Man.

According to the ratio found in recent Man the capacity of the (entire) Neandertal-skull would have been 1288 cm^3, in concordance with the earlier and with SCHWALBE's estimations; that of Spy I would at least be 1260, and that of Spy II at least 1470 cm^3.

But at the skulls of Apes (Gorilla gorilla, Simia satyrus, Hylobates agilis, Semnopithecus entellus, Macacus cynomolgus) I found that the ratio of these capacities, which were again as homologous as possible and deviated little inter se, is 1.6 on an average. In the

---

[1]) EUG. DUBOIS, Remarks upon the Brain-Cast of Pithecanthropus erectus. Proceedings of the Fourth International Congress of Zoology. Cambridge 1898. p. 85–86. There too with regard to the same investigation made on skulls of apes and on the calvaria of Pithecanthropus erectus. The results were in detail as follows:

Pithecanthropus		Gorilla ♂	Anthropopithecus ♀	Simia satyrus ♀
Capacity	—	540	356	346
Calvaria	570	334	255	219
Ratio	—	1.61	1.43	1.58

Hylobates agilis ♂	Symphalangus ♂	Semnopithecus entellus ♂	Macacus cynomolgus ♂
114	128	116	77
73	62	72	48
1.56	2.06	1.61	1.60

The measurements of the capacities with rape-seed yielded average results equal to those with water; the values found can in view of this, be considered as the true capacities.

very flat-headed Siamang it has even risen to 2 (in contrast with 1.56 in Hylobates agilis); in a female Chimpanzee I found on the other hand 1.43.

These last ratios give rise to doubt whether the comparatively small capacity of the upper part of the skull (calvaria) and the platycephaly are really an indication in general of a low development of the brain; they make it probable that here m e c h a n i c factors lying outside the brain, which are in connection with the comparatively great size of the jaws or the poise of the head, if they are not the only ones, at least preponderate. Actually the jaws of the Siamang are comparatively much larger than those of the small Hylobatides (the ratio capacity: palatal area was 6.7 : 1 in Symphalangus syndactylus, 9.5 : 1 in Hylobates leuciscus); also the female Chimpanzee has comparatively small jaws. And undoubtedly the head poise of Homo neandertalensis was different from that of Homo sapiens.

The ratios found in skulls of Apes might have led us to expect that in the platycephalic skulls of the Neandertal type the lower part of the skull, hence the whole capacity of the skull in comparison with the calvaria, was more spacious than in skulls of the Homo sapiens type.

This has actually appeared, after in 1909 BOULE[1]) with VERNEAU and RIVET, through direct measurement with millet-seed, had determined the (total) skull capacity of the fossil man of La Chapelle-Aux-Saints, and had found the considerable amount of 1626 cm^3 Broca-measure, i.e. 1530 cm^3 real capacity[2]).

SCHWALBE[3]) then concluded from this skull that it would not do to calculate the missing part of the capacity of the Neandertal-skull from the comparison with a skull of Homo sapiens, as he had done before, and found that the Neandertal type is sharply distinguished from that of Homo sapiens by the much more considerable relative height of the lower part of the skull, measured by the perpendicular of the basion to the glabella-inion line. He states from photograms published by BOULE that the height of the lower part of the skull constitutes a relatively much larger part of the total height (normal to the glabella-inion line) than for instance in Australian skulls. The calvarial height of the La Chapelle skull is 82 mm. according to

[1]) Comptes rendus. Académie des Sciences, loc. cit.

[2]) According to E. SCHMIDT's Reductionstabelle für die Broca'sche Schrotmessungen. Archiv für Anthropologie. Band 13. Supplement, p. 78. Braunschweig 1882.

[3]) G. SCHWALBE, Kritische Besprechung von BOULE's Werk: „L'Homme fossile de La Chapelle aux-Saints" mit eigenen Untersuchungen. Zeitschrift für Morphologie und Anthropologie. Band 16, p. 593 - 594. Fig. 1—3. Stuttgart 1914.

his measurement, which I, too, find from BOULE's figures 24, p. 34 and 1, Pl. III. The rest of his statements are difficult to follow; this calvarial height is for instance as "Unterschädel" added to an "Oberschädel (Kalottenhöhe)" of 130 mm., which yields an (impossible) total height of 212 mm. In reality the basion-bregma height is, according to BOULE's statement[1]), 131 mm., from which I find 135 mm. for the total height, hence 53 mm. for the height of the lower part of the skull, or 39.3 °/₀ of the total height, which latter result is after all in good agreement with SCHWALBE's 38.7 °/₀. He gives 12.7 °/₀ height of the lower part of the skull for an Alsatian man, 27.6 °/₀ for an Australian. In his figures 1 (Alsatian) and 2 (Australian) I, however, measure ratios of 24.1 and 21.6 °/₀. Two other Australians have 22.8 and 27.7 °/₀. The skull of Wadjak I gives the ratio 28.6 °/₀. I find 21.2 °/₀ in a Javanese skull, 25,6 °/₀ in a Dutch skull of unknown origin. SCHWALBE finds 50 °/₀ height of the lower skull part for a full-grown chimpanzee, and 55 °/₀ for a Macacus nemestrinus. I determined the ratio 46.5 °/₀ in a skull of Hylobates agilis, and 60 °/₀ in that of the Siamang, Hylobates (Symphalangus) syndactylus. SCHWALBE calculates 38.5 °/₀ for the Neandertal skull; but on comparison with the total height of 135 of the La Chapelle-skull I find with SCHWALBE's 80.5 mm. calvarial height of the Neandertal man, 37 °/₀.

In this connection the comparative height of the lower part of the skulls of Frisians of old mounds ("terpen") and of the island of Marken in the Zuiderzee, which have been excellently described by BARGE, gets particular significance[2]).

In this BARGE has proved conclusively, what had already been

---

[1]) L'Homme fossile de la Chapelle-aux-Saints, p. 37.

[2]) J. A. J. BARGE, Beiträge zur Kenntnis der niederländischen Anthropologie I. Friesenschädel. Zeitschrift für Morphologie und Anthropologie. Band 16, p. 329—396. Stuttgart 1913. II. Schädel von der Insel Marken. Ibid., p. 465—521, Stuttgart 1914. With reproductions and tables. — From the island of Marken originates also Blumenbach's "Batavus genuinus", at whose forehead SCHAAFFHAUSEN, SPENGEL and R. VIRCHOW thought they could detect neandertaloid characteristics. On the evidence of the "Batavus genuinus" RUD. WAGNER was even led to pronounce the sentence: "Der Neanderthalschädel ist von einem alten Holländer", with the attenuating circumstance: "bis zum Gorilla hat es doch noch entsetzlich weit hin". (H. SCHAAFFHAUSEN, Der Neanderthaler Fund, p. 21, footnote. Bonn 1888). This large Marken skull cannot be called platycephalic, because the calvarial height index is 54.8 (G. SCHWALBE, Neanderthal Schädel und Friesenschädel. Globus. Band 81, p. 173. Braunschweig 1901), which is about equal to the mean of Australians and Tasmanians. Also the shape of the forehead should sooner be called austroloid. The height of the lower part of the skull is 19.4 °/₀ of its total height (measured on SCHWALBE's Abbildung 3, p. 172).

82*

observed by BOLK, that the Frisian skulls of the island of Marken, and more particularly the female skulls, have become artificially deformed, platycephalic through a particular kind of children's caps; their calvarial height index is on an average 55.4 (in three female skulls 52.5) as against 59.4 in the naturally formed old Frisian skulls from mounds.

It is very remarkable that also this artificial flattening is accompanied by an increase in height of the lower skull. On 28 of BARGE's median curves of skulls of mound-Frisians the comparative height of the lower skull (vertically below the glabella-inion line) can be measured; I find the following values for this in percentages of the total height of the skull: 25.1, 13.9, 21.3, 19.5, 19.9, 10.3, 23.0, 23.5, 21.2, 25.2, 21.5, 21.1, 24.0, 25.7, 23.7, 24.7, 24.8, 15.7, 22.6, 17.7, 12.8, 19.3, 16.5, 21.4, 25.5, 20.0, 24.4, 18.8. The mean of these Frisian skulls is **20.8**.

From 9 median curves of Marken skulls I find: 28.4, 28.4, 27.0, 23.0, 26.0, 26.3, 25.4, 22.9, 21.1. The three first, largest, values are of female skulls. The mean of the nine Marken skulls is **25.4**, of the six male ones(?) alone 24.1, of the three female ones 27.9.

It thus appears that this artificial platycephaly is attended with greater height of the lower skull. This can hardly be imagined in another way than that through the pressure from above part of the brain mass was forced downward. Therefore to the slight depression of the upper part of the skull corresponds a proportionally slight rise of the lower part of the skull; in the skull of La Chapelle-aux-Saints to 40.5 calvarial height index 39.3 °/₀ height of the lower part of the skull.

Now the greater height of the lower part of the skull, below the glabella-inion line in skulls of the Neandertal-type and in skulls of Apes can certainly partly be accounted for by the relatively high situation of the inion. In Spy I I found this point 12 mm., in Spy II 14 mm. above the middle of the right sulcus transversus, while in skulls of the present type the two points lie mostly on the same level [1]). In the skull of a ♀ chimpanzee the inion lies 23 mm., in

[1]) J. FRAIPONT and M. LOHEST (Recherches Ethnographiques sur des ossements humains découverts dans les dépôts quaternaires d'une grotte à Spy. Archives de Biologie. Vol VII, p. 622. Gand 1887) say that the protuberantia occipitalis interna "est située plus bas et en avant à un centimètre de distance environ".
K. GORJANOVIĆ—KRAMBERGER (Der diluviale Mensch von Krapina in Kroatien, p. 112. Wiesbaden 1906) found the protuberantia occipitalis interna „etwa 2 cm. abwärts vom Torus", M. BOULE (loc. cit., p. 47) between the same points, "inion interne" and "inion externe", the distance of 24 mm. at the skull of La Chapelle-aux-Saints, and SCHWALBE (loc. cit., p. 50) in the Neandertal-calvaria the external inion opposite the internal "nur um ein Geringes verschoben".

that of a ♀ orang utan 32 mm., of a ♂ Hylobates agilis and of a ♂ Siamang 5 mm., of a ♂ Semnopithecus entellus 14 mm., and of a ♂ Macacus cynomologus 18 mm. above the right sulcus transversus.

But in this way the great height of the lower part of the skull in the Neandertal type can only be accounted for for about a third part, and there exists a considerable difference in the relative height of the lower part of the skull between the two Hylobatides, though the inion is situated at the same distance above the sulcus transversus. It should be pointed out here that the platycephaly of the Siamang is by no means to be explained by the greater size of its body, for its weight is only the half more than that of the smaller Hylobatides. In the development of the brain they are certainly all about on a line, and yet the skull of the Siamang is in comparison with the other Hylobatides as much flattened as that of the Neandertal Man in comparison with recent Man (Fig. 1 and Fig. 2).

It may, therefore, be assumed that the homologous lower part of the skull in relation to the whole is more capacious in Homo neandertalensis than in Homo sapiens, not or not chiefly on account of the upper part of the brain being less large in itself, but in consequence of similar external causes as make the lower part more spacious in the platycephalic Siamang than in his smaller relative. Also in the skull of the Neandertal Man the flattening above must have caused part of the brain to be displaced downward. In fact for the physiological function of the brain the place which it occupies in the skull is very indifferent; it is not so with the bone- and muscle substance at the skull, whose function is directly dependent on the place. This leads to the insight that the peculiar shape of the skull of the Neandertal type was not determined, at least not chiefly, by the comparatively small size and low stage of development of the encephalon, but by external mechanic factors, chiefly in connection with the position and poise of the skull on the spinal column — which I have referred in my communication of September 25, 1920 on the "Protoaustralian Fossil Man of Wadjak, Java" — just as the platycephaly in the Siamang, in contrast to the other Hylobatides, can only be explained by its comparatively large jaws.

The capacity of the skull of 1288 $cm.^3$ to be calculated for the man of the Neander-valley from the calvaria, in accordance with the proportion in the recent human type, must then be much too small. According to the ratio which exists in Apes between the calvaria and the total capacity of the skull this fossil man would have possessed a brain capacity of 1472 cm^3. BOULE [1]) calculated

[1]) M. BOULE, L'Homme fossile de La Chapelle-aux-Saints, p. 189.

1408 cm.^{3} Broca (i.e. 1320 cm.^{3} real capacity) from the comparison of the greatest length and breadth and a corresponding height of the endocranial plaster casts of the Neandertal calvaria and the

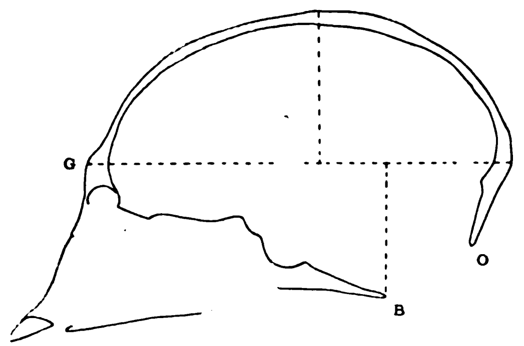


Fig. 1. Median cross section of a skull of Hylobates agilis. $^2/_3$ natural size.

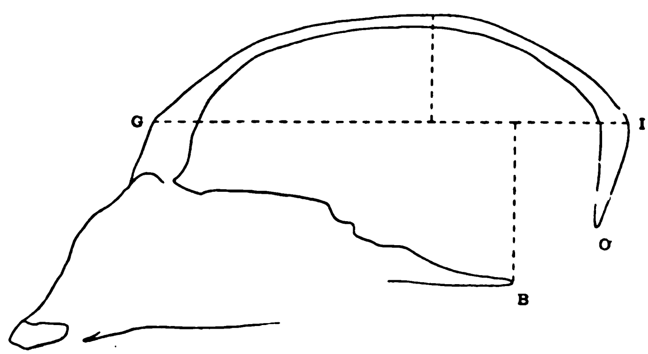


Fig. 2. Median cross-section of the skull of Hylobates (Symphalangus) syndactylus. $^2/_3$ nat. size.

La Chapelle skull in relation to its capacity. In this the relatively more considerable breadth of the Neandertal calvaria in the frontal region was not taken into account. Perhaps some measure did not exactly correspond. Assuming similarity of form, the capacity as computed from the relation of the calvarial heights of these skulls, is 1450 cm^3. On the strength of these and of the foregoing considerations it seems to me that an estimation of the capacity of the entire Neandertal-skull at 1400 cm^3. at least cannot be far from the truth. That of Spy I can have been but little smaller, and Spy II must, in the same ratio, have reached a true capacity of 1600 cm^3. By the method of the "cubic index" J. Fraipont had calculated for Spy I 1562 cm^3. Broca-capacity (which corresponds to 1470 cm^3. real volume), for Spy II 1723 cm^3. Broca (i.e. 1620 cm^3. real volume),

unexpectedly high results, so much so that he was perplexed ("effrayé") by them, and deterred from publishing these values; he communicated then, however, to BOULE in a letter [1]). At present these calculated capacities do not seem improbable to us at all; for the more highly vaulted skull of Spy II exceeds the La Chapelle skull only by from 70 to 90 cm³.

SOLLAS [2]) calculated the capacity of the Gibraltar skull at about 1260 cm³. from the right half, which had been partly reconstructed, and of which he had measured the capacity with millet seed. Comparison of the endocranial plaster cast (of this right half of the skull) with that of the La Chapelle skull gave BOULE [3]) 1296 cm³. Broca (= 1214 cm³. real capacity), and by direct determination of the capacity of such a cast KEITH [4]) found about 1200 cm³. cranial capacity. No great value can be attached to these estimates from the very incomplete fossil. More trustworthy is the result obtained from the skull of La Quina, whose capacity BOULE [5]) put 1367 cm³. Broca (= 1282 cm³. real capacity) from the less incomplete endocranial plaster cast.

The two last-mentioned skulls are generally considered to be female, the other skulls of the Neandertaltype are probably all male. As the mean real capacity of the Europeans can be put about 1450 cm³. for men, and 1300 cm³. for women, the absolute capacity of the Neandertal Man appears to have been no less than that of Europeans.

But the relative capacity must certainly have been greater then, for Homo neandertalensis was a small type of men. After a full discussion of the length dimensions of the skeleton BOULE [6]) arrives at the estimate of 154 or 155 cm. for the body length of the fossil man of La Chapelle-aux-Saints in life, which was probably also the mean male length of the species, hence as much as or a few centimeters less than those of the smallest present human races, except the "pygmies", and 14 or 15 cm. less than the mean of the male Europeans. It is true that the Neandertal Man through his compact stature, must have been comparatively heavy, but it is not probable that this made him reach the mean body weight of the so

[1]) M. BOULE, loc. cit., p. 187.

[2]) W. J. SOLLAS, On the Cranial Characters of the Neandertal Race. Phil. Transactions Roy. Society. Series B. Vol. 199, p. 329. London 1908.

[3]) M. BOULE. loc. cit., p. 189.

[4]) A. KEITH, Antiquity of Man. (London 1920). p. 124.

[5]) M. BOULE, loc. cit., p. 189.

[6]) Loc. cit., p. 115—118.

much taller European; we may, therefore, assume that this **b r a i n q u a n t i t y, a l s o c a l c u l a t e d i n r e l a t i o n t o t h e b o d y-w e i g h t, e x c e e d e d t h a t o f t h e p r e s e n t E u r o p e a n.**

This high cephalisation of Homo neandertalensis can, in my opinion, be explained by the fact that he was in possession of particularly powerful muscles, which may be inferred from the robust character of his bones and the comparative shortness of his limbs, especially of his legs [1]). In this respect the Neandertal Man resembles the Japanese, the Eskimos, probably also the Chinese and Javanese, in general the Mongolian race [2]).

MANOUVRIER [3]) was the first to point out that the cranial capacity of men with thin limbs (as the Hindus and the Australians) is comparatively small, of men with "carrure" which are "trapus" and "robustes" (mountaineers, Eskimos) comparatively large. About the "carrure" he says: "Ce facteur me paraît avoir une importance considérable d'après mes propres observations. Il est certainement plus important que la longueur du corps, et cela s'expliquerait par le fait que l'énergie motrice des muscles est bien plus en rapport avec leur section transversale qu'avec leur longueur". (p. 686). He sees a connection between the great cranial capacity of the Eskimos and the fact that they are "trapus et actifs". (p. 219). I lay particular stress on the last word.

Later MATIEGKA [4]) has demonstrated from Prague section reports that there exist relations between the brain weight and muscularity and also the more or less powerful build of the bones.

These relations of the brain weight and its dependence on the build of the body, especially on its breadth, can be much better studied now than formerly, by comparison of the human races.

In the first place it may now be considered as certain that among the present human races it is not the Europeans, but the Mongoloids that possess the greatest relative quantity of brain. The best data

---

[1]) M. BOULE, loc. cil., p. 125—170 and p. 120.

[2]) After what precedes it will be self-evident that it is not my intention, to have recourse here to the well-worn path of relationship.

[3]) L. MANOUVRIER, Sur l'interprétation de la quantité dans l'encéphale. Mémoires de la Société d'Anthropologie de Paris. 2me série. Tome 3, p. 217—219. 1885, — and under "Cerveau" in Dictionnaire de Physiologie par CHARLES RICHET, p. 686—687. Paris 1898.

[4]) H. MATIEGKA, Ueber das Hirngewicht, die Schädelkapacität und die Kopfform, sowie deren Beziehungen zur psychischen Tätigkeit des Menschen. Sitzungsberichte der Kön. böhmischen Gesellschafft der Wissenschaften. Mathem.-Naturw. Classe. Jahrgang 1902. XX, p. 13—14 and 44. Prague 1903.

about the latter refer to the Japanese. They were supplied by TAGUCHI's [1]) researches referring to no less than 421 male and 176 female Japanese, of whom most had died in the hospitals. The mean brain weight of 374 adult men was 1367 grams, of 150 adult women 1214 grams. These are quantities that pretty closely agree with the means of the Europeans obtained in the same way. But on an average the body weight of the Japanese men is 8 kg., their length 10 cm. less, and the Japanese women are on an average 7 kg. lighter and 10 cm. shorter [2]).

Accordingly these East-Asiatics have more brain-weight than the Europeans, both per cm. body length and in proportion to the body weight. Still greater is the difference with regard to the muscle length, with which, strictly speaking, the brain quantity can be better compared than with the body length. The Japanese are built more compactly; their arms, and especially their legs, are shorter in proportion to the trunk and exceedingly muscular; to the great strength of the muscles corresponds their considerable cross-section, and also the robust build of the long bones is in connection with this. In proportion to the muscle length the brain-mass is, therefore, still considerably greater than in proportion to the body length; the brain-mass is evidently proportional to the cross-section of the muscles. KAGUCHI showed that, later than in Europeans, this great brain quantity of the Japanese is not acquired until after childhood and first youth, and according to BAELZ the Japanese are later full-grown in body-length and weight. Hence the large relative brain quantity and the greater muscular power of the Japanese is certainly not owing to a greater n u m b e r of the neurones and of the muscle fibers, but to larger separate c r o s s - s e c t i o n s of these, larger separate v o l u m e of those.

Still somewhat shorter than the Japanese are the Eskimos, and also still broader and more compactly built, still shorter of limbs, especially of legs, and more muscular. Judging by the few determinations of their brain weight, which we owe to the determinations of CHUDZINSKI, HRDLICKA, SPITZKA [3]), this mean is certainly no less

---

1) E. A. SPITZKA, The Brain-Weight of the Japanese. Science. New Series, Vol. 18, p. 371—373. Philadelphia 1903.

2) E. BAELZ, Die körperlichen Eigenschaften der Japaner. Mittheilungen der deutschen Gesellschaft für Natur- und Völkerkunde Ostasiens. Erster Teil. Band III (1880—1884), p. 330—359. Berlin und Yokohama. — Zweiter Teil. Band IV (1884—1888), p. 35—103). Higher weights and greater body lengths do not refer to means for the whole people, but for definite classes or selected individuals.

3) E. A. SPITZKA in American Journal of Anatomy. Baltimore. Vol. II (1902—1903), p. 26—31. Three male brains of an average weight of 1457 grams

high, probably higher than in the Japanese. From the many available determinations of the cranial capacity, which, however, mostly refer indifferently to male and female skulls, the same statement may be deduced.

The brain-weights of the Chinese which are out of proportion high to the length of the body, have been very striking in each of the few determinations that could be made, and it was ascertained many times that the mean cranial capacity is great. [1])

KOHLBRUGGE [2]) showed that also the Javanese, whose large cranial capacity was already known, belong to the peoples with relatively high brain weight. In this respect, too, they may be placed side by side with the other mongoloids mentioned.

In the Australians, Negroes, Hindus on the other hand, a slender figure, with long and thin legs and arms, is accompanied with a brain weight which is low in proportion to the body length, and small cranial capacity.

Comparison of the Neandertal Man with these present human races renders it exceedingly probable, that also in him the great brain-quantity was in relation with the thickset, strongly built body and the short limbs, hence with great muscular force. We are particularly justified in this assumption, because such a relation is frequently met with in Mammals.

Thus the Bears are distinguished from the other land-Carnivora by their heavy, massive shape, and thick limbs, which are short in proportion to the body, and with which they can exert a tremendous force. The long bones of the limbs in the Bears are thicker with respect to their length, in part somewhat prismatically shaped, and the surfaces of attachment of the muscles still more developed in cristae and apophyses, — in a similar way as in the Neandertal Man.

(1398—1503), two female brains of an average weight of 1242 grams (1227—1256). Also body lengths.

[1]) CROCHLEY—CLAPHAM: eleven male brains of an average weight of 1430 grams (1310—1587), cited in P. TOPINARD's, Eléments d'Anthropologie générale, p. 571. (1885). — KURZ in Zeitschrift für Morphologie und Anthropologie. Bd. 16. (1913), p. 284: of a man of a body weight of 160 cm., 1454 grams; of a woman, 155 cm. long, 1200 grams.

[2]) J. H. F. KOHLBRUGGE, Die Gehirnfurchen der Javanen. Verhandelingen der Kon. Akademie van Wetenschappen te Amsterdam. 2de Sectie, Deel 12, N°. 4 (1906), p. 13. The mean weight of 16 adult male brains (of the 19 determinations I exclude one of exceptionally high, and one of exceptionally low weight, and one of a child of seven years old) was 1301 grams (the extremes were 1101 and 1458). This is a high brain weight with 50.27 kg. (living) body weight, which is probably not reached by European men of equal living body weight. (Compare: EUG. DUBOIS, Ueber die Abhängigkeit des Hirngewichtes von der Körpergrösse beim Menschen. Archiv für Anthropologie. Band 25, p. 432. Braunschweig 1898).

According to the data about brain weight and body weight of Bears, supplied by MAX WEBER, AL. HRDLICKA, W. T. BLANFORD and others, and capacity determinations of my own, the cephalisation of Ursus arctos, horribilis, tibetanus, and maritimus may be indicated about by 0.5, i.e. one and a half times as high as of Felides (0.33), and Canides (0.37), which means that in this ratio a Bear species in the adult state with equal body weight, exceeds a Cat- or a Dog species.

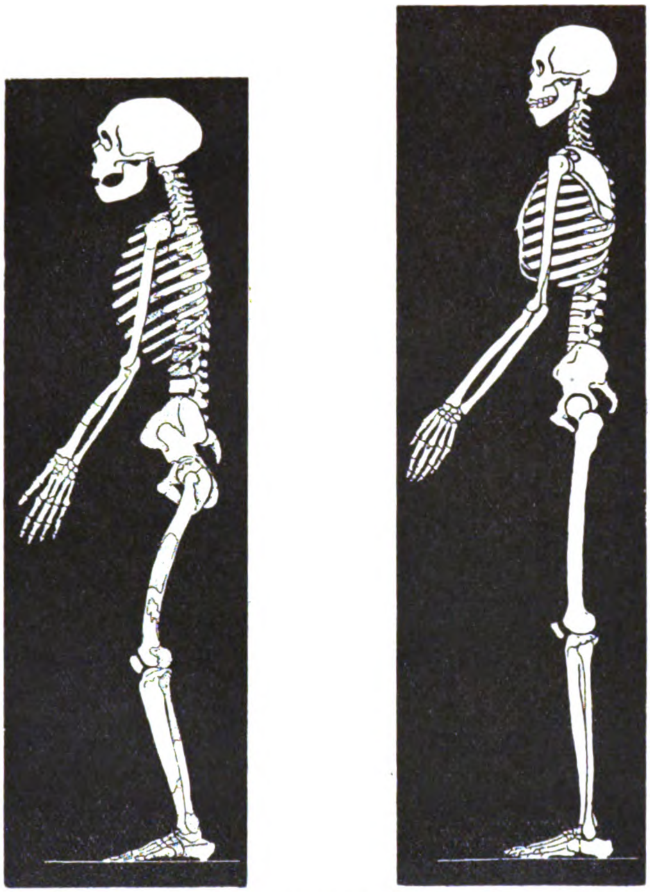


Fig. 3. Skeleton of the man of La Chapelle-aux-Saints by the side of that of an Australian. (From BOULE). [1])

As regards their brain quantity the said Ursides are on a line with the Monkey genus Semnopithecus, but Ursus malayanus is even equal with the Anthropoid Apes. I see in this a very striking proof of the truth of the conception that the quantity of the brain is determined by the functional mechanism.

[1]) L'Homme fossile de la Chapelle-aux-Saints, Fig. 99 (p. 232), Fig. 100 (p. 233). Paris 1913.

The Malay Bear or Bèruwang is of comparatively small build, and has still disproportionally shorter, at the same time still more muscular limbs than the other Bears. He uses his out of proportion enormous claws as dexterously as powerfully. Considering the size of his body he is by far the strongest of his race; he is also the best climber and the swiftest runner. As regards motor mechanism he may be called the most perfect of the Bears.

His very marked macrocephaly results from the very considerable size of the encephalon, which also manifests itself in the brain-weight and cranial capacity.

WEBER[1]) determined the body weight of a male Bĕruwang of 114 cm. body length (from nose to rump), which had died in the Amsterdam Zoological gardens, and was probably much too light, at 20 kilograms, the brain weight at 325 grams. HRDLICKA[2]) found for the brain weight of a female specimen from the Washington Zoological Park, weighing 45.02 kilograms, 385.5 grams. According to records by BLANFORD[3]) the weight of a female bear of Borneo was 60 lbs. or 27.215 kilograms, with 36 inches or 91.5 cm. body length (from nose to rump). The male body length is averagely 4 feet or 122 cm., and probably never becomes greater than $4\frac{1}{2}$ feet or 137 cm.

I have been able to measure the capacity of five adult skulls from the Museum of Natural History at Leiden, placed kindly at my disposal for this purpose by the director Prof. E. D. VAN OORT:

N°. 1. ("b. Sumatra-Reinwardt"). Male skull. Basal (basion-inion) length (Flower) 214 mm. Greatest breadth, across the zygomatic arches, 190 mm. Middle-aged from the degree of wear of the teeth. [BLANFORD measured at a "very old and large skull" 8.5 inches basal length or 216 mm., and 8.3 inches or 211 mm. breadth]. Capacity (measured with mustard seed)[4]) 373 cm^3.

N°. 2. ("f. Borneo). Male skull. Middle age. Basal length 214 mm. Greatest breadth 188 mm. Capacity 355 cm^3.

N°. 3. ("c. Borneo. S. Müller 1827"). Female skull. Middle age, Basal length ± 187 mm. Greatest breadth, across the zygomata, 163 mm. Capacity 325 cm^3.

[1]) MAX WEBER, Vorstudien über das Hirngewicht der Säugethiere, (Festschrift für CARL GEGENBAUR), p. 113. Leipzig 1896.

[2]) AL. HRDLICKA, Brain Weight in Vertebrates. Smithsonian Miscellaneous Collections. Vol. 48, p. 94. Washingtón 1905.

[3]) W. T. BLANFORD. Mammalia. The Fauna of British India including Ceylon and Burma, p. 199. London 1891.

[4]) With shot, by Broca's method, I get 380 cm^3. Such a ratio applies also to the following measurements, which have all been made with mustard seed.

N°. 4. ("a. Borneo. Reinwardt"). Female skeleton. Basal length of skull 169 mm. Breadth across the zygomata 150 mm. Length of the skeleton from alveolar point to caudal basis 89 cm. (measured along the back). Somewhat below middle age, the cranial sutures only commencing to obliterate. Capacity 278 cm^3.

N°. 5. ("907"). Balik Papan, Borneo. Female skull. With skeleton, allowing to measure the length of the skeleton (along the back) from alvesolar point to caudal basis, 109 cm. Full-grown and middle-aged, according to skeleton and skull. Basal length 199 mm. Breadth across the zygomata 167 mm. Capacity 341 cm^3.[1]).

I determined the cranial capacity of a young female bĕruwang, whose teeth, with the exception of the canini, had all erupted, and which weighed 12 kilograms according to my estimation, when kept in captivity in its native country at Bua in Sumatra for some time, at 305 cm^3.

I find 325 cm^3 for the capacity of a large male tiger, killed there, the same value as for that of the female bear N°. 3 from Borneo, with probably four times greater body weight. The skull of a female orang utan of this island has a capacity of 380 cm^3. The animal probably weighed as much as a large Bornean bear with the same cranial capacity.

With these data about brain weights and body weights, and longitudinal dimensions of body and skeleton, determined directly, and with the brain weights calculated by means of CORNEVIN's comparisons[2]) I find that the cephalisation-coefficient of Ursus malayanus may be put at least at 0.75, equal with that of the Anthropoid Apes.

The relation between the muscular power (which is determined by the cross-section of the muscles) and the rapidity of motion which depends on it, and the quantity of brain manifests itself in a very striking way in American Monkeys. The Howlers (Mycetes) have much less brains in proportion to the size of their bodies than

---

[1]) No. 4 and 5 were not mentioned in the Dutch version of this communication. Accordingly the coefficient of cephalisation of Ursus malayanus found here is somewhat different from that in the *Verslagen.*

[2]) CH. CORNEVIN, Examen comparé de la capacité cranienne dans les diverses races des espèces domestiques. Journal de Médecine vétérinaire et de Zootechnie publié à l'École de Lyon, 3me Série. Tome 14, p.8-31 and Étude sur le poids de l'encéphale dans les diverses races des espèces domestiques. p. 248—262. — From his recorded values I calculate 88 % brain weight for capacities of a mean of 650, and 93 % brain weight for an average capacity of 100 and less, say 91 % for the capacities mentioned above.

the species Cebus and Ateles living in the same country. FLOWER [1]) determined the brain weight of an exceedingly emaciated old male Mycetes seniculus, which had died in the London Zoological Gardens, at 48 grams (740 grains), the body weight at 3444.5 grams (9 lbs. $9^1/_2$ ozs. avoirdupois). SPITZKA [2]) found the brain weight of a female Mycetes ursinus (which species is somewhat larger) to be 54 grams, LECHE [3]) found 63 cm^3. for the cranial capacity of Mycetes ursinus in an adult male, and 54 cm^3. in an adult female specimen. The body weight was only known of atrophical zoological garden individuals. From Surinam I received the skull and other parts of the skeleton of a male Mycetes seniculus, killed in the natural state, which weighed 6750 grams, though judging from the condition of the skeleton, it was only almost full-grown. The cranial capacity is 54 cm^3., from which a brain weight of 50 grams can be calculated and a cephalisation-coefficient 0.37, about the same as that of Macacus cynomolgus. For the entirely full-grown state a still somewhat lower value would certainly have been found [4]).

The Howling Monkeys, now, are described as being, in their free state, exceedingly indolent animals, which remain very much at the place where they are. All their movements are slow, almost creeping; they never play with each other, climb deliberately, and never jump far — in sharp contrast to the lively, rapid movements, the leaps and swings of the agile rovers of the genera of Cebus and Ateles. The cephalisation coefficient of these is more than three times as great as that of Mycetes.

Here, therefore, the same contrast as between swift and slow species of Reptiles and Amphibians. Thus Hyla arborea has double the cephalisation of Rana fusca. And as it is demonstrated there (e.g. between Phrynosoma and Sceloporus) it may be assumed here that the nerve fibers (and the muscle fibers) are thicker, the neurones more voluminous in the more vigorous and quicker species [5]).

[1]) W. H. FLOWER, On the Brain of the Red Howling Monkey (Mycetes seniculus Linn.) Proceed. Zool. Soc. London. 1864, p. 335—338.

[2]) E. A. SPITZKA, Brain-Weights of Animals with Special Reference to the Weight of the Brain in the Macaque Monkey. Journal of Comparative Neurology. Vol. 13, p. 13, Philadelphia 1903.

[3]) W. LECHE, Ueber Beziehungen zwischen Gehirn und Schädel bei den Affen. Zoologische Jahrbücher. (SPENGEL). Supplement XV, Band 2, p. 17. Jena 1912.

[4]) What is urgently required is more data of body weights in the free state. Especially nimble animals get much lighter in captivity; the brain weights change less, and can also be calculated pretty accurately from the cranial capacity.

[5]) EUG. DUBOIS, "The Significance of the Size of the Neurone and its Parts." These Proceedings, Vol. XXI, No. 5, p. 711.

A contrast of the same nature, but not so great, exists between the Orang utan and the Chimpanzee. The slow, clumsy, deliberate movements, without jumps, of the Malay anthropoid are indeed sharply distinguished from the mode of moving of his African relation, the Chimpanzee, which is an excellent climber, swings over large distances from one branch to another, and jumps with wonderful agility. But though the body weight of the Orang utan is certainly a third greater than that of the Chimpanzee, the brain weight of the two species is the same in the females, in the males that of the Orang utan is only little more.

SELENKA[1]) determined the mean capacity in the sexes of the Orang utan at 455 and 390 cm^3., and of the Chimpanzee at 420 and 390 cm^3. To him in the Anthropoids "Muskelmasse und Hirngrösse" seem "daher in direkter Beziehung zu stehen", because the "rein geistigen Fähigkeiten wohl als nahezu gleich angenommen werden dürfen." It also strikes him that in Orang utan "Skelet und Muskulatur des Männchens" are "ausserordentlich viel stärker als die des Weibchens." It is now remarkable that according to FICK's[2]) research the total muscle weight in reference to the body weight is much less, the fat percentage on the other hand, greater in Orang utan than in Man. We meet here with the same difference in the composition of the body weight as between woman and man, and here too we see this accompanied on one side by a brain weight low in comparison with the body weight; for we may assume that the Chimpanzee, like most other Apes, is more muscular than the Orang utan.[3])

Among the American Monkeys, Saimiri (Chrysothrix) is further much quicker and nimbler in its movements than Leontocebus (Midas) and Callithrix (Hapale); accordingly its cephalisation coefficient is considerably higher.

In conclusion attention may still be drawn in this connection to the high cephalisation of the Seals and to the considerably higher cephalisation of the Toothed Whales than that of the Whalebone Whales. For Balaenoptera musculus I calculated the coefficient 0.384[4]).

---

[1]) EMIL SELENKA, Menschenaffen. Zweite Lieferung, p. 99—100. Wiesbaden 1899.

[2]) R. FICK, Vergleichend anatomische Studien an einem erwachsenen Orang-Utang. Archiv für Anatomie und Entwickelungsgeschichte. (W. HIS). Leipzig. Jahrgang 1895, p. 68—69 and p. 73. The examined specimen was a male Orang utan.

[3]) EUG. DUBOIS, Comparison of the Brain Weight in Function of the Body-Weight, between the Two Sexes. These Proc. Vol. XXI, No. 6 and 7, p. 850 seq. 1918. — H. WELCKER (loc. cit. p. 41) found the relative muscle weight of a male "Inuus cynomolgus" greater than the mean of the male in Man.

[4]) The Significance of the Size of the Neurone and its Parts. These Proc. Vol. XXI, No. 5, p. 724.

Through accurate determination WEBER[1]) found 1886 grams for the brain weight of a fullgrown female specimen of Tursiops tursio, a toothed whale of the Delphinidae family, the body weight being 432 times as much. From this the cephalisation-coefficient 0,981 can be calculated. The Odontocetes, among them especially the Delphinidae, swim with extraordinary dexterity and swiftness, faster than the fastest steamer, they even swim round a steamer at full speed; the Mysticetes, on the contrary, cannot reach the speed of an ordinary steamer. In connection with this the dorsal muscles of the former are much more powerful, which is to be seen by the great thickness of the back part of the body.

Thus the muscle apparatus of Homo neandertalensis was also stronger than that of Homo sapiens, and among the races of modern Man the Mongoloids possess the most powerful muscle apparatus. In agreement with this Homo neandertalensis and the Mongoloids possess also the relatively largest encephalon.

[1]) loc. cit, p. 113. The body weight with the brain ratio 1 : 432 is 815 kg. The value 278 is given, evidently a misprint.

Botany. — "*On the influence of circumstances of culture on the habitus and partial sterility of the pollengrains of Hyacinthus orientalis*". By Dr. W. E. DE MOL. (Communicated by Prof. A. H. BLAAUW.)

(Communicated at the meeting of February 26, 1921).

I. *Introduction.*

When, in the spring of 1919, it had become evident to me that the nuclei of the single-flowered, rose-coloured hyacinth-variety *Nimrod* possessed 19 chromosomes [1]), I thought it advisable to examine the fertility of the pollen and to compare it with that of the Dutch varieties with 24 chromosomes in the somatic cells, which number I at that time still considered as diploid. I chose for that purpose the closed anthers, taken from growing *Nimrod*-plants that belonged to the same grower as those of which I had fixed the root-tops in behalf of my chromosome-examination. To my surprise the pollen-grains in these anthers differed greatly from the aspect which hyacinth-pollen had always shown to me. I did not find *one* normal fertile grain. The sterile pollengrains were elliptic, round or triangular in shape and had various dimensions. The wartlike protuberances on the exine, which in normal cases cause the pollengrains, when plunged into a drop of some liquid, to stick together to some extent, were undeveloped, so that the pollen dispersed very easily. Apart from these sterile pollengrains, there appeared in the preparations many that were much larger and globe-shaped, and were full of large starchgrains. If the pollen was put into a diluted solution of jodine in jodide of potassium, one saw at once the abnormal pollengrains lying like intensely blue-black globes among the yellow, shrivelled exines of the sterile pollengrains. In a drop of water the exine usually burst rather soon and the starchgrains

[1]) Over het optreden van heteroploide Hollandsche varieteiten van *Hyacinthus orientalis L.* en de chromosomengarnituur van deze plantensoort.

Verslagen van de Koninklijke Akademie van Wetenschappen te Amsterdam, Wis- en Natuurkundige Afdeeling, Deel XXIX, p. 513.

Nieuwe banen voor het winnen van waardevolle varieteiten van bolgewassen, p. 19.

then floated loose in the preparation causing a bluish spine when the light fell through it. There appeared to be starchgrains among them, reaching the size of a small sterile pollengrain.

But moreover, numerous elliptic pollengrains occurred that had germinated, not with a normal pollen-tube however, but with a tube that had rather the shape of a bubble. This bubble was sometimes smaller than the pollengrain itself, but often it was much larger. It contained 2, 3, 4, 5 or more globe-shaped nuclei.

The closed anthers were slightly shrivelled, and the violet anthocyanin in their walls was slightly discoloured.

This observation stimulated me to try and find out whether in literature any mention was made of phenomena like those observed by me. I then, indeed, found that Němec (1898) had already observed the same thing, also in the hyacinth. He however, had discovered it in the partly petaloid anthers of a variety with double flowers. The anthers were taken by him from young closed flower-buds, fixed, bedded in paraffine, coloured, and made into microtome-series. He also found the small sterile pollengrains without reservesubstance, and the large, globe-shaped ones full of starchgrains. And here, too, many pollengrains had developed in the closed anthers pollen-tubes that often looked like large bubbles. He sometimes saw 8 nuclei in them.

Němec supposed these deviations to be the consequence of the anthers being petaloid from the fact that I found them in the anthers of a single-flowered variety, one could infer that they were due to other causes than the flowers being double. I therefore thought the phenomenon was to be ascribed either to the peculiar bastardlike nature, of the variety *Nimrod* — as the latter is supposed to be a product of cross-fertilization between a French and a Dutch variety! — or to the occurrence of the deviating number of chromosomes in the somatic cells.

I now resolved to examine the pollen of a large number of varieties Carefully I considered in what way this examination was to be performed, in order to derive the most favourable results from it. In the month of April and May 1919 I managed to perform it at Lisse, the centre of the Dutch hyacinth-cultures, in the following way. I chose varieties with the most diverging shapes, dimensions and flowering-times, double-flowered as well as single-flowered ones. Moreover I collected several times many racemes of one and the same variety, cultivated by different growers under greatly diverging circumstances. In this way I was able to come to palpable results. One of these results, in my eys the most important, may be mentioned here.

II. *On the occurrence of pollengrains with* 3 *and more nuclei.*

SCHÜRHOFF (1919) has drawn our attention to the fact that among *Monocotyledons* as well as *Dicotyledons* pollengrains with 3 nuclei — 1 vegatative and 2 generative — occurs before the germination. He wonders whether this early division of the male sexual nucleus has a biological or a systematical importance and concludes on p. 147, after giving a general view of the orders of plants where pollengrains with 3 nuclei have been noticed: "Es ergibt sich also aus dieser Aufstellung, dass dem Vorkommen von dreikernigen Pollenkörnern keine besondere systematische Bedeutung zukommt. Eine derartige Bedeutung liesze sich zur Not für die *Monokotylen* konstruieren, da bei den ersten Ordnungen das Vorkommen dreikerniger Pollenkörner die Regel bildet, während sie bei den letzten Ordnungen der *Monokotylen* fehlen". Of the *Monocotyledons*, according to him, the *Helobiae*, the *Glumiflorae* and part of the *Spadiciflorae* are characterized by pollengrains with 3 nuclei. The orders of the *Enantioblastae*, the *Liliiflorae*, the *Scitamineae* and the *Gynandrae*, *with the exception of the Juncaceae and other isolated cases*, do not possess them.

If we examine more closely some cases of the *Monocotyledons*, we notice that ELFVING saw, as early as 1878, 3 kernels in the pollengrains of *Andropogon campestris*, and that STRASBURGER (1884) saw that in many cases the generative nucleus was divided in the pollengrain. GOLINSKY (1893) found 2 sperm-nuclei in the pollengrains of *Triticum*, SCHAFFNER (1897) in *Sagittaria variabilis*. "The division of the generative nucleus before pollination", he says on p. 254, "seems to be quite common in monocotyledons, and it is probable that this condition will be found to be the rule rather than the exception in this group". According to the researches of CHAMBERLAIN (1897) the generative nucleus of *Lilium aurantiacum* and of *Lilium tigrinum* was divided in the pollengrain, "a condition not uncommon, in monocotyledons", he says. In 2 cases he also observed in *Lilium aurantiacum*, that the divisions went still further, so that 3 generative nuclei were present. In *Lilium Philadelphicum* the early division of the generative nucleus occured seldom. In *Lilium martagon* it is perhaps out of the question. GUIGNARD (1891) at least notices that the generative nucleus is here only divided in the pollentube. The vegetative nucleus is never divided. In 1899 this naturalist saw 3 nuclei in the pollengrains of *Najas major*. WIEGAND (1899) observed 2 generative nuclei in *Potamageton foliosus Raf.* and gives an enumeration of the cases at that time observed in *Mono-*

cotyledons. MÜRBECK (1902) observed 3 nuclei, 1 vegetative and 2 generative in *Ruppia rostellata Koch*. SCHÜRHOFF (1919) describes explicitly the mechanism of division in the generative nucleus of *Sagittaria sagittifolia*.

In *germinated pollengrains* of *Monocotyledons* more than 2 generative nuclei have sometimes been observed. Thus STRASBURGER (1884) saw 4 of them in the pollen-tubes of *Ornithogalum* and of *Scilla*. THIES (1901) caused ripe pollengrains of *Scilla sibrica* to germinate in a 5 °/₀ solution of cane-sugar and says: "ausnahmsweise wurden in einem gekeimten Pollenkorn 5 Kerne beobachtet".

In hyacinth-varieties, too, one may observe under particular circumstances, that they contain pollengrains with 2 generative nuclei. I found them i.a. in the single-flowered white variety *La Neige*. The vegetative nucleus was large and round. The exine of the pollengrains was very transparent here, the wartlike protuberences were almost entirely absent. This made it possible to observe the nuclei closely, without having to colour them green first in a drop of methylgreen acid of vinegar. Not nearly all pollengrains possessed 2 generative nuclei. That I ascribe the early division of the generative nucleus to external circumstances, may appear later on.

III. *Further particulars concerning the occurrence of pollengrains with several nuclei in Dutch hyacinth-varieties.*

By way of introduction I mentioned that I found pollengrains with several nuclei in the variety *Nimrod*. When composing my extensive tables, I indicated not only the percentage of sterility, but also the origin of the racemes, the latter by indicating each particular category with a capital. Moreover I nearly always gave, with each numeration a short description of the habitus of the pollengrains. In this contribution I think it sufficient to take out of the tables in question those varieties in which I found *in the closed anthers pollengrains, germinated with abnormal pollen-tubes, in which lay several nuclei*. At the same time I mention the other numerations, bearing on the same varieties, but in which no pollengrains with abnormal tubes were found. Because NĚMEC (1898), as I said before, observed before me in the pollen of double-flowered hyacinths the same phenomenon, shall henceforth indicate it, in his honour, by the name: "NĚMEC's phenomenon". In so far as I have fixed the number of chromosomes of the varieties named below, I shall mention this.

A. Single-flowered varieties.

Charles Dickens, single-flowered red.

1. The upper flowers of the raceme are much smaller than the others and have green points at the lobes of the corolla. The pollen of these flowers show NĚMEC's phenomenon. The pollengrains in the lower flowers of the raceme are normally formed, 21 % are sterile. The pollen of the upper flowers disperse at once in a drop of water; of the lower flowers it sticks together.

2. All flowers contain normal pollengrains, 27 % of which are sterile. The pollen sticks together.

Général Pélissier, single-flowered red, 16 chromosomes.

1. The upper flowers of the raceme, are small and green coloured, show NĚMEC's phenomenon. The other flowers contain normal pollen, 14 % of which is sterile.

2. Ibid. The stickiness of the pollen is as in *Charles Dickens*.

3. The sterility amounts to 6 %.

Lady Derby, single-flowered red, 24 chromosomes.

1. The pollen in all flowers of different racemes gives the same impression. The flowerbuds are still quite green and closed. The pollengrains do not stick together. Only a few sterile grains are present. They at once swell very strongly. In most of the pollengrains many starchgrains are present, which are often very large. There occur pollengrains showing NĚMEC's phenomenon. The various nuclei are clearly visible.

2. The pollengrains only swell after a long time; 4 % are sterile.

3. The sterility is 3½ %.

4. The racemes originate from water-cultures: the sterility is 6 %.

5. The pollen is sterile for 4 %.

Moreno, single-flowered red.

1. The anthers are shrivelled and discoloured and contain very few pollengrains. These do not swell in water. They have a turbid content. Further there are large globe-shaped pollengrains present which soon burst. The preparation is then full of loose starchgrains. Several normally formed pollengrains show NĚMEC's phenomenon. In some normal non-swelling pollengrains 2 globe-shaped nuclei of equal size are clearly visible. I do not find *one* normal fertile grain. Some starchgrains are as large as a small pollengrain.

2. The colour of the flower is orange-red. The dark streak that runs over the middle of the lobes of the corolla is clearly outlined. The sterility is 24 %.

3. The colour of the flower is rose-red. The dark streak is dimly

outlined. The sterility amounts to 40°/₀. (The bulbs of these flowers and those of the former had been subjected to high temperatures).

4. The pollengrains stick together. They swell very soon; 11°/₀ are sterile.

5. Like 4, but 15°/₀ are sterile.

6. The pollengrains are taken from the flowers of bulbs that have not been planted. The flowers lay shrivelled up in the bulb. The pollengrains are slightly smaller than usual for the rest they look quite normal; 13°/₀ are sterile.

7. The pollengrains are taken from a budvariety-in-colour of *Moreno*. The colour of the flower has become dark red from rose-red; 59°/₀ of the pollengrains were sterile in the only raceme I had at my disposal.

8. The pollen is not sticky, the sterile pollengrains are ellipsoidical, triangular or globe-shaped; the sterility is 40°/₀.

9. The sterility is 8°/₀.

10. The sterility is 35°/₀.

Nimrod, single-flowered red, 19 chromosomes.

1. The anthers are shrivelled and give the same impression as those of the variety *Moreno* of the same grower (c.f. 1). The pollen does not stick together. I do not find *one* fertile grain. The sterile ones are elleptic, round or triangular.

2. The flowers are taken from a lot that blooms very early. The anthers are badly developed. The pollen shows beautifully NĚMEC's phenomenon.

3. The flowers are taken from a lot that blooms late, coming from the same grower as those under 2. They are not yet open. The pollen shows, as in 2, NĚMEC's phenomenon.

4. All pollengrains, fertile as well as sterile, are normally formed; 26°/₀ are sterile.

5. The flowers are taken from a lot that blooms early. The bulbs have been planted medio October, and on April 4th 1919, the date when I cut the racemes, they have long roots. The anthers are normally formed, not shrivelled. The pollengrains at once swell in water, but they are sticky. Among the fertily pollengrains these are some that contain a more or less wide zone of water round a rounded mass of protoplasm. Some pollengrains have developed a short, wide pollen-tube; 36°/₀ are sterile.

6. The flowers are taken from a late-flowering lot of the same grower. The bulbs were planted on November 7th, and on April 5th the gemma rises only 2 cm. above the ground. The roots of the bulbs are still very short. The sterile pollengrains are all ellipsoidical,

not triangular. Some fertile pollengrains of a normal size contain starchgrains. I find some large globe-shaped pollengrains, full of starchgrains; 29 % are sterile. I perform many more enumerations in pollen of flowers from other racemes, out of this same lot. I always find the same phenomenon and a sterility, from 29 to 30 %.

7. As in 6.

8. The pollen is not sticky. The nuclei are clearly visible. The sterile pollengrains are all ellipsoidical. In the preparations float some starchgrains, 66 % are sterile.

City of Haarlem, single-flowered yellow, 23 chromosomes.

1. The pollen is not sticky. The anthers are large and normally developed, but they contain hardly any pollengrains. The few there are, nearly all have abnormal pollen-tubes with several nuclei or they are large, globe-shaped and filled with starchgrains. I examine several racemes. Always the phenomenon is the same.

King of the Yellows, single-flowered yellow.

1. I examine the pollen in the anthers of green buds. Everywhere the pollen shows NĚMEC's phenomenon.

2. All pollengrains are normally formed. Only 2 % are sterile.

Yellow Hammer, single-flowered yellow, 16 chromosomes.

1. The pollengrains are all normal; $17\frac{1}{2}$ % are sterile.

2. Id. 14 % are sterile.

3. The pollen is sticky; some pollengrains have a wide, abnormal pollen-tube; 87 % are sterile.

4. The pollen is not sticky. The various preparations never show a normal fertile pollengrain. The pollen shows perfectly NĚMEC's phenomenon.

5. I examine the pollen in all flowers of *one* raceme.

Undermost flower: the anthers contain only few pollengrains, which are all sterile.

Next flower: there are sterile pollengrains, others with abnormal pollen tubes and others again that are large and globe-shaped, full of starchgrains.

Next flower: pollen as in the preceding flower; besides there are triangular, sterile pollengrains, while the 2 former flowers had only ellipsoidical sterile ones.

Next flower: pollen as in the former, but besides some fertile pollengrains present.

Next flower: as in the former, but more fertile pollengrains present, i.e. $27\frac{1}{2}$ %.

Next flower: as in the former, but the fertility is $35\frac{1}{2}$ %.

So if we compare the pollen of the various flowers, going from

below to the top, we observe a gradual transition from pollen which is sterile or which shows NĚMEC's phenomenon, to pollen which consists of normally formed sterile and fertile pollengrains.

6. The examination again refers to all flowers of *one* raceme.

Undermost flower: very few normal, fertile pollengrains. For the rest: NĚMEC's phenomenon. The anthers are shrivelled and contain only little pollen.

Next flower: like the preceding one; in the abnormal pollen-tubes more than 3 nuclei are clearly visible.

Next 12 flowers: the number of normal fertile pollengrains preponderate.

7. The pollen is not sticky, but it gives a normal impression; 36 °/₀ are sterile.

8. Besides normally formed fertile and sterile pollengrains there occur large, globe-shaped ones and others, of normal size, full of starchgrains; 15 °/₀ are sterile.

Marchioness of Lorne, single-flowered orange, 16 chromosomes.

1. In the great number of preparations only sterile pollengrains occur and others that have developed abnormal pollen-tubes.

2. More than 90 °/₀ of the pollengrains are sterile.

3. Id.

B. Double-flowered varieties.

La Virginité, double flowered white.

1. In a drop of water the pollengrains burst very soon. The grey colour they show in their protoplasm, has then disappeared and the empty, bright yellow coloured exines, on which the wartlike protuberances are clearly visible, are left. I make some preparations and leave the pollengrains in water for half a minute. If I look at them then, it appears that some grains have germinated with an abnormal, wide pollen-tube. The protoplasm is now entirely in the pollen-tube and encloses several nuclei.

Noble par Mérite, double-flowered red.

1. The flowers are doubled to such an extent that I find only few pollengrains. 18 °/₀ of these are sterile.

2. The sterility is difficult to make out, because most of the pollengrains are in a transition-stage from fertile to sterile.

3. Most of the pollengrains are sterile. In the fertile ones 2 nuclei are clearly visible. I observe almost exclusively pollengrains that stick together in groups of four. These tetrads consist generally of 1 fertile and 3 sterile pollengrains or of 4 sterils ones. The sterile ones are fullgrown. Among the sterile as well as among the

fertile ones I observe some that are filled with starchgrains. There also occur pollengrains with abnormal pollentubes. Not in all anthers I find germinated pollengrains. The sterility amounts to at least 50 °/₀.

Bloksberg, double-flowered blue.

1. I examine the anthers out of a green bud. The pollen is not sticky and clearly shows NĚMEC's phenomenon. *Van Speijk*, double-flowered blue, 21 chromosomes.

1. The anthers of a green bud are examined. The preparations are entirely in accordance with the drawings NĚMEC has made of the abnormal pollen-tubes with many nuclei, of the large globe-shaped pollengrains full of starch and of the sterile pollengrains.

2. The petaloid anthers contain little pollen. but this normal; 25°/₀ is sterile.

To this I join a table, which renders, in a surveyable form, the content af the descriptions. Under the figures 1° — 10° the result of the numeration is indicated for every variety, in accordance with what stood behind the same figure in the descriptions. These figures indicate the percentages of sterility; by the letter *n* NĚMEC's phenomenon is indicated. The capital letter in parenthesis shows the origin of the raceme from which the pollen was taken.

Further I put the number of chromosomen occuring in the somatic cells, behind the varieties of which I know this number.

The principal conclusions down from the preceding examinations, with a view to the purpose of this publication, are the following:

1°. The pollengrains in *normally formed anthers*, as well as those in *petaloid anthers*, may germinate with abnormal pollentubes with several nuclei. So NĚMEC's opinion that this phenomenon should only occur in double flowers, is inaccurate.

2°. The phenomenon has nothing to do either with the quaestion of the hyacinth-varieties being heteroploid or not. It may be observed in diploids as well as in heteroploids.

3. From the fact that in one and the same raceme or in different racemes of the same variety we now find quite normal pollengrains, now pollen showing NĚMEC's phenomenon, we may infer that the abnormally germinated pollengrains with several nuclei are caused by *external* and not by *internal* influences.

After a more superficial examination one would possibly be inclined to ascribe this kind of deviations simply to *under-* or *overfeeding*, causes that are also so often named to explain the abnormal increase of the number of chromosomes. NĚMEC thought that as the cause of the existence of the abnormal pollentubes with several nuclei was to be considered the *overfeeding* of the petaloid anthers. The finding

Name of the variety.	Number of chromosomes.	1°	2°	3°	4°	5°	6°	7°	8°	9°	10°
A. Single-flowered varieties.											
Charles Dickens		n+21 (C.)	27 (E.)								
Général Pélissier	16	n+14 (C.)	n+14 (C,)								
Lady Derby	24	n (C.)	4 (G.)	$3^1/_2$ (D.)	6 (D.)	4 (D.)					
Moreno		n (B.)	24 (D.)	40 (D.)	11 (G.)	15 (G.)	13	59	48 (A.)	8 (B.)	35 (B.)
Nimrod	19	n (B.)	n (C.)	n (C.)	26 (D.)	n+36 (H.)	29 (H.)	29 (H.)	66 (K.)		
City of Haarlem	23	n (C.)									
King of the Yellows		n (C.)	2 (D.)								
Yellow Hammer	16	$17^1/_2$ (A.)	14 (A.)	n+87 (B.)	n (B.)	n (B.)	n (B.)	36 (C.)	15 (D.)		
Marchioness of Lorne	16	n (C.)	90 (C.)	90 (E.)							
B. Double-flowered varieties.											
La Virginité		n (D.)									
Noble par Mérite		18 (A.)	? (D.)	n+50 (E.)							
Bloksberg		n (C.)									
Van Speijk	21	n (C.)	25 (E.)								

of the abnormal pollengrains in the uppermost flowers of the racemes of *Charles Dickens* (1°) and *Général Pélissier* (1° and 2°) might be an argument in favour of the fact that *underfeeding* is the cause, as the uppermost flowers are always in a far worse condition than the undermost of the raceme. However, what I have observed in *Yellow Hammer* (5° and 6°) is not in accordance with this. For here the *undermost* flowers of the raceme are developed in the same way as the *uppermost* of *Charles Dickens* and *Général Pélisier* though the same rule holds good here as well, that the undermost flowers are in a better condition with regard to the feeding than the uppermost. So at any rate the causes must be considered to be of a more complicated nature; of which presently more.

5°. The abnormal pollen-tubes may developed when the bud is still green. In other cases they do not develop until the flowering-time has set in.

6°. From the descriptions given, as well as from the large tables not published here, we may conclude that between the aspect shown by normal pollen, and NĚMEC's phenomenon, there is as it were a gradual transition, manifesting itself as follows:

a. Normal aspect of the pollen; fertile and sterile grains are present; the latter large and elliptic.

b. Besides the large ellipsoidical pollengrains there occur smaller elliptic ones and others that are round or triangular.

c. the sterile pollengrains of various shapes and sizes get the upper hand, which probable indicates and early dying off, partially; besides these there occur large, round pollengrains, filled with starchgrains.

d. NĚMEC's phenomenon complete.

IV. *On the conditions, under which* NĚMEC's *phenomenon may come to exist.*

I may here remind the reader that all the pollen grains examined originated from plants which, on the ground of the various conditions under which they had been cultivated, were classed by me in 13 different categories, from *A* to *M*. That I owe a considerable part of my results to this arrangement, may appear from a glance at the last table. This table shows clearly, that NĚMEC's phenomenon confined itself almost exclusively to the categories *C* and *B*. Of the 20 times that this aspect was found it occured 11 times in *C*, 6 times in *B* and once in *D, E,* in *H*. Besides it immediately struck the eye, that the pollen-aspects which showed one of the

transitions outlined above of a normal aspect to NĚMEC's phenomenon likewise manifested themselves most in the categories *C* and *B*.

As I had acquainted myself as carefully as possible with the circumstances under which the plants belonging to the 13 categories, had lived, from the moment when the bulbs were dug up in 1918 till the moment when the pollen was examined by me, it was not difficult for me to decide under what conditions of culture hyacinth-varieties are able to produce pollen that exhibits NĚMEC's phenomenon.

It may be considered as a well-known fact, that the growers dig up their hyacinth-bulbs after the leaves have died off, towards the end of June and in July, then lay the bulbs in artificially heated barns to be dried, and plant them again in September. So, for instance the bulbs, — I mention this in outline only — which were classed by me under category *D*, were dug up in 1918 between July 1 and July 25. The barn was heated from August 1 till November 1, during the first weeks to 65° F., afterwards the temperature was allowed to rise to 75° F. Between September 20 and November 1 the bulbs were planted again.

All bulbs which (1) were treated in this way, be it that duration of heating and degree of heating diverged a little, besides others which (2) were housed in barns where there was no heating; (3) were not dug up, so passed the resting-time outside; (4) were not planted or placed upon glasses; (5) were cultivated on glasses; (6) differed in age or size, never produced flowers the pollen of which showed NĚMEC's phenomenon. It was different with those plants which *partially* had suffered what is called "preparation". The bulbs chosen for that purpose, *are dug up in an unripe state*, heated pretty strongly in the barn, and afterwards planted in pots or placed upon glasses. When the bud begins to rise a little above ground, the plant is exposed to a higher temperature a second time, the consequence of which is that the flowers bloom very early. See for this: A. H. BLAAUW: *On the periodicity of Hyacinthus orientalis p.* 51, *Vol. XVIII of the "Communications of the Agricultural University"*, and my publication: *"On the occurrence of heteroploid varieties of Hyacinthus orientalis L. in the Dutch cultures". Arch. Néerl.* 1921 *and Genetica* 1921.

So the bulbs of the variety *Nimrod* (category B) in the flowers of which I first found pollen-grains which showed me NĚMEC's phenomenon, were dug up on June 10th, *the leaves still being a fresh green not showing a trace of dying off.*

They were exposed for 21 days to a temperature varying on an indented line from 90° F. to 78° F.; afterwards till October 26 to

a temperature fluctuating between 70° F. and 60° F. They were not placed in pots, and afterwards not forced to early florescence, but between October 26 and November 1 planted in the open ground outside. In this way all bulbs had been treated which are grouped under category B and developed abnormal pollen-tubes.

Very remarkable was the phenomenon that I observed in the plants belonging to category C. These were all cultivated by the same grower, of whom I examined 25 varieties (19 single-flowered and 6 double-flowered), with which I effected 31 numerations. Of six varieties 2 numerations were noted. These were varieties which had been grown in 2 lots, under different conditions. In 21 of the 31 cases the pollen diverged more from the normal aspect than that of the same varieties classed under all other categories. In 5 cases I could not compare it with that of other categories because for them I had no racemes at my disposal. In the remaining cases it was found in the same condition or in one a little better than that of any other category.

In a very striking manner it now appeared to me that the racemes with pollen showing NĚMEC's *phenomenon or transitions from the normal aspect to* NĚMEC's *phenomenon, always originated from bulbs dug up in an unripe condition, between July* 1 *and* 15. Immediately after the digging up artificial heating was started, till September 20, from 75° to 80° F.; afterwards till the planting time, which was in October, from 70° to 75° F. The planting was done in the open ground.

My observations have induced me last sommer to purposely subject several hyacinth-bulbs to various exterior influences. From these

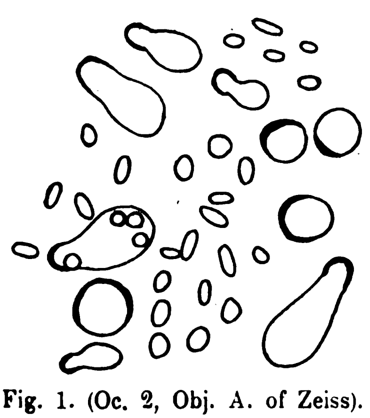


Fig. 1. (Oc. 2, Obj. A. of Zeiss).

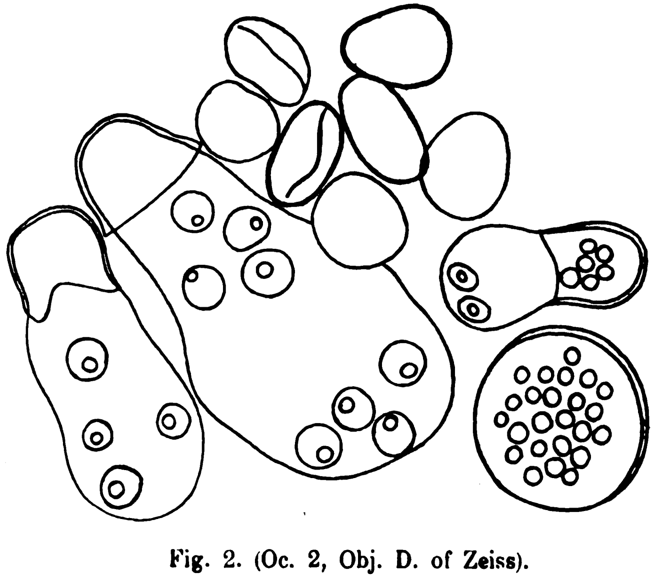


Fig. 2. (Oc. 2, Obj. D. of Zeiss).

experiments it has now, on the 3rd February, become evident to me *that it is possible, to cause the growth of pollen-grains that exhibit* NĚMEC's *phenomenon. To the great importance of deliberately producing pollen-grains with more than one nucleus* — also SAKAMURA (1920, p. 145) by his discovery of several nuclei in the pollen-grains of *Allium Cepa* has come to the opinion that they owe their origin to a modification of the exterior conditions of life -- I hope shortly to draw attention.

The diagrams 1 and 2 picture forth the pollen-grains of the variety *Yellow Hammer*, as I found them now in all anthers of the plants which are exposed to the same particular exterior circumstances.

REFERENCES.

CHAMBERLAIN, CH. J., Contribution to the life history of *Lilium philadelphicum* (The pollengrain), Bot. Gaz,, 23, 1897.

ELFVING, FR., Studien über die Pollenkörner der *Angiospermen*, Jenaische Zeitschr. f. Naturwiss., 13, 1878.

GOLINSKI, Ein Beitrag zur Entwicklungsgeschichte des Androeceums und des Gynaeceums der *Gräser*, Bot. Centralbl., 55, 1893.

GUIGNARD, L., Le développement du pollen et la réduction chromatique dans le *Najas major*, Arch. d'anat. microsc., II, 1899.

MURBECK, SV., Ueber die Embryologie von *Ruppia rostellata Koch*. K. Sv. Vetensk. Handl., 36, 1902.

NĚMEC, B., Ueber den Pollen der petaloiden Antheren von *Hyacinthus orientalis L.*, Academie des Sciences de l'Empereur François Joseph I, V, Bulletin Internal, (Sc. math. et nat.), 1898.

SAKAMURA, F. Experimentelle Studien über die Zell- und Kernteilung mit besonderer Rücksicht auf Form, Grösze und Zahl der Chromosomen, Journal of the College of Science, Imperial Univ. of Tokyo, Vol. XXXIX, Art. 11, March 20th, 1920.

SCHAFFNER, J. H., Contribution to the Life-History of *Sagittaria variabilis* Bot. Gaz., 23, 1897.

SCHNIEWIND—THIES, J., Die Reduction der Chromosomenzahl und die ihr folgenden Kernteilungen in den Embryosackmutterzellen der *Angiospermen*, Jena, 1901.

SCHÜRHOFF, P. N., Ueber die Teilung des generativen Kerns vor der Keimung des Pollenkorns, Arch. f. Zellf., 15, Heft 2, 1919.

STRASBURGER, E., Neue Untersuchungen über dem Befruchtungsvorgang der Phanerogamen usw., Jena, 1884.

WIEGAND, K. M., The development of the microsporangium and microspores in *Convallaria* and *Potamogeton*, Bot. Gaz., 28, 1899.

From the Laboratory for the Physiology of Plants of
Prof. ED. VERSCHAFFELT, *Hortus Botanicus.*

Amsterdam, February 1921.

Botany. — *"A new method of recording the modifications in aperture of stomata."* (First Communication). By M. PINKHOF. (Communicated by Prof. F. A. F. C. WENT).

(Communicated at the meeting of October 30, 1920).

§ 1. *Introduction.*

For organisms which, like the "higher" plants, live as a rule in the (gaseous) atmosphere, the regulation of the gas interchange is an essential point in the organisation. In connection with this the fact must be pointed out, that among the vegetative organs, common to all plants, the only part that is capable of quick response in consequence of its specialised structure, and thus deserves the name of "apparatus", is the very part that has to regulate the gas interchange. This organ is the stoma (Spaltöffnungs*apparat*).

Just as essential as the stomata themselves are for the plant, the study of them is for the plant-physiologist. The finding of methods to get acquainted with the behaviour of stomata by means of experimental researches, has been indeed a subject of constant care in physiology. Much has already been done in this department, but of course there are always improvements to be made and as an attempt in that direction should be regarded the conception of a self-recording type of an existing apparatus treated below.

After the exhaustive discussion, which VAN SLOGTEREN, in the introduction to his dissertation [1]) has devoted to the numerous direct and indirect methods, invented to judge the aperture of stomata — I think it to be superfluous to mention these methods again and I restrict myself to quoting what VAN SLOGTEREN says about the porometer-method of DARWIN and PERTZ and its advantages. [2])

"... it is based on the following principle: a glass chamber is fixed air-tight to a leaf and through a tube connected with this chamber, the air is sucked out, so that the pressure in the chamber is diminished. After the side-tube has been closed, the difference in pressure inside and outside the chamber, can only be annulled, when air is sucked through the leaf. From the time, necessary for making equal the pressure, the degree of aperture of the stomata is judged.

[1]) E. V. SLOGTEREN, De gasbeweging door het blad in verband met stomata en intercellulaire ruimten. Groningen 1917, p. 1–13.

[2]) l.c., p. 14.

A great advantage of this method is, that the values found are not directly connected with the transpiration . . .".

"An advantage offered by the porometer-method above all other methods is, that it enables us to examine the stomata in the living leaf, which remains attached to the plant, in circumstances *more* normal than with the other methods. It enables us to observe the same leaf a very long time consecutively without its experiencing any injurious consequences".

Finally VAN SLOGTEREN [1]) points out, that the porometer of DARWIN and PERTZ should also be preferred, because it likewise indicates small modifications in the aperture accurately and makes known the average of thousands of stomata at the same time.

It might be expected, that after the method of DARWIN and PERTZ [2]) got known, attempts should be made, to make it capable of self-recording. It may be said, that the phenomenon to be studied asks for a continuous picture projected of its changes. Indeed some three self-recording porometers have already been described (resp. by BALLS [3]), JONES [4]) and LAIDLAW & KNIGHT [5]) of which BALLS's stomatograph may be called the most successful. About these apparatus VAN SLOGTEREN says [6]): "There is a danger, that either a too great pressure is used, or the apparatus is made so complicated that it may give rise to all kinds of sources of error. A great advantage of the original porometer of DARWIN and PERTZ is its very simplicity, through which the influence of external factors exercised on the apparatus, is so easily judged. Besides the plant is placed in quite abnormal circumstances, if a continuous current of air is sucked through the leaf, as is necessary in these methods".

With the recording porometer, that I am now going to describe, all these objections have been avoided and the circumstances in which the plant remains during the experiment, are even more favourable than in the case of the original porometer.

§ 2. *General description.*

When a selfrecording apparatus is made, the purpose is a.o. to

---

1) l.c., p. 16.

2) F. DARWIN and D. F. M. PERTZ. On a New Method of Estimating the Aperture of Stomata, Proc. Roy. Soc. Lond Serie B. Vol. **84**, 1912.

3) W. L. BALLS, The Stomatograph, Proc. Roy. Soc. Lond. B, **85**, 1912, p. 33.

4) W. NEILSON JONES, A Selfrecording Porometer and Potometer, New Phytologist, XIII, 1914, p. 353.

5) C. G. P. LAIDLAW and R. C. KNIGHT, A Description of a Recording Porometer etc., Annals of Botany, XXX, 1916, p. 47.

6) l.c., p. 16.

replace the person, who works a similar non-recording apparatus. In the simplest case his work consists in the reading and recording the position of a hand and of a time designer (e.p. aneroid-barometer and barograph). With the porometer of DARWIN and PERTZ however the working of the apparatus is more complicated, for besides the fact that for each observation *two* positions and *two* moments have to be read and recorded, the apparatus has to be brought into its original position after each observation. So there are several functions here, which will have to be done automatically by a selfrecording porometer. In constructing the automatic apparatus I have tried to keep the principle of the original porometer unaltered as far as possible.

The apparatus (see fig. 1 lower half) consists of a glass chamber cemented to the leaf (**1**) [provided with the side-tube (**2**) indicated by VAN SLOGTEREN [1]), *which must for the present be considered closed*, just as the tube **18** in fig. 1], connected with a U-shaped manometer, with distilled water. The closed limb $\mathbf{M}_1$ has a side-tube with rubber tube (**3**). In the case of the ordinary porometer the observer would, in bringing about a certain pressure, push open the elastic clamp-cock suck at the rubber tube and then let the elastic clamp recoil. In this case however the rubber tube leads to the water-jet-air-pump (**P**) and passes a compression cock, which as a rule closes it, but at the required moments is pulled open by an electro-magnet $\mathbf{K}_1$, so that the connection between air-pump and porometer is brought about. When the required air-rarefaction has been attained, the current of the magnet is broken, the clamp recoils and the rarefaction can only be adjusted via the stomata.

Since the air-pump works continuously a very great rarefaction would soon arise in the space outside the cock, which on opening the cock would also appear in the porometer. To prevent this, the tube leading from the air-pump is branched outside the cock and a second rubber tube (**4**) runs between the movable part (**5**) of the cock and a fixed metal block (**6**).

In the position of rest of the cock, this rubber tube is open and connects the pump with the atmosphere (at **7**). When however the cock is pulled open, **4** is closed off, so that now the pump can only work on the porometer (through **3**). So the water-jet-air-pump and the electro-magnetic clamp replace the sucking with the mouth and the clamp cock of the ordinary porometer.

Now it should be looked to, that the electro-magnet works at the right time. The open limb $\mathbf{M}_2$ of the manometer is wider than $\mathbf{M}_1$

---

1) l.c., p. 18.

and on the water floats a glass floater, suspended on a cord, passing over a pulley, and consequently in case of a change of water-level

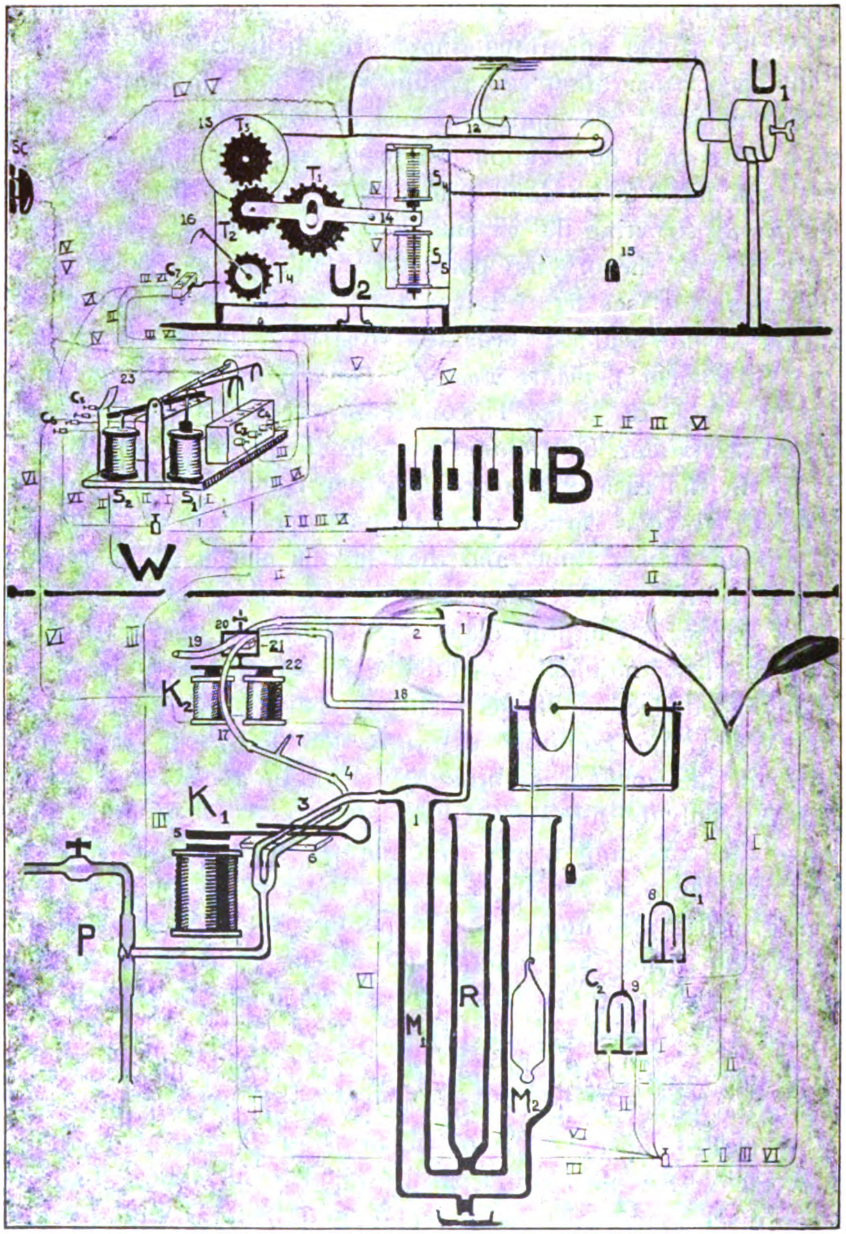


Fig. 1. *Scheme of the arrangement of the self-recording porometer*. Explanation of letters and figures in the text.

sets the pulley in motion. The force, with which this takes place, is exclusively dependent on the floating-power of the floater and *quite*

independent of the difference in pressure to be adjusted. Therefore pressures differing but little from the atmospheric can be employed, which makes circumstances more natural for the plant. Fig. 1 refers to a moment, when the difference in pressure is being adjusted, so the water is falling in $\mathbf{M}_1$ and rising in $\mathbf{M}_2$ and the pulley is moving as the arrow indicates. Now there turns on the same shaft a second pulley and over this a cord passes, carrying at either of its extremities a ∩-shaped fork of nickel wire. Such a fork moves, when the pulley turns, in a vertical direction, with each of its limbs in a glass tube, at the bottom of which is mercury, which has been inserted into a circuit.

If the fork **8** dips into the mercury of contact $\mathbf{C}_1$, circuit **I** is closed and the current passes through the solenoid $\mathbf{S}_1$ of the automatic electric mercury-switch **W** (upper half of Fig. 1). The core of soft iron is attracted, the switch is changed over and a.o. one of the insulated iron forks dips into the mercury cups of contact $\mathbf{C}_3$, in consequence of which circuit **III** is closed, the electro-magnet opens the cock $\mathbf{K}_1$, the porometer is brought in connection with the pump and the water rises in $\mathbf{M}_1$ and falls in $\mathbf{M}_2$. The air-rarefaction and with it the movement of the water and of the pulley continues till the fork **9**, which is sinking now, reaches the mercury in $\mathbf{C}_2$, for in consequence of that circuit **II** is closed, solinoid $\mathbf{S}_2$ works and, the switch changing over, circuit **III** is broken, the porometer-space is closed by the clamp and the current of air passing through the stomata rebegins the adjustment of the difference in pressure. It is evident, that the degrees of air-rarefaction between which we want to work, can be regulated by altering the place of the mercury-contacts along their stand.

The preceding may be summarized as follows:

There are two alternate periods in the working of the apparatus:

Period I, of indefinite shorter or longer duration, in which the position of the switch is such, that the current through the electro-magnet is broken; the porometer-space consequently is closed by the cock and the pressure is increased by the current of air through the leaf from the lowest fixed limit to the highest.

Period II, of a short duration, in which the position of the switch is such, that the current passes through the electro-magnet; the porometer-space consequently is connected with the pump, which diminishes the pressure again to the lowest limit.

Each period begins with a contact, made by one of the forks (taking part in the movement of the water in the manometer) through which the switch is drawn in the required position.

Now the purpose is to record the duration of period 1 every time. The simplest method would be, to make the switch give a mark on a revolving drum at each changing over, and next to measure the distances of those marks and make a graphic of them. A similar method has already been applied by BALLS and by LAIDLAW and KNIGHT, who employed drums, revolving rather quickly and falling simultaneously, because otherwise the marks would come too close together for them to be able to judge the distances.

The recording-apparatus to be discussed here, has been constructed thus, that it gives the required data directly in a survey on a slowly revolving drum, so that the modifications in the degree of aperture of the stomata, for instance during 24 hours can be read, without measuring or copying.

It consists firstly of a recording drum, turned round by the clockwork $\mathbf{U}_1$ in 24 hours (see fig. 1 at the top).

The tracing-pencil (**11**) is fastened to a trolley (**12**), which can be moved on rail parallel to the descriptive line of the cylinder, being attached with a cord to the disk (**13**), which is fixed on one shaft with the cog-wheel $\mathbf{T}_3$ of the second clockwork $\mathbf{U}_2$. $\mathbf{T}_3$ however is not constantly in connection with the rest of the cog-wheels of $\mathbf{U}_2$. In fact, the cog-wheel $\mathbf{T}_2$, that has to transmit the motion of the clockwork to $\mathbf{T}_3$, does not turn in the frame of the clockwork, but in a separate lever, which can be moved round the pivot **14** to a small extent by alternate attraction of the cores of soft iron of the solenoids $\mathbf{S}_4$ and $\mathbf{S}_5$. If $\mathbf{S}_5$ pulls down, $\mathbf{T}_2$ goes up, the motion of $\mathbf{T}_1$ is transmitted to $\mathbf{T}_3$ and the trolley is drawn to the left with a certain speed. If next $\mathbf{S}_4$ pulls, $\mathbf{T}_2$ goes down, $\mathbf{T}_3$ is released and the trolly is drawn back to zero by the weight **15**.

The intention with this arrangement is, that the trolley is only taken along by the clockwork during the period 1 of the porometer, which is to be recorded, and that during period 2 it has an opportunity of running back to zero, to recommence its uniform movement with the next period 1. The longer period 1 lasts, the longer the clockwork continues pulling uninterruptedly and the longer therefore grows the line, traced by the pencil towards the left. In this way the left extremities of the lines thus obtained, give a distinct picture of the course of the durations of the periods 1, i.e. of the degree of aperture of the stomata.

For the changing over of the cog-wheels at the exact time the automatic switch **W** cares, which itself is indeed changed over at the alternation of the two periods. To its shaft namely some other iron forks have been fixed insulatedly and these serve to close circuit

V at the beginning of period 1 in the mercury-contact $\mathbf{C}_5$, in consequence of which the solenoid $\mathbf{S}_5$ (at the clockwork) works and makes $\mathbf{T}_2$ catch into $\mathbf{T}_3$ — while at the beginning of period 2 in $\mathbf{C}_4$ circuit **IV** is closed, which by means of $\mathbf{S}_4$ pulls down the cog-wheel $\mathbf{T}_2$.

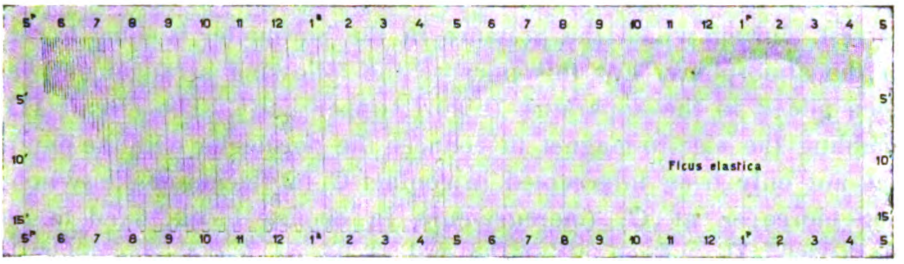


Fig. 2. Course of the stomatal aperture in a leaf of *Ficus elastica* for nearly 24 hours. The lengths of the lines denote the time, in which the pressure in the porometer was increased from 5 to 1 cms. of water below the height of the barometer.

Fig. 2 is a reproduction of a diagram, obtained with the aid of the described arrangement. It is hardly necessary to point out, that the distance between the lines is the smaller, the shorter the periods 1 are. If period 1 lasts longer than 16 minutes, the pencil goes no farther, so that the line is, as it were cut off. There is however no objection to this, because in such long periods the length may be accurately read from the time-axis down to half-minutes.

The picture given by fig. 2 clearly shows how on May 13th at 5.30 p.m. a slow closure of the stomata had already set in, between 7 and 8 o'clock the closure went quicker and quicker, at 8.30 the period was 5 times longer than at 5.30. Next the stomata remained in the strongly-narrowed condition, till about midnight, then they began to re-open, first very slowly, but between 4 and 6 o'clock in the morning very fast.

From 7 to 9 o'clock they were more open than the previous afternoon at 5.30. Between 9 and 2 the degree of aperture showed fairly strong oscillations, which, in connection with observations I shall no further discuss here, should probably be attributed more to the influence of temperature than the light. At the maximal aperture at 2 o'clock, period 1 was 16 times shorter than at the greatest closure at 10 o'clock in the previous night. After 4 hours

a slow closing set in, so that at about 5 o'clock the condition was almost equal to that of 24 hours ago.

At first sight the results obtained in this way, seem satisfactory. Undoubtedly the general course of the aperture of the stomata may be read from the diagram. Yet the apparatus discussed above meets with the same objection as the other recording porometers: the uninterrupted flow of air through the leaf. This objection can be removed, by introducing between every two observations a period of rest, during which the pressure in the porometer is equal to that of the atmosphere. For this purpose the following arrangement has been added to the apparatus. (Fig. 1. upper half). The circuit **III**, enabling the electro-magnet to connect the porometer with the pump, has not only been interrupted in contact C_3 (at the automatic switch), but moreover at C_7 (mercury-cups in ebonite block, fixed to the clockwork U_2). Even if at the end of period 1 the connection in C_3 is brought about, yet the current will not pass through the magnet before C_7 has likewise been closed. This may be attained by a fork (**16**), fixed to the cog-wheel T_4. In the position as illustrated (during period 1), this cog-wheel is free from the clockwork and is kept in this position by a weight, that keeps the fork **16** at a special distance from C_7. At the end of period 1 T_3 goes down and its teeth catch into those of T_4. Now this latter turns slowly back and after some time **16** reaches C_7, the circuit **III** is entirely closed and the reduction of the pressure in the porometer may begin. As the rod, to which **16** has been fixed, insulated, can be turned round the shaft of T_4 with friction, the angle through which it has to be moved by the clockwork, may be taken arbitrarily and in this way the duration of the period of rest can be determined.

Our purpose, viz. to have atmospheric pressure in the porometer during the period of rest has not yet been attained, for at the beginning of the period of rest there is still a difference in pressure and it would take long to adjust this quite.

The porometer-space therefore should be brought into direct connection with the atmosphere and this happens by means of the rubber tube **19** (Fig. 1. bottom half), attached to the side-tube **2** of the glass chamber and which is not — as is the case with VAN SLOGTEREN's porometer — closed off with a common clamp, but runs through an electromagnetic clamp. Tube **19** had been closed off between the frame **20** and the fixed block **21**, during period 1, because the armature **22** was attracted by the electro-magnet K_2. Circuit **VI** serves this purpose, which can be interrupted at the mercury-contact C_6 of the switch,

but in the illustration (period 1) is just kept closed by a fork. Is the switch changed over at the end of period 1, **VI** is interrupted and the frame **21** is pushed upward by the elasticity of the rubbertube itself so that **19** is opened and the pressure inside and outside the porometer becomes equal. When the period of rest is finished, **19** must be closed, otherwise the pump cannot reduce the pressure. In order to attain this end, circuit **VI** has also been conducted past C_7, so that **19** is closed off, as soon as **16** dips into the mercury. As T_4 (thus **16**) is not released by T_3 before the switch has taken the position illustrated in fig. 1, consequently the current **VI** is also closed in C_6, **VI** keeps going and the rubber tube **19** remains closed, also during the transition from period 2 to 1.

The arrangement is such, that circuit **III** is entirely closed, when *both in* C_6 *and in* C_7 contact has been made, whereas the current already goes through **VI**, when in *either of the contacts* C_6 of C_7 the fork dips into the mercury.

It had appeared in practice, that during the period of rest, in consequence of the transpiration of the leaf, the air in the glass-chamber was saturated with water-vapour, the result of which was, that the condition of the glue-rim did not remain trustworthy. Therefore it was necessary to renew the air during the rest and this could be attained, by giving the waste rubber-tube **4**, with an open end at **7**, (where during period 1 and the rest-period air is sucked in by the pump from outside,) a side-limb (**17**) passing clamp K_2, next past **18** and ending at the bottom of the glass-chamber. If K_2 is open, a current of air enters at **19**, and passes through **2**, **1**, **18**, **17**, **4** to the pump. Since **7** always remains open, the current of air through the chamber is not too strong. When the period of rest is over, **17** in K_2 is closed by the clamp at the same time as **19** and the circulation stops.

With the apparatus thus modified 3 periods are to be observed:

Period 1. Porometer-space closed. The air enters through stomata.

Period of rest. Porometer-space connected both with atmosphere and pump. The air circulates.

Period 2. Porometer-space only connected with pump. The air is rarified.

Fig. 3 is a photograph of a diagram, showing the periods of rest between the observations. The laboratory possesses two recording-porometers, writing on one drum, enabling us to examine two plants simultaneously.

In this case *Ficus elastica* and *Peperomia maculosa* were treated. It is striking, that in the case of *Ficus*, since the introduction of

the period of rest, a much larger amplitude is to be noticed, than in the case of uninterrupted recording, which proves the necessity

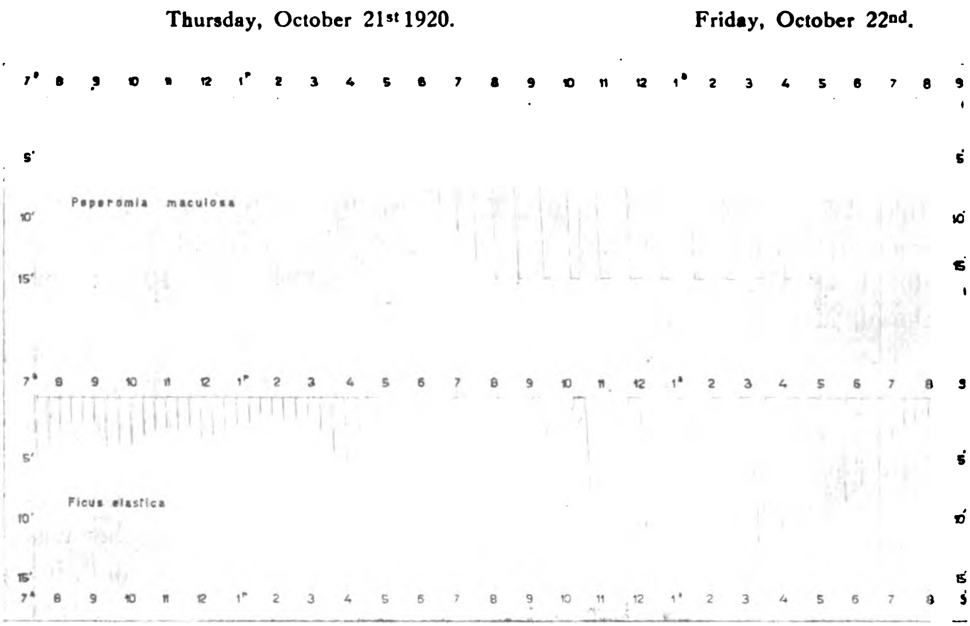


Fig. 3. Course of the stomatal aperture in leaves of *Peperomia maculosa* and *Ficus elastica* recorded simultaneously with intervals of 20 minutes for nearly 26 hours. The lengths of the lines denote the time, in which the pressure in the porometer is increased from 7 to 4 cms. of water below the height of the barometer.

of the period of rest (cp. fig. 2 and fig. 3). If from 13 to 14 May the rate of time between "most open" and "most closed" was 1 : 16, in fig. 3 it is 1 : 85. It is also remarkable, that even during the night opening and closing may be noticed, which proves, that other factors than light act a part.

Comparison between *Ficus* and *Peperomia* shows, how with the latter the stomata check the current of air but very little in the middle of the day, towards evening the closure is much quicker with *Ficus*. What lasts still $5\frac{1}{2}$ min. at 4 o'clock, is done in 125 min. between 5 and 7 o'clock. In the evening *Peperomia* gave a slower closure than *Ficus*, in the morning however an opening at least equally quick. The ratio of open and closed is particularly strong in the case of *Peperomia*. At one o'clock in the afternoon the period of fall is not yet $\frac{1}{4}$ minute, about midnight 32 minutes. Here it is not the place for a further inquiry into these results, which immediately tempt to further physiological contemplations.

§ 3. *Particulars.*

The practical execution of the system discussed has given various experiences, which I will communicate here. In fig. 4 and 5 the

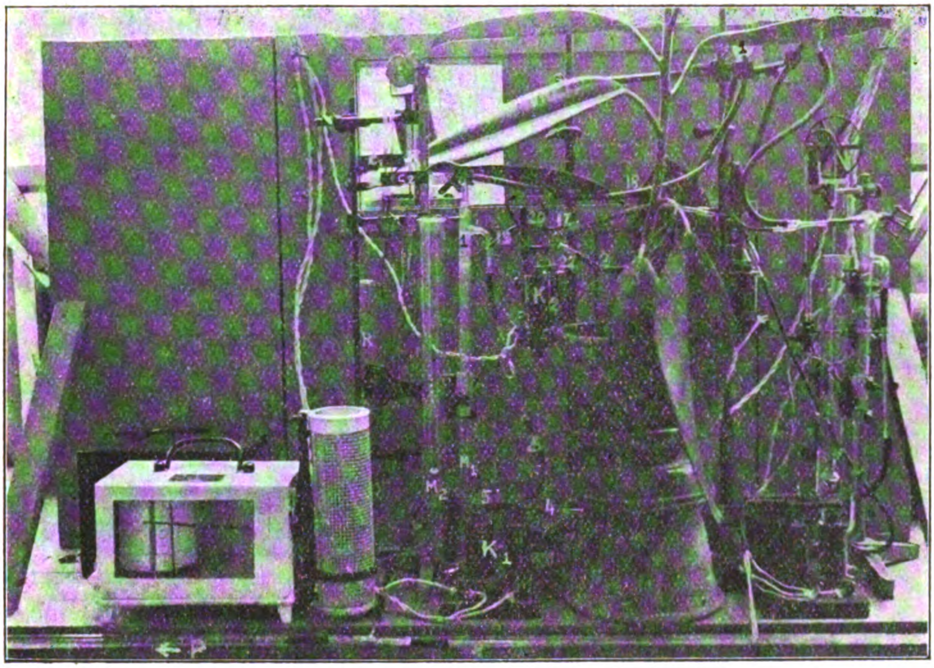


Photo J. QUELLE.

Fig. 4. *Arrangement of two recording porometers*, the chambers of which have been fixed to one leaf of a *Ficus elastica* on either side of the middle-vein. Explanation of letters and marks in the text. To the left a hygrograph and behind it a thermograph. Through the opening at the top in the middle something of the sunshine-autograph may be seen.

real arrangement of the parts has been illustrated, deviating here and there from the schematic fig. 1. The signification of letters and figures is the same throughout.

1. *The glass chamber.* At present I always use the model, as illustrated by VAN SLOGTEREN on p. 18 of his dissertation, so the one with the side-tube attached near the edge. In the case of *Ficus elastica* I use chambers with wide mouths (diameter 4 centims), because by doing so a greater number of stomata is set to work and the falling of the water in the porometer goes quicker. Because I use pressures, little below the atmospheric pressure, (viz. between 7 and 4 millims underpressure or between 5 and 1 millims under-pressure) to make the condition of the plant differ as slightly as possible from the natural condition, the fall-period is longer than

for instance in the case of most of VAN SLOGTEREN's experiments. That is why measures such as the use of the large-surface-chambers

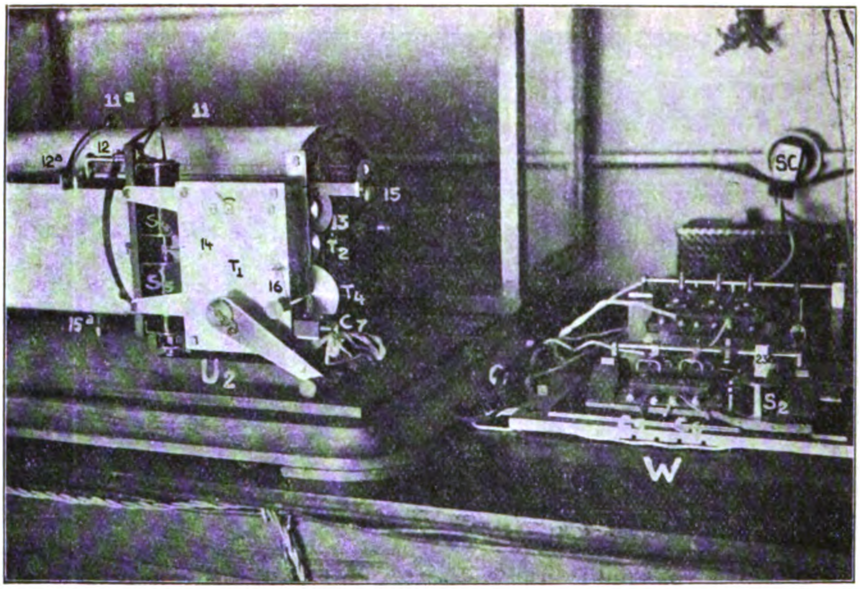


Photo J. QUELLE.

Fig. 5. *Arrangement of the recording apparatus.* To the left the recording drum and in front of it one of the two writing clockworks (U_2). The second clock conformable to the first was too much to the left to be seen in the photo only some parts belonging to it (11^a, 12^a, 15^a) are visible. The clockwork moving the drum should also be thought to the left. To the right the two automatic mercury switches (the one in front of the other) and behind those a resistance. Further explanation of the figures in the text.

and the diminution of the gas-contents of the porometer are to be recommended.

The fixing of the chamber to the leaf remains a difficult question. In 1914, when I was occupied with ordinary porometer-observations, I tried a number of glues, and just as VAN SLOGTEREN [1]), I came to the conclusion, that ordinary Arabic gum is by far the best. In 1919 on the advice of my colleague J. HEIMANS, I tried mixing a small quantity of *Sesame-oil* with the gum and indeed this makes the substance somewhat tougher and cracks do not so soon appear in drying. Besides I always add a trace of *Thymol*, to prevent its decay. Too much thymol is injurious to the experimental plant.

---

[1]) l.c., p. 20.

It is my experience, that well-fixed chambers may remain fit for use for 4 weeks, but they should be ventilated regularly between the experiments. It is essential, that the air in the hothouse at least in the first hours after attaching the chamber is fairly dry, otherwise the glue remains soft too long, which may cause the leaf to get slightly removed from the chamber.

II. *The manometer*. The two limbs are of unequal diameter, therefore the manometer should be tested before use in order to know what distance in the narrow tube conforms with a change in pressure of 1 cm of water. Fixing the water-level at zero is a very simple thing and quickly done, in consequence of the presence of a water-reservoir **R** (attached to the manometer by means of a rubber tube with clamp) and of a drain with clamp. Practice teaches that hardly any water evaporates from the manometer, so that the apparatus can work 48 hours at a stretch, without the water-level having to be fixed anew [1]).

III. *The electro-magnetic clamp* K_1. The whole apparatus is *really based* upon the theorem, that the elastic clamps close the rubber tubes absolutely and that at the points where the rubber tubes pass over the glass tubes, leakage is impossible. What fig. 3 shows for Ficus elastica between 5 and 7 p. m. is the best proof for the practicability of this theorem.

About the structure of the clamp itself there is not much more to be said, than can be seen in the picture. The electric current however should be further discussed. It is yielded by a battery of accumulators of 7 cells, therefore with a voltage of 14 Volt and the force of the current is for the clamp $\mathbf{K}_1$ only 1 Ampère. There is a very small loss of electricity, for the time used for sucking up the water in the manometer is at most 4 seconds. The current of the battery is also used for the solenoids of the switch (circuit **I** and **II**) and for the electromagnetic clamp $\mathbf{K}_2$ (circuit **VI**). Part of the apparatus being in the hothouse (everything below the horizontal line in fig. 1) and the rest in a room of the laboratory (switch, recording-apparatus and battery) an amount of connecting wire is necessary. A tube, containing 16 wires, joins the two localities and the wires end on terminal boards with numbered terminals, from which flexible wires lead towards the different apparatus. Two series-switches have been inserted, which can close and break the current if necessary both in the hothouse and in the room.

---

[1]) The U-shaped manometer is therefore also to be preferred in case of the non-recording porometer to the ordinary glass tube placed in a vessel of water.

IV. *The electro-magnetic clamp* **K_2**. Except during the period of rest, this clamp must remain closed and consequently is in circuit. Therefore it has been arranged in such a way, that a current of 0,45 Ampère is sufficient. For the rubber tubes **17** and **19** has been used common bicycle valve-tube.

V. *The pulley-contact-arrangement.* The pulley-shaft runs very lightly between two conical pivots. The floater is made of glass and contains mercury at the bottom. A thin cord runs over one pulley and bears a weight at its other end. In fig. 4 to the right is shown, that in the newest model the second pulley is larger, which magnifies the movement of the forks **8** and **9** and enables us to fix the contacts **C_1** and **C_2** more accurately.

As appears from the description, I have always used mercury in the contact-arrangements, which was done because mercury-contacts are absolutely trustworthy, contrary to brushes of other metals and because force is not wanted for bringing about the connection.

It would not be an easy matter to keep the mercury in the glass tubes **C_1** and **C_2** clean, when soiled by sparks. With the amperage used sparks only occur on breaking the current, therefore it was necessary to break the circuits I and II somewhere else before the forks **8** or **9** rise out of the mercury. This end has been attained by adding to

VI. *the automatic mercury switch,* an arrangement of levers (not shown in fig. 1), by which, again with the aid of mercury-contacts, the circuits I and II can be broken, when the switch has reached the required position and further action of the solenoids **S_1** or **S_2** is superfluous. With this arrangement a double purpose has been attained: 1st the spark of breaking is removed to a place where the mercury must not so anxiously be kept clean; 2nd the loss of electricity in the solenoids is limited to a minimum. — At the first trials of the switch it appeared, that in consequence of the elastic fall of the iron forks in the mercury-cups, the switch recoiled halfway after a moment. To prevent this, the clockmaker J. Messias devised a brake, consisting of a couple of metal springs, checking the movement of the lever. One of them has been rendered in the diagram as No. **23**.

The mercury-cups of the contacts **C_3** to **C_6** are bored out in ebonite blocks. At the bottom of the cavities there end the iron screws of the "terminals" put up at the sides, carrying the current to the mercury.

VII. *The clockwork* **U_2**. The proportion of the cog-wheels and the

outline of the pulley **13** is such, that the trolley is drawn on with a speed of 1 mm. per minute.

It is essential, that the lever, in which T_2 turns, is kept steadily in the required position. Therefore after every movement it shoots behind an elastic pawl. Consequently the solenoids S_4 and S_5 must possess great strength in order to draw the lever back along the pawl. The electric current yielded by the battery could not perform this, and therefore use has been made for this arrangement of the alternate current of 220 Volt taken from the plug-switch **SC**. The circuits **IV** and **V** (drawn as an undulating line in the illustration) carry the alternate current. Mr. MESSIAS invented an arrangement, which prevented the current to pass longer than strictly necessary. In fact, the vertical shaft, to which the cores of soft iron are fixed in S_4 and S_5, ends on both sides in a bone tip and this breaks the current in a gold-contact when the extreme position is reached. (This is not indicated in the picture).

VIII. *The tracing-pencil.* The tracing is done in recording-ink i.e. with red, because the common lilac ink is not well reproduced in an ordinary photograph. The glass needles are made according to a model, used in the Royal Dutch Metereological Institute at De Bilt [1]). Every needle consists of a bulb, (capable of containing ink enough for recording during 48 hours), from which issues a pointed capillary tube, which rests on the paper. This tube is fixed to the bulb with the aid of sealing-wax. The whole is seized in a small clamp at the end of the shaft of the trolley.

IX. *The recording-drum* revolves in 24 hours and has an outline of 72 cms. Each hour is therefore 3 cms. and the minutes of $\frac{1}{2}$ mm. are easily read. For paper millimeter-paper is used, so that afterwards a scale has not to be added and the data can immediately be accurately read. The paper is fastened round the drum by means of narrow brass hoops.

The lines, written by the pencil when the trolley is drawn on by the clockwork U_1, form an angle with the ordinate, the tangent of which is: the quotient of the distance, travelled by the recording-drum in one minute and that which the trolley travels along its rail in one minute, or $\frac{1}{20}$.

As has already been mentioned above, the trolley is stopped after having run for 15 minutes. The clockwork need not stop, because the shaft of T_1 turns with friction. During that time the pencil writes a line parallel to the abscis (see fig. 2 and 3).

[1]) Dr. C. SCHOUTE, Director of the Institute, was kind enough to explain this method to me.

X. *General remarks*. The idea of using a tracing-pencil periodically moving backwards I got from the selfrecording anemometers used in meteorology. There the distance travelled is automatically differentiated as to time by the backward movement of the stiletto every 5 minutes, so the velocity is recorded. In the case of my apparatus the differentiation does not take place as usual according to time, but the velocity is taken as "the time needed to travel a certain distance".

In this connection I want to point out, that the recording apparatus by itself, therefore apart from the porometer, may be used for all kinds of other physiological inquiries, in which velocities have to be recorded. I hope soon to provide the "arc-indicator of growth" with an apparatus, enabling us to record the growth in the way mentioned above. Then the alterations in *rate* of growth, which are really wanted, may be directly read, while with the usual method, they must be derived from the inclination of the curve.

It may not be undesirable to consider the advantages of recording-instruments in general, apart from the special method followed here. In the first place we have the uninterrupted observation, next a great saving of time, for, when the construction is sufficiently sure, an apparatus as the one under discussion can work for 24 hours, without our having to look at it. Finally the accuracy with which everything can be regulated. For instance it is impossible for an experimentator to observe in the way followed here: an observation, during which he must patiently wait, then waiting for exactly 20 minutes, another observation, another 20 minutes waiting — that is slow work and not very encouraging which cannot be kept up very long. Moreover with slow movement of the meniscus it cannot be precisely stated, when the exact point is reached. The mercury-contact tells us this with minute accuracy. Indeed working with such slight differences in pressure (i.e. in circumstances so slightly unfavourable for the plant) as is possible here, is almost impracticable for an ordinary observer.

Let us finally consider what advantages the apparatus discussed here offers above the existing recording porometers and we find as general advantage: the possibility of introducing periods of rest with renewal of air between the observations.

Let us next consider separately the real porometer and the recording apparatus. The porometer itself may be built almost equally compactly as BALLS' stomatograph, and is therefore equally fit for use in hothouses or outside among the plants. The mechanism, which has to be in the immediate vicinity of the plant however is simpler

and, on account of the mercury-contacts, works more accurately. As a complication however it should be added, that the apparatus must be connected with an air-pump by means of a tube. The porometer of LAIDLAW and KNIGHT is in consequence of the necessity of placing a bath of a constant temperature around the *jar of Mariotte*, only practicable for laboratory-experiments of no long duration and is consequently inferior to the apparatus of BALLS and of myself, in spite of the pressure being almost constant. Moreover the forming and getting loose of a drop of water seems to me a less accurate measuring of time, than the change of level in a manometer. All methods mentioned however are to be preferred to that of JONES, in which the recording-apparatus forms one inseparable whole with the porometer.

The advantage of the electric connection between porometer and recording-apparatus, rendering an arbitrary distance between the two possible, cannot be valued too high. The plant can be examined in the most different circumstances, while the recording occurs in a fixed place and the apparatus suffers no injury from uncommon temperature or moisture.

The special adventages of the recording-method used have already been mentioned: 1°. The fact that the readings can be taken much more accurately, also of very short periods, without the unities along the time-axis having to be particularly large; 2°. writing a directly practicable graphic of the course of the phenomenon to be studied; 3°. the fact that it can be used for recording other physiological phenomena respecting velocities and gives a better survey of those, than has been done hitherto. These properties of the recording-apparatus make up for the fairly high costs of procuring, and make it an apparatus that can be regularly used in the laboratory.

The observations with a recording-apparatus that may be left alone for a whole day, require however, that the circumstances in which the plant finds itself, are recorded uninterruptedly. Otherwise it cannot be controlled what the occurring phenoma are due to. It would be ideal to record temperature, moisture etc. on one recording-drum with the vital phenomenon to be examined. Until this has been achieved, we must be satisfied with the existing apparatus for this purpose, the only inconvenience of which is, that their drums revolving in one week are too small for us to read the time accurately even for a period of 5 minutes. Through the kindness of Prof. E. VAN EVERDINGEN, Chief-Director of the Royal Dutch Meteorological Institute, I had some apparatus in loan during my experiments. At present the Laboratory of Plant Physiology possesses a thermograph, a hygrograph and a sunshine-autograph.

The results mentioned above would not have been attained, if I had not been sustained by the confidence in the success of the undertaking, which the director of the Laboratory, Prof. E. VERSCHAFFELT, has always shown, and by the aid in word and deed, I experienced from Mr. J. VAN DER ZWAAL, instrumentmaker, and Mr. J. MESSIAS, clockmaker, in the technical execution of the plans.

October 1920.

(From the Laboratory of Plant-Physiology of the University of Amsterdam).

Hydrology. — "*On the Motion of Ground Water in Frost and Thawing Weather.*" By Prof. EUG. DUBOIS.

(Communicated at the meeting of January 29, 1921).

From small pools, from detached ditches, especially with high sides, from wheel tracks, water is seen to disappear on prolonged frost from under the ice formed, so that beneath the ice there are air-filled spaces. The vanished water-layer can be from a few centimeters to some decimeters thick. The phenomenon is universally known, but the question what happens to the water, has not been answered as yet.

Other, equally common phenomena, are observed in thawing weather. Before the frost the soil may have been fairly dry near the surface, but without previous snow or rain it is found to be muddy on the still frozen substratum, as soon as thaw has set in. Not until the frost has quite gone from the ground, the superficial soil resumes its former, less wet condition, because then the excess of water sinks away. Whence this excess of water?

When the frost has gone from the ground, new-set plants that had not yet properly taken root, may be found "frozen up", that is partly, in some cases of small plants entirely, uprooted. By what cause?

Some winters, especially that of 1917/18, I had an opportunity to make observations in the "sand-diluvium" of central Limburg, which, I think, can throw some light on the causes of these phenomena.

The most important fact found, was that *in thawing weather the ground water rises.* Without snow or rain, and without superficial inflow of water, the level of the water, among others in ditches, after the ice in them had melted, is seen to rise in the district mentioned to such an amount as 1 cm. per twenty-four hours.

Hence there is displacement of ground water, during frost upwards, and during thaw downwards, and I imagine this to take place as follows.

The pressing action of the surface tension of the water that surrounds the ground grains, decreases with increasing diameter. Hence in the state of equilibrium the coarse ground grains are covered with thicker water layers than the fine ground grains. And just as

the liquid moves from a small soap-bubble towards a large one, in the same way water that surrounds the ground grains moves from the fine towards the coarse grains when the state of equilibrium has not yet been reached. So far as I know, E. RAMANN was the first to point out this important influence of the grain-size on the moisture of the soil.[1]) In the capillary spaces between the grains the height to which the water rises is also in inverse ratio to the cross-section of those interstitial spaces.

Now it seems to me that the same influence manifests itself in these phenomena of freezing and thawing of the ground.

For as soon as the frost penetrates into the ground, the ground grains become larger in consequence of their water-envelopes getting frozen, and then suck the water to them from the neighbouring, still unfrozen grains; this water freezes again, and thus the diameter of the solid grains gets greater and greater. In the same way the capillary spaces get narrower, so that ground water rises in them. The quantities of water that thus can be retained in the frozen parts of the ground, must be very considerable.

This appears in thawing weather from the muddy state of the ground at the surface, which thaws first. When also the lower layers are thawed, the water that has risen during the frost, can sink away, and return to the ground water.

Plants are not found uprooted through frost until it thaws. This may be explained in this way: when the ground thaws, differences of tension arise directed from below upward, through which the plants that have not yet firmly taken root, are ejected.

---

[1]) In the third edition of his "Bodenkunde", p. 332, (Berlin 1911).

Mathematics. — "*Two Representations of the Field of Circles on Point-Space.*" By Prof. JAN DE VRIES.

(Communicated at the meeting of January 29, 1921).

1. In 1917 these Proceedings (Vol. 19, p. 1130) contained a paper of Dr. K. W. WALSTRA on the representation of the circles of a plane on the points of space. In this representation a pencil of circles is replaced by a point-range, a net of circles by a field of points, and two orthogonal circles are represented by two points that are harmonically separated by a paraboloid of revolution, the points of which are the images of the point-circles of the field of circles.

Lately this representation has been investigated more closely and applied further by Dr. J. SMIT in his thesis entitled: "A Representation of the Field of Circles on Point-Space" (Utrecht 1920). We arrive also at this representation in the following way. Let A be a point outside the plane Φ of the circles c; through c and A a sphere is passed. If we consider its centre as the image of c the representation defined in this way shows all the above mentioned peculiarities.

2. In order to arrive at another representation of the field of circles we transform in the first place the plane Φ by inversion with centre N into a sphere β; the circles c are in this way replaced by circles c' of β. Now we consider the pole C of the plane γ' of c' as the *image* of the circle c. The *point-circles* P of Φ are, evidently, represented by the points P' of β. A *straight line* l of Φ is transformed by the inversion into a circle λ through N, is therefore represented by a point L of the plane ν touching β at N. N is apparently the image of the straight line at infinity of Φ.

3. A *pencil of circles* (c) is transformed by inversion into a "pencil" (c'), i.e. a system of which there passes one circle through any point of β, so that the planes γ' of the circles c' form a pencil, pass therefore through a straight line r'. But then the poles C lie in a straight line r (the polar line of r' with respect to β). Also in this representation a *pencil of circles* is therefore transformed into a *point-range*.

If (c) has the base point B_1 and B_2 also the circles c' pass through two fixed points and the image r of (c) lies outside β.

If on the contrary (c) has two point-circles P_1 and P_2, r' is the intersection of the planes touching β at P'_1 and P'_2 and the image r cuts the sphere.

The image of a *pencil of concentrical circles* is evidently a straight line r through the point N.

A *parabolical pencil of circles* is represented by a tangent of β. Any two circles of such a pencil touch at a point P; the images of two touching circles are therefore joined by a tangent.

4. A *net of circles* $[c]$ is transformed by inversion into a "net" $[c']$; the planes γ' pass through a fixed point S, consequently the images C lie in a *plane* σ, the polar plane of S.

The image of a net with base-point P is the plane touching β at P'.

All the circles c cutting a circle c_0 *at right angles*, form a net $[c]$; to this belong the points P of c_0. As these points may be considered as circles touching c_0, they have their images in the points of contact of the tangents of β meeting in the image C_0. Consequently the net is represented by the polar plane of C_0. The images of two *orthogonal circles* are therefore *harmonically separated* by β. The sphere β plays here the same part as the paraboloid in the above mentioned representation.

All the circles intersecting c_0 *diametrically* also form a net, $[c^*]$. As $[c^*]$ has no circle in common with the net of the circles intersecting c_0 at right angles, the image σ^* is parallel to the plane σ of c_0. To $[c^*]$ belongs also the circle c_0: hence σ^* passes through C_0.

5. An arbitrary conic δ^2 contains the image of a *system* $(c)_2$, with *index two*; for the tangent plane ϱ of a point R' has two points in common with δ and these are images of two circles c through the point R. The system $(c)_2$ belongs to the net $[c]$ which is represented by the plane of δ^2. [1])

If the plane ϱ touching β at R' also touches δ^2, R' is the central projection of a point R of the curve enveloped by $(c)_2$. Now let L be the image of the straight line l in Φ; the enveloping cone of β the vertex of which is L, has four tangent planes ϱ in common

[1]) The orthoptical circles of a pencil of conics form a system $(c)_2$. For through a point of the straight line at infinity pass the degenerate circles consisting of l_∞ and the director lines of the two parabolas. The point-circles of the system are found in the double points of the three pairs of lines and in the centra (the orthogonal circle of the net to which $(c)_2$ belongs) having its centre in the point of intersection of the two director lines.

with the cone projecting σ out of L. From this it ensues that the system $(c)_2$ is enveloped by a *curve of the fourth order*.

The points of intersection of σ^4 with β are the image of four point-circles belonging to $(c)_2$; the points of intersection of σ^4 with the plane ν represent the two straight lines of $(c)_2$.

6. A twisted cubic σ^3 is the image of a system $(c)_2$ with *index three*. At such a system we can arrive in the following way. Let us consider three projective pencils of circles (c_1), (c_2), (c_3) in Φ; let c be the circle intersecting the homologous circles c_1, c_2 and c_3 at right angles. The image C of c is the point of intersection of the polar planes γ_1, γ_2, γ_3 of the images C_1, C_2, C_3. These planes revolve round the polar lines r_1, r_2, r_3 of the straight lines r^*_1, r^*_2, r^*_3 containing the images C_1, C_2, C_3; the locus of the point C is accordingly a twisted cubic, σ^3. Apparently we can inversely choose for r_1, r_2, r_3 three arbitrary chords of a given curve σ^3; their polar lines with respect to β define in Φ three projective pencils of circles, which in their turn define the system $(c)_2$ which has σ^3 its image.

7. A plane curve σ^3 is the image of a $(c)_2$ belonging to the net that is represented by the plane σ of σ^3. A tangent plane ϱ of β intersects σ^3 in three points of the straight line $\sigma\varrho$; as a second tangent plane to β can be passed through this straight line, the system $(c)_2$ is characterized by the property that the three circles through a point P have another P^* in common. If $\sigma\varrho$ is a tangent to β the three circles touch in a point P; this point of contact lies evidently on the orthogonal circle (diametrical circle) of the net. In a special case the orthogonal circle can be replaced by a straight line, containing in this case the centres of the circles of the system $(c)_2$.

8. Let O be the centre of the sphere β. If the images C_1 and C_0 of two circles c_1 and c_0 are such that $\angle OC_0C_1$ is a right angle, c_0 is intersected diametrically by c_1 (§ 4). If C_1 is fixed C_0 remains on the sphere Γ having OC_1 as diameter. This sphere is apparently the image of the twofold infinite system of circles c that are intersected diametrically by the fixed circle c_1. The intersection of two tangent planes of β has two points in common with Γ; hence through two points there generally pass two circles of the system. A pencil of circles contains also two circles of the system.

9. We arrive at *another representation* of the field of circles in the following way. In the plane Φ of the field there are assumed

three arbitrary points K, L, M; the powers of the circle c with respect to these points are considered to be the coordinates x, y, z of a point C with respect to an orthogonal system of axes.

The plane $x=0$ contains the images of the circles passing through K. As a *pencil of circles* (c) sends one circle through K, the image of (c) has one point in common with $x=0$ and is therefore also in this case a *straight line*. As further a pencil (c) has one circle in common with a net $[c]$, a *net* is represented by a *plane*.

A pencil (c) has two *pointcircles*; the locus of the images of the pointcircles P is again a *quadratic surface* Φ^2. We find its equation by making use of the well known relation between the sides of the complete quadrilateral $PKLM$.[1]) Substituting there $KL^2=h$, $LM^2=f$, $MK^2=g$; we find after some reduction,

$$fx^2+gy^2+hz^2+(h-f-g)(xy+hz)+(f-g-h)(yz+fx)+ \\ +(g-h-f)(zx+gy)+fgh=0.$$

The plane $x=0$ contains only the image of the point-circle K; from this follows that Φ^2 touches the coordinate planes.

Any circle concentrical with the circle KLM, has equal powers relative to K, L and M, is therefore represented by a point of the straight line $x=y=z$; as a concentrical pencil contains only one point-circle at finite distance, Φ^2 must be an *elliptical paraboloid* the diameters of which make equal angles with the three coordinate axes.

If we choose K, L, M in the angular points of an equilateral triangle, so that $f=g=h$, Φ^2 becomes apparently a *paraboloid of revolution*.

---

1) See e.g. SALMON-FIEDLER, Anal. Geom. des Raumes I (1879) p. 74.

KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN
TE AMSTERDAM.

PROCEEDINGS

VOLUME XXIII

N°. 9.

President: Prof. H. A. LORENTZ.

Secretary: Prof. L. BOLK.

(Translated from: "Verslag van de gewone vergaderingen der Wis- en Natuurkundige Afdeeling," Vols. XXVIII and XXIX).

CONTENTS.

Anatomy. — "*On the character of morphological modifications in consequence of affections of the endocrine organs*". By Prof. L. BOLK.

(Communicated at the meeting of February 26, 1921).

By endocrine organs are meant a group of organs whose right functioning is requisite for the regular development of the body. Besides these organs exercise a regulating influence on the metabolic processes in the body. This latter function, although hardly separable from the former, bears a more physiological character, and therefore is beyond the scope of the following considerations which treat mainly of the significance of these glands for morphology.

The principal of these glands are: Hypophysis, Epiphysis, Glandulae suprarenales, Glandula thyroidea, Thymus and the so-called interstitial gland.

The influence of these glands appears from the fact that, if their right functioning is disturbed, whether by abnormal development, or by affections at a later age, the regular morphogenesis of the body is disturbed in some respect or other, or the adult body shows characteristic variations in form. These variations differ greatly as to their nature, and clinical observations as well as experimental research have of late years enabled us to reduce definite metamorphoses to affections of one or more of these organs. The morphological modifications are so divergent in character, that any relation or connection between them seems wanting. And still, in this communication, the facts will be set forth in such a way, that a relation will become evident, and at the same time a new perspective will be opened on the part that these organs play, or rather have played.

For the better understanding what follows, it is desirable, in this place already, to set forth this perspective, and to announce the conclusion, to which the facts point. This conclusion runs as follows: with abnormal functioning of the endocrine organs, such morphological properties are developed again in the human body, as it had lost during the last phase of its phylogenetic development. The latest phase of his phylogenesis was that in which man got his morphological specifically human characteristics, such as distinguished him from his nearest relations, the Anthropoids and the other Pri-

mates. Well then, affection of the endocrine organs asserts its influence in the first place on these specifically human characteristics.

From an à posteriori point of view, this is really quite natural and perfectly logical. The endocrine organs regulate the development of the form up to the most trifling details, and least considerable properties, they control the morphogenesis. In the man of to-day they co-operate in such a manner as to originate the Homo recens. But in the ancestor of man, with his rather pithecoid characters, their co-operation was such as to originate a more primitive form. During evolution the activity of the endocrine organs must therefore have undergone a modification, a morphological character that in a primitive form acquired a certain degree of development, in a higher form must have altered its development. That alteration may be either a regressive one — so retardation or suppression of the development — or a progressive one, so a more forcible development. What I will call the architectonic function of the endocrine organs in man must be somewhat different from, say, that of Gorilla, or Chimpanzee. If the endocrine organs control the morphogenesis, then they must also adapt themselves in their function to the phylogenetic metamorphoses. Is it, however, really beyond all doubt, that we have to speak of adaptation in this case? Did the endocrine organs indeed play only the passive part of adaptation, in this process, or was that part rather an active one, if not a directing, then, at least, a regulating one? The question is one of biological principle, for it is immediately related to the question of even wider bearing, whether evolution has only been brought about by the influence of external circumstances, or rather by such influence in connection with an internal agent. In a former publication I wrote in favour of the last named opinion and I then pointed out the possibility that the endocrine organs have played an important part in evolution. It was then impossible to me to arrive at any notion as to the nature of that relation between their function and evolution. I believe now to have advanced a step, as it has become evident to me, that, at least in the bringing about of the specifically human characteristics, that influence has always acted in the same direction.

The differences in form between related species may be brought about either by something existant developing more strongly, or being suppressed in its development. In this latter case, therefore. the influence of the endocrine organs must have been a retardative one. Well then, I propose to demonstrate that the development of the specifically human characteristics is the result of a retardative

or suppressing activity of the endocrine organs, and that, in case of affection or abnormal development of those organs that suppressing influence is lost, so that the modifications which then appear in the body, have a pithecoid character.

I hope further to draw your attention to phenomena from which we may conclude that that retardative influence not only corcerns the merely morphological faculties, but that the entire process of development, the developmental rate of the human individual, as compared to that of the other Primates, is retardated.

One of the first characteristics by which man is distinguished from all other Primates is his hairlessness. On the cause of this Darwin is not always of the same opinion. At one time he is inclined to consider the nakedness of man as the result of sexual selection, at other times he leaves scope for the possibility of climatological influences having been at work. I have no space for a criticism of these opinions. I shall restrict myself to the observation that the foetus of all Primates is hairless and that therefore, in the genesis of man, the part played by the endocrine organs consisted in suppressing the potentiality for hair-development, with the exception of that part of the skin which covers the skull. The hairlessness of man is therefore the persistance of a foetal character. It is a well-known fact that now and then this suppressing influence on the development of the hair-covering is removed and individuals are born with a more or less complete hair-coat, the so-called hair-men.

Which of the endocrine organs was it, that had a suppressing influence on the formation of the hair-covering in man? We can indicate it with a good deal of certainty. This organ is the Hypophysis. For we know that with hyperfunction of this organ in man as well as in woman, a hair-covering may be developed in an extremely strong degree, so that the limbs and trunk in particular are thickly covered with long hairs.

So we have here our first instance of an affection of an endocrine organ influencing a specifically human character, and that in such a way, that a pithecoid condition returns. However, an affection of Hypophysis is present in yet another series of phenomena, which bear the same character and manifest themselves in particular at the skull.

The skull of apes, especially of anthropoids, is distinguished from man's, by the marked development of the jaws, the so-called supraorbital-ridges also attaining an extraordinary size. The anthropoids are what is called markedly prognathic, man is orthognathic, whereas his supraorbital-ridges have hardly come to any development at all.

Again orthognathy is a foetal character, not only of man, but

also of apes. It is only gradually, that in the ape-foetus and the young ape-child, the jaws extend forward. In man this growth is again suppressed, he retains his foetal character. And this retardation also takes place under the influence of hypophysis. For we know that when this organ is affected, both jaws, but particularly the lower jaw can enlarge considerably; the suppressing influence therefore, which proceeded from this organ — and which, in normal circumstances, keeps this potentiality of development in a latent condition, for the whole of life — is removed, and the old pithecoid factor of development tries to re-assert itself. Not only, however, for the jaws, also for the frontal-ridges. Do not many historia morbi of such patients mention a gradually stronger development of these ridges?

Not all human races however are equally orthognathic; the stronger prognathy of the "black" races is well-known. From this it might be inferred that that suppressing influence acted less strongly in the latter races than in the white ones. This assertion is supported by another character to which I want to draw your attention, viz. the colour of the human skin, and especiably that of the white races.

The colour of the skin is defined by the development to a more or less degree of the so-called skin-pigment. This is developed in all Primates, and as regards man, particularly in the black races. So the hardly if at all pigmented skin of the white races has again been brought about by a retardation or suppression of the development of skin-pigment. And here again we have a persistence of a foetal character. The white skin of man, therefore, is not a new-acquired property, for it is a condition found temporarily in all fetal primate individuals, the further developmental phase, which follows in the other Primates and also in the black races, being suppressed in the white races. And that this influence is active already in negroes, is evident from the fact that negrochildren are born with a white skin, and only become black shortly after birth. Therefore the white has, in a way, retained a foetal character, which the negro has lost, consequently the retardation process in the white is stronger than in the black, in complete accordance with what could be ascertained with regard to the development of the jaws. So human evolution has, at least with regard to the morphological properties, progressed further in the white race than in the black races. How far the same applies to the mental qualities may here be left out of discussion.

So the whiteness of the human skin is again an instance of retardation. Which endocrine organ is responsible here? To this

question we can also give a rather exact answer. It is the adrenals. Since long we have known a disease first described in 1855 by ADDISON, and also named after him, the main symptom of which is an abnormally strong development of pigment in the skin, so that the latter acquires a colour between bronze and dark-brown. This pigment is not of haematogeneous origin, for it is without iron, as is the skin-pigment of the coloured races. With autopsy of persons who have died of this disease, we find, almost without exception, degeneration of the adrenals, generally of a tuberculous nature.

I proceed with another phenomenon. A typical difference between man and the other Primates consists in the fact that the sutures of the skull in primates close very early. With anthropoids, where they remain longest open, they still disappear before the individual is adult. In man, on the other hand, the sutures remain much longer, and the closing bears more of the character of a symptom of senility. The fact of the sutures remaining open is not to be ascribed to the considerable development of the brain in man. If this were the case, they might also close in man immediately after the termination of the growth of the brain, that is, at a comparatively early age. The fact of the sutures remaining open in man, is again to be considered as the persistence of a foetal character, to which I drew attention, a few years ago, in a monograph on this subject. This persistence is again the consequence of a retardation or suppression of the growing-together-process, which manifests itself in the other Primates in an earlier or later phase of their development. This process consists in the ossification of the fibrous tissue that separated the parts of the skeleton, the adjoining parts of the skeleton becoming one whole. This ossification-process is retarded in man and shifted to a later phase of his life. Even in a centenarian WALDEYER found sutures. Now by the influence of which endocrine gland was this retardation brought about? The answer to this must be: under the influence of Thymus. The significance of this organ is not yet known in all respects, but we do know, that it is in the first place the skeletogeneous processes in the body which are influenced by this gland.

When thymus is removed experimentally in youthful animals, or when this gland has from necessity to be removed surgically in young human beings, serious disturbances appear after a short time in the ostegeneous processes. Mostly affection of a rachitical nature are the result. Now one of the symptoms of rachitis is the so-called premature closure of the sutures of the skull, that is to say, the sutures disappear even before the individual is adult, sometimes

even at a very youthful age. Even in the rachitical hydrocephalus, in whom the skull under the influence of the increased intracranial pressure, may strongly increase in size, this premature ossification may manifest itself. So here again we state that with affection of one of the endocrine organs, by the removal of a retardation-process, a specifically human character is lost, the bones of the skull growing together, as is normal for other Primates.

So now we have met with four specifically human characters which are to be considered as persisting foetal characteristics, and which, when any of the endocrine organs is affected, can undergo alterations in a pithecoid direction: his hairlessness, his orthognathy, his white skin, his persisting sutures of the skull. Each of these characters is the result of a retardation in the development and this retardative influence is removed with degeneration of any of the endocrine organs. I could bring under this group another phenomenon, viz. the slight development and slender build of the hands and feet in man, in contradistinction to those of the anthropomorphous apes. If sufficient time were at my disposal I would prove by a somewhat detailed anatomical demonstration, that in outward appearance as well as in structure, especially the foot of man has retained a foetal character. So here again we are confronted with a retardation in development. And this retardative influence is bound at the function of the Hypophysis. For with affection of this organ this retardative influence can again be removed, hand and foot become inelegant in shape, grow, and may considerably increase in size.

We now come to a second group of phenomena, in which that retardative influence of the endocrine organs manifests itself in a somewhat different manner.

The genital organs, especially in woman, differ from those of the female individuals of the other Primates in a very particular way by the presence of the so-called mons veneris and Labia majora. These form therefore a specifically human characteristic and a typical difference with the apes. The absence of these characters in apes is still of some historical importance as far as, at the time, BISCHOFF the Munich anatomist, adduced this difference, as well as the absence of a hymen in apes as a great argument against the Darwinian theory about the relation between man and the other Primates. Yet BISCHOFF was wrong, for though it may be true, that the organs mentioned are absent in born individuals, if we examine the foetal stages of development of these animals, we obtain a different view. The organs mentioned originate from a wall-shaped eminence, which is formed round the genital

orifice, not only in man, but in all Primates. In the lower apes this wall soon disappears, in the foetus of anthropoids on the other hand it continues to exist, as appears from the observations of DENIKER, SPERINO, myself, and others, so that the older foetus of these apes has, in front of the vestibulum a mons veneris and at the sides labia majora, be it that they are not nearly so strongly developed as in man. After birth these characters seem to disappear soon, also in anthropoids, in man they continue to exist. So we have here again a typical human character, which is nothing but a persisting foetal property.

And here also we are struck by the fact that the difference between man and the apes is the result of a developmental retardation, but now of a somewhat different character from the preceding cases. For though in the other Primates the genital wall disappears at last, and the *disappearance* is therefore the normal process, it continues to exist in man and increases in size in accordance with the general growth. So here the retardation did not mean the suppression of the coming about of a morphological character, but the prevention of the disappearance of a very early condition. Which endocrine organ may be thought of in relation to this phenomenon? By many clinical observations we are able to answer also this question. It is almost certain that we must here think of the influence of glandula thyroidea. Congenital absence, or too slight development of this gland is accompanied by an insufficient development of the exterior genitals, the labia majora and the mons veneris are sometimes missing. The genital apparatus therefore, has in such cases, been formed as in the other Primates, the specifical human character having developed in an insufficient manner. The influence of Glandula thyroidea on this human character appears further from the fact that after extirpation of this endocrine organ atrophy of the genitals is resulting, adhibition of the extract of this gland brings atrophied or insufficiently developed external genitals to a stronger development.

The case that we have last discussed forms a welcome transition to a specifical human charactar, that I now want to discuss briefly and with which I shall conclude the casuistic part of my contribution. *The* specifically human character par excellence is the great weight of the brain. Is this also the result of a retardative influence, you will ask somewhat sceptically? I reply, yes, however paradoxical this may sound.

Let us therefore briefly take a nearer view of the part played by the endocrine organs in morphogenesis. This part consists in these organs ensuring a harmonious development of the form. An har-

monious development is brought about when the rate of development of each part of the body is exactly dosed and the duration of development exactly limited. This rate and this duration are different for each part and so in every stage of development there will be a different correlation between the organs. As an instance I mention the heart. In the first phase of embryonal development this organ grows very rapidly, and is already completely formed when other organs are stil more or less far removed from their final stages. For each organ comes the moment when it has reached its definitive size-relation to the whole, after which it further increases in size in accordance with the general growth. If, for instance, the size-correlation of the heart or the kidneys, as it exists in the beginning of the second month of development in the human foetus, would be made permanent, then individuals would be born with monstrously developed heart or kidneys. But in the course of the further development heart and kidneys decrease again in relative size.

Now this same point of view also applies to the brain. It also increases enormously in size in the first developmental phase, not only in man, but in all Primates. Of all Primates the foetus passes a developmental stage in which with regard to the whole body the brain has a monstrous degree of development. It afterwards decreases relatively in size until a definite phase of development, after which regularly to take part in the general growth. Now the difference between man and the apes is, that this happens in man in an earlier stage of development than in the other Primates, therefore the correlative relation in him was fixed in a younger phase when the brain was relatively larger. So here again the principle of retardation finds expression. The sooner a limit is put to the relative diminution in size, the larger the brain will become, the longer this relative diminution lasts during the development, the smaller the definitive weight of the brain becomes.

I have done my best to give my view in as succinct a form as possible; I hope that the intelligibility has not suffered too much by doing so.

Under the influence of which organs stood this retardative process? In order to answer this question let us examine what is the consequence if the correlation is not fixed at the point of time normal for man, but, as in the other primates at a more or less later moment in the development, when the brain in regard to the entire organism has become relatively smaller.

Then also in man, therefore, a decreased mass-relation of the brain to the whole is brought about, the wellknown microcephaly

with the accompanying idiocy. And with this the answer has been given to the question asked, for clinical observations of this disease point most decidedly in the direction of glandula thyroidea.

It will not have escaped the attentive hearer that in the preceding, not one, but several points of view have been discussed collectively. The first point of view I could summarize in this thesis: the specifically human characters are persisting foetal properties; the second point of view in this: the specifically human characters originate in the general developmental process of primates having been retarded in certain parts; the third view in this; this retardation is caused by the various endocrine organs, and finally the fourth in this: with affection of the endocrine organs this retardative influence is removed and the human body recovers pithecoid properties. Now I cannot conclude this contribution before, on the basis of the theses just mentioned, opening up before you another point of view of more general biological interest. The specifically morphological characteristics of man are a result of retardation; well then, this influence of the endocrine organs not only concerns the development of morphological properties, but has stamped the entire developmental process of man as such, his rate of development has been slackened, his youth, as compared with the other Primates, has been lengthened, the adult phase of his life also, and perhaps also his phase of senility. This is an idea which I had carried about with me for years, but for which I had really never been able to find a proper correlation to other views or observations, until I at last found it in my view of the significance of the endocrine organs for the bringing about of the morphological properties during evolution.

Still this idea was formerly not a merely intuitive one, it was founded on a fact. This was supplied by the peculiar behaviour of the genital glands in human beings, particularly in woman. These glands show the remarkable phenomenon that after that the specifical elements have developed, and no longer increase by division — which is already the case in the second year — a phase of rest begins. The histological differentiation of the genital glands is finished in the second year, but their function only begins at a later age — say between the 12th and the 14th year. Between the histological final stage and the beginning of the function a pause, a latent period is shoved in. This latent phase in human beings has always seemed something very remarkable to me. And, to put it briefly, it has always seemed to me that a discongruency has arisen between the sexual and the somatic development. How far this also occurs in other mammals may here be left undiscussed; so much is certain

however, that it is not the normal relation in mammals. This discongruency can only have been brought about by a retardation of the somatic development, not by an acceleration of the development of the genital glands, as this would be absolutely useless. But with this retardation of the rate of the somatic development the function of the genital glands had of necessity to be suppressed so long, till the body was sufficiently developed to be able to pass through the whole physiological process which follows a conception. So I see here again the activity of two influences: retardation in the developmental rate of the body as a whole, and temporary suppression of the function of the genital glands.

Here also I touched but slightly on my line of thought, to conclude with the question: have we to do in this case also, with the influence of an endocrine organ? The clinical observations gives us again the answer to this, and refers us to those, happily rare, but sad cases of so-called premature sexual development. Children in whom puberty begins at the age of four or five, girls, as cases are known, who conceive at the age of seven or eight.

How are these abnormal conditions to be explained? In my opinion here also the solution of the problem is to be sought in the removal of the retardative influence on the developmental rate of the sexual development, which, as appears from what was said above, exists in man. And like all those mentioned before, this retardative influence also originates with an endocrine organ, for, as is proved conclusively by autopsy, this premature sexual development is the result of a degeneration of the pineal gland. With this I wish to conclude my exposition. The abundance of matter has compelled me to give my contribution in as succinct a form as possible. That the subject-matter is not exhausted with what is here discussed, will be clear to anybody more or less expert. For the perspectives here opened up go in two directions. In the first place they give to the physician a wider view of the character of the conditions observed by him, and to the physiologist they submit the question to examine whether in the province of metabolic phenomena, with affection of the endocrine organs, he can make observations which are parallel to the morphological consequences of these affections. Personally I am more attracted by the second direction in which the facts point. That is the general biological one. In the exposition given it has been shown that the endocrine organs influence the coming about of the evolutional development of a higher organism, it is therefore, in my opinion, no longer to be denied, that internal factors play a part in evolution.

Still I must warn against a possible one-sided interpretation of the part that these organs can play in the bringing about of morphological differences. In the evolution of man this part was, say, pre-eminently a retardative one. I have not been able to find any indication of an activating influence. But this is not saying that in other cases an activating influence may not exist, causing an accentuation in younger forms of characteristics which existed in principle in their ancestors.

And, to conclude, another question, or rather a problem. I have advanced a ground to prove that the developmental rate of man has been retarded, a retardation which, by the way, is stronger in the male than in the female sex. To this retardation-process he owes it, I would observe, that he is born "nudus et inermis", that, in contradistinction to the other mammals, it is only rather a long time after his birth, that his consciousness of self awakes, followed by the longer infantile, puerile and juvenile phases.

This is a privilege which man has over other organisms. But — would it be overbold to attach to this view the remark, that he has had to pay for this privilege with: greater sensitiveness to disease-causing influences as a consequence of a weakened capacity for resistance? A retardation of the vital processes means a decrease of their intensity. A decreased intensity of the vital processes means weakening of staying-power; weakening of staying-power means increased sensitiveness to noxious influences. And is there any other organism to be found, so sensitive to noxious influences, so subject to disease as man? With this question I wish to close my contribution.

Physiology. — "*Concerning the Influence of the Adrenal Cortex upon the Growth and the Reproduction of Lower Organisms and its presumable Antitoxic Action*". By Miss M. A. VAN HERWERDEN. (Communicated by Prof. C. A. PEKELHARING).

(Communicated at the meeting of March 26, 1921).

I have been studying the influence of various tissue-extracts on the cyclic reproduction in a race of Daphnia pulex bred for many years in the laboratory. I thereby hit upon a phenomenon which led me to make further experiments. Although these experiments are still in progress, I consider it worth while to say a few words about this problem here, as it is one of general importance.

The first time when I added a small quantity of dried adrenal cortex from the ox to my Cladocera cultures, it appeared to produce a peculiar effect. First of all the fecundity of parthenogenetic females is considerably intensified, sexual maturity commences sooner than in the control-animals (for which I always selected sisters from one and the same brood) and the generations follow each other in quicker succession. Not infrequently as many as three broods have been deposited, while the control-sisters are still bearing their first. It also often happens that the first generations have reached maturity while the ovary of the untreated sisters is still infantile. This great difference may be seen as well when the control-culture contains only ditch-water with unicellular algae, as when for purposes of comparison an equal quantity of adrenal medulla was added insteadof cortex.

Furthermore there is another very remarkable influence of the adrenal cortex upon the cultures, which is not exerted by the medulla. It is to this phenomenon that I wish to call special attention. It is well-known that Daphniae do not tolerate multicellular algae in the culture-glasses. The presence of these long filaments invariably occasion in my cultures depression, and ultimately induce death, if the animals are not transmitted in time to fresh ditch-water with unicellular algae. It now appeared that the addition of a small quantum of dried adrenal cortex, or an aqueous extract of it, was sufficient to ensure a healthy life to the Daphniae in a tangle of

long algae even without cleaning the glasses regularly. More remarkable still is that different mould-mycelia, (otherwise invariably destroying a culture within a short time), do not at all affect the health of Daphniae and do not interfere with their rapid reproduction in the adrenal-*cortex* cultures where the moulds are speedily developed.

I have availed myself of the above phenomenon in keeping my insufficiently cleaned cultures alive during the holidays. Such a densely populated culture, which contains per 25 c.c. of ditchwater, say, 5 mgrms of dried adrenal cortex, mocks at all the care generally bestowed upon an organism like Daphnia pulex, which needs much oxygen and is otherwise greatly affected by impurity and thrives best when transmitted to fresh ditchwater every week.

The fresh adrenal cortex of the ox was carefully separated from the medulla, minced up, dried for 24 hrs in an incubator at a temperature of 60° and then pulverized. By due caution one really succeeds in getting cortex tissue free from the medulla.

Moreover an admixture of a minimal quantity of medulla gives a pink colouration of the ditchwater [1]), so that any contamination may be directly recognized.

Besides it has appeared that the medulla (if sufficiently free from cortex tissue) [2]) lacks the influence upon Daphniae described above and its prolongated action even seems often to be noxious.

The quantity of dried adrenal cortex, just giving a positive result, amounted to 1 mgr. per 20 cc. of ditchwater. No experiments were made with smaller quanta.

When we distribute sisters from the same brood over culture-glasses to which respectively an equal quantum of fresh aqueous extract [3]) of dried adrenal cortex, adrenal medulla, thyroid gland and hypophysis of the ox is added, the medulla appears to have the least favourable effect. Thyroid-gland extract is often tolerated less in the beginning than after prolonged administration (tachycardia), the reproduction of Daphniae is much less intensive than in the adrenal cortex cultures, stronger though than in the control-cultures, and the same holds for the hypophysis-cultures.

The extremely rapid growth of algae in the culture-glasses to which adrenal-*cortex* is added, is very striking. By placing the

---

1) This is a very sensitive adrenalin-reaction, appearing in the presence of oxygen (Biedl I, p. 527).

2) It is much more difficult to obtain adrenal medulla free from cortex-tissue than vice versa.

3) 1 cc. of aqueous extract (prepared 24 hrs previously by adding 0,5 gr. to 25 c.c. of water at room-temperature was added to 10 c.c. of ditchwater.

animals during the experiment in a badly lighted space it may be observed, that only the addition of the adrenal-cortex tissue is answerable for the favourable condition of the cultures, but *not* the rapid growth of the algae. It has, for that matter, already been stated that the growth of multicellular algae is just a factor which, on the contrary, inhibits the development of Daphniae.

An aqueous extract of the dried adrenal-cortex (0,5 gr. per 25 cc.), which had been heated for 2 hours at 110° in the autoclave, proved to have the same favourable influence as the non-heated extract on the growth of the algae. Judging from the preliminary experiences it also seemed to promote the reproduction of the Daphniae. Spontaneous growth of moulds does not occur in these cultures, so that they have still to be infected in order to ascertain whether the prolonged heating leaves intact the substance, wich counteracts the effect of moulds in the cultures.

In the spring of the year 1920 I noted an extraordinary difference in the growth of eggs of the watersnail (Limnaea ovata), according as they were treated with dried adrenal-cortex, or not. A mass of jelly containing eggs, originating from a single female, has been cut in two halves, each of which has been put in a glass filled with ditchwater. During the experiment an equal amount of algae was added to each glass. After the lapse of three weeks the size of all the young snails, contained in the glass, to which a few milligrams of dried cortex of the suprarenal capsule had been added, surpassed several times the size of those of the controlculture or the adrenal medulla cultures.

It has been shown, that the adrenal cortex contains a substance, soluble in water, which in these invertebrates exerts an influence upon their fecundity and upon their health. Factors, which are in other circumstances noxious to Daphniae, such as caused by overpopulation, inadequate supply of oxygen, the growth of moulds and of multicellular algae, are removed by the addition of very small quantities of dried *cortex* of the suprarenal gland. This influence seems to exist also among lower plants, witness the highly intensified growth of unicellular and multicellular algae. Sexual maturity is accelerated, the embryonal development is promoted, the broods follow each other in quicker succession. In the snail the rate of growth of the tissues has largely increased during the embryonal and the post-embryonal stage.

The study of adrenalin has pushed the question of the significance of the adrenal-*cortex* tissue into the background for a considerable length of time, until it was brought under our notice again, notably

by A. Biedl. In addition to the connection between the function of the cortex and that of the sexual organs, also the possibility of an antitoxic action of the adrenal cortex has been emphasized. In the latest edition of his Manual of Internal Secretion Biedl still considers it as an open question whether, just as Cobra-poison in vitro, also endogenous poisons can be counter-acted by adrenal-cortex.

It seems to me that the result of adding small quantities of adrenal cortex to the Daphnia cultures — in the case of over-population and of mould infection — indeed points to an inhibitory influence on the action of normal metabolic products. It is notable that this occurs in lower organisms, which, so far as we know, do not possess an organ corresponding to the suprarenal capsule of the vertebrata.

Our deficient knowledge of the normal function of the adrenal cortex justifies a further inquiry of this problem in different directions.

Physiology. — "*The Interchange between Blood-plasm on one hand and Humor aqueus and cerebro-spinal fluid on the other hand, studied from their sugar-percentages and in connection with the problem of combined sugar.*" By J. DE HAAN and S. VAN CREVELD. (Communicated by Prof. H. J. HAMBURGER.)

(Communicated at the meeting of March 26, 1921).

Humor aqueus and cerebro-spinal fluid are two very remarkable tissue-liquids. They are so in the first place because they are almost entirely free from colloids and secondly because of the great similarity between their chemical composition and that of the blood. A large part of the present-day investigators are of opinion that these liquids must be regarded as formed by "active secretion" by certain layers of cells, namely the epithelial layer of the corpus ciliare and the chorioid plexus. HALLIBURTON and DIXON [1]) a.o. have accepted this secretion especially for the cerebro-spinal liquid on account of (among other things) the discongruity which would exist between the specific action of certain substances on the secretion of the liquid and the action of those substances on the *pressure* of the blood. It is, however, exceedingly difficult to establish experimentally and without any doubt, an increased formation of the liquid; and equally difficult is the exact determination of the blood-pressure in the vascular system, connected with the formation of the liquid. We here only refer to the recent elaborate publications of BECHT [2]) in which from his own researches this author reaches the conclusion that all the phenomena which at first sight point to secretion, may very well be explained in a mechanical way. For the rest we will leave out of consideration the significance of the blood-pressure, nor will we, in what follows, discuss the value of the argument that the histological gland-structure of the said epithelial cells should furnish a proof for the secretion. Leaving alone the way the liquid is formed and the place where it originates, in other words leaving alone the direction of its movement, we wish to take as starting-point for our researches the chemical composition of these liquids.

And then he who speaks of secretion here, will admit that the

[1]) Journ. of Physiology **47**, p. 215, 1913 and **48**, p. 128, 1914.

[2]) Americ. Journal of Physiology **51**, 1, 1920.

secreting cells "produce" a substance which remarkably resembles the blood-liquid and which therefore those cells hardly alter actively: osmotic pressure, concentration of the various salts and crystalloids and of the H-ions, they all oscillate within narrow limits round values which are practically those always indicated for the blood.[1]) When, however, one states that the secreting membrane actively checks crystalloids (fluorescin for the liquid of the eye-chamber, aceton and other substances in the cerebro-spinal fluid), then the researches, justifying these conclusions, are nearly always open to criticism both on account of the method used and on account of their varying results[2]). The fact that these liquids contain only traces of proteins and that ferments, immune bodies are practically absent from them, does not tell at all for an active, vital stopping. For every well-functioning dialysing-membrane, every ultra-filter will do this too.

What follows here is the provisional brief communication[3]) of the results of an investigation to find an answer to the question: in how far can the said liquids be regarded as ultra-filtrates or, rather, as dialysates? For the name ultra-filtrate indicates a liquid which is pressed through by means of a super-pressure (in this case the pressure of the blood) and with which a not unimportant speed of circulation is supposed. Now it is very probable, especially for the liquid of the eye-chamber that under physiological conditions the movement of the liquid is very slow. When these liquids follow the fluctuations in the composition of the blood, this will be the consequence of a process, of diffusion for the greater part, and further of direct filtration; consequently in our view a combination of dialysis and ultra-filtration. In how far does the composition of these liquids correspond to what we must expect if the separating layer between the latter and the primary fluctuating liquid (blood) acts as a simple dialysing-membrane? We need not find complete similarity: for the liquid interacts not only with the blood, but also with the remaining surroundings (cerebral tissue, tissues round the eye-chamber). But it will be especially interesting to trace, which changes should be attributed to the last-mentioned factor.

For the present we have limited our investigations to one of the substances which occur normally in the blood, namely: *glucose.*

---

[1]) See a.o. OSBORNE: Journ. of Physiology, **52**, p. 347, 1918—1919.

[2]) We hope to have an opportunity of returning to this question in further researches.

[3]) A more elaborate publication will appear in the Biochemische Zeitschrift.

Our choice is explained by the fact that one of us [1]) (when he was comparing the sugar-percentage of blood-serum and that of the ultra-filtrate of this serum), made the unexpected discovery that in the process of ultra-filtration a considerable portion of the substance, causing the reduction, remains behind. The above-mentioned difference in the sugar-percentage between serum and its ultra-filtrate [2]) has been described at about the same time by RUSZNYÁK [3]). The problem of combined sugar now again came to the fore. By the researches of v. HESS and MC. GUIGAN [4]), using ABEL'S [5]), method of vivodiffusion, and of MICHAELIS and RONA [6]) it seemed to have been solved in this sense that all the sugar in the blood occurred in a free state. Here we can leave out of consideration (as unsolved) the question whether the reducing substance which remains behind in ultra-filtration, is really combined sugar or whether it must be accounted one of the substances which give the so-called "rest-reduction". We would only observe here that we should have to accept that what does not pass the ultra-filter, is really glucose, if we relied on the investigations of EGE [7]). This author found that of the total reduction of the blood, determined by a slightly modified method BANG (also used by us), only a very small part should be ascribed to this rest-reduction. For convenience sake we shall call the difference which was found, "combined sugar".

When the difference in sugar-percentage between the serum (containing colloids) and the ultra-filtrate (containing no colloids) had been established, this problem presented itself to us: in how for can those liquids of the body, containing like the ultra-filtrate only insignificant quantities of albumen and colloids in general (such as humor aqueus, cerebro-spinal fluid, amnion-liquid) be compared with ultra-filtrates of the blood, also as regards their chemical com-

[1]) S. VAN CREVELD: Communication "Physiologendag" 16 December 1920, Amsterdam. Report to appear in Arch. Néerl. de Physiologie.

[2]) We wish to draw attention to a communication by HAMBURGER and BRINKMAN (Biochem. Zeitschr. **88**, 103, 1918). These authors did not find a difference between the serum and its ultra-filtrate. But their results were based only on provisional investigations, the chief aim of their researches lying in a totally different field. Undoubtedly HAMBURGER and BRINKMAN would by continuing their investigations have found the difference which is mostly present and considerable.

[3]) RUSZNYÁK. Biochem. Zeitschr. **113**, 52, 1921.

[4]) ABEL, ROWNTREE and TURNER, Journ. of Pharmac. and Exp. Ther. **5**, 275 and 611, 1914.

[5]) v. HESS and MC. GUIGAN. Ibid. **6**, 45, 1914.

[6]) MICHÄELIS and RONA. Biochem. Zeitschr. **14**, 476, 1908.

[7]) EGE. Biochem. Zeitschr. **107**, 229, 1920.

position? And especially: Is the "combined" sugar kept back here? (We shall have to investigate whether the sugar is combined with protein, or phosphatides or cholesterin). This we have investigated most fully in the case of the liquid of the eye-chamber. The values for the sugar-percentage of this liquid, mentioned in the literature of the subject, did not help us much. We had to rely on our own researches. In the first place one nearly always finds indicated (as e.g. by OSBORNE[1]) that the sugar-percentage of the liquid of the eye-chamber is about equal to that of the blood. Nearly always however it is omitted to investigate blood and eye-chamber liquid simultaneously. This may be called the first requisite, on account of the important fluctuations of the sugar-percentage in the blood which are found even under physiological conditions; and, where this simultaneous investigation was performed, as in the very detailed communication of ASK[2]) of comparatively recent date concerning the eye-liquid, these researches have lost much of their significance in the light of our present state of knowledge. For, the liquid with which the humor aqueus must be compared, is not the total blood, but only the blood-plasm, the sugar-percentages of which are quite different. By investigations of most recent date, a.o. of one of us[3]), it has been established without doubt that in the case of a number of animals (in any case with man and the rabbit) the corpuscles are free from sugar. The value found when determining the sugar-percentage of the total blood, is therefore considerably lower than the actual concentration of sugar in the plasm. And, consequently, if one compares the sugar-percentage of the liquid of the eye-chamber with that of the total blood or with that of a blood-liquid the identity of which with blood-plasma is not entirely without doubt, one arrives at conclusions which are quite wrong. Serum obtained by the coagulation of blood, plasm obtained by blood-coagulating means (hirudin, oxalate) show a lower blood-sugar percentage than the plasm proper, because in these operations in an exceedingly short space of time part of the sugar disappears in the corpuscles, this being due to changed permeability-relations. In this way are to be explained ASK's results (differing from ours) and his conclu-

---

[1]) OSBORNE, l.c.

[2]) ASK, Biochem. Zeitschr. **59**, 1 and 35, 1914.

[3]) S. VAN CREVELD and R. BRINKMAN: Proceedings of the Royal Acad. of Sciences Section of Dec. 17 1920. Further BRINKMAN and Miss VAN DAM: Arch. Internat. de Phys. XV, p. 105, 1919. In these articles elaborate bibliographies.

sions[1]) based on them, although this investigator, besides examining the total blood, has also tried to investigate bloodplasm.

The bloodplasm required for our purpose was obtained in the way indicated by one of us[2]). Into a small paraffined tube one drops rapidly a small quantity of blood from a punctioned ear-vein of a rabbit, one centrifugates for a few moments and takes away the topmost fluid plasm with a paraffined glass-pipette. The sugar-percentage in this was determined by the latest method of BANG. This method was used by us for all sugar-analyses and gave complete satisfaction. The double-determinations agreed well. We believe that this method, used for a long time already in this laboratory, gives reliable results, provided some precautions are observed. These precautions were that with each experiment the reduction of a 0.1 °/₀ glucose-solution was determined and besides it a blind-determination in duplicate of all reagents used. This was done especially with a view to the varying titre of the thiosulfate-solution.

Using this method with a number of rabbits we have first of all compared the sugar-percentage of the blood-plasm and that of the aqueus humour, which was taken at about the same time (difference in time 10 minutes at most).

The chamber-liquid was obtained very easily (after cocain-anaesthesia) by inserting a glass capillary tube with ground point into the anterior eye-chamber. In general we performed (besides the sugar-determination) also a determination of the refraction of plasm and aqueus humour (refractometer of ABBÉ) to obtain an idea of the albumen-percentage of these liquids.

In Table I the values found for the sugar-percentage in blood-plasm and that in aqueus humour (examined at the same time) are laid down.

In considering this table we must bear in mind the just-mentioned fact that the chamber-liquid flows very slowly under normal conditions, so that we can almost neglect filtration as a factor for establishing the equilibrium in the components of the liquids, but that this equilibrium is a consequence of the slower process of diffusion of the various dissolved components. It follows that (from this point of view) we can expect any change (increase or decrease) in sugar-percentage of blood-plasm to be followed somewhat more slowly by a similar change in the sugar-percentage of the chamber-liquid. Where it is known that important changes in blood-sugar-percentage may take place in a very short time, there may be

---

1) We shall return to these conclusions in detail later on.

2) S. v. CREVELD, l.c.

TABLE I.

Nº. of Rabbit	Sugar-percentage Blood-plasm	Sugar-percentage Primary aqueus humour	Difference	Mean Difference
1	0.20	0.19	+ 0.01	
2	0.28	0.19	+ 0.09	
3	0.2	0.15	+ 0.05	
4	0.2	0.19	+ 0.01	
5	0.28	0.16	+ 0.12	
6	0.27	0.21	+ 0.06	
7	0.24	0.22	+ 0.02	
8	0.22	0.19	+ 0.03	$\frac{0.66}{15} = 0.044$
9	0.25	0.17	+ 0.08	
10	0.20	0.19	+ 0.01	
11	0.22	0.19	+ 0.03	
12	0.21	0.19	+ 0.02	
13	0.26	0.24	+ 0.02	
14	0.32	0.25	+ 0.07	
15	0.22	0.18	+ 0.04	

moments that the difference in sugar-percentage between plasm and chamber-liquid does not correspond to what we should expect after the analogy of what was stated with ultra-filtration. With a rapid drop in the sugar-percentage of blood it will therefore be quite possible for the relations to be temporarily reversed, so that the chamber-liquid shows the greater percentage. In this way we can satisfactorily explain the very diverging differences in table I. But when they are compared in a large number of experiments, we may expect the chamber-liquid to show the lower figures in the majority of cases. We may further expect that in the mean values of a large number of figures the same relation between chamber-liquid and plasm will be shown which we should expect if this chamber-liquid was, not a dialysate, but a quickly-flowing ultra-filtrate. And when we find a mean difference of 0.044 °/₀ between blood-plasm of the ear-vein and chamber-liquid, we have a right to conclude: *Provisionally the phenomenon observed in vitro of the combined bloodsugar which behaves as a colloid, is confirmed in vivo*:

the sugar-percentage of chamber-liquid corresponds to the free plasma-sugar and follows its fluctuations.

When we look at Table II it becomes still more probable that we have a case of "colloidal sugar" remaining behind. In Table II we have examined not only plasm and primary chamber-liquid, but also the so called "secondary chamber-liquid" which rapidly (in a few minutes) after the punction regenerates. As is known, it has more direct connection with the composition of the blood; it contains e.g. more protein: WESSELY[1]) gives one to two percent; we found (like HAGEN)[2]) much higher values, (on account of refraction-figures), namely from 3 to 5 %. The secondary liquid therefore, as regards albumen-percentage, approaches blood-plasm and the more so, as the primary liquid had been taken away more completely. Now it appeared that this secondary liquid (coagulating rapidly in the case of the rabbit), also "qua" sugar-percentage, must be regarded as a kind of rapidly entering blood plasm, consequently blood of which only the cellular elements are kept back. For the sugar-percentage of this regenerating liquid corresponds strikingly with that of blood plasm, investigated at the same time.

In considering table II we must bear in mind that through "psychic" stimuli (sympathicus-stimulation) during the experiment, the sugar-percentage in the blood of the rabbit mostly increases. As between 1 and 3 (see table II) generally 20 or 30 minutes elapse, the sugar-percentage of 3 (as appears from the table) can no longer be compared with the bloodplasm of 20 minutes before, but with that of 4. Then there appears to be nearly complete correspondence between bloodplasm and secondary chamber-liquid: *the secondary chamber-liquid therefore has obtained the "combined sugar" together with the plasma-colloids.*

To get a further insight into the manner and the rapidity with which fluctuations in the sugar-percentage of bloodplasm are followed by the aqueus humour, we have, in the case of a number of rabbits, traced the sugar percentage of the chamber-liquid at different times during severe hyperglycaemia, caused by subconjunctival injection of 0.75 c.c. of a 1 ‰ adrenalin-solution in both eyes. We shall here mention a few brief results of a long series of experiments. The sugar-percentage of the bloodplasm rises rapidly after the injection, after 45 minutes already it reaches 0.6 to 0.7 % and remains thus for one to two hours; then it decreases again rather

[1]) Ergebnisse d. Physiologie 4[1], p. 565, 1905.

[2]) Klin. Monatsbl. f. Augenheilkunde 64, p. 187, 1920.

TABLE II.

Nº. of Rabbit	1 Bloodplasm at beginning of experiment		Time passed between 1 and 2	2 Primary aqueus humour		Time passed between 2 and 3	3 Secundary aqueus humour		Time passed between 3 and 4	4 Bloodplasm at end of experiment	
	Sugar-percentage	Refraction		Sugar-percentage	Refraction		Sugar-percentage	Refraction		Sugar-percentage	Refraction
1	0.20	—	—	0.19	1.3345	10 min.	0.24	1.3436	—	—	—
2	0.20	—	10 min.	0.15	1.3333	10 min.	0.20	1.3424	5 min.	0.21	1.3458
3	—	—	—	0.19	1.3339	15 min.	0.20	1.3409	15 min.	0.20	1.3461
4	0.28	1.3449	15 min.	0.16	1.3341	10 min.	0.19	1.3412	10 min.	0.21	—
5	0.27	1.3477	10 min.	0.21	1.3350	10 min.	0.25	1.3444	20 min.	0.29	1.3472
6	0.24	1.3479	7 min.	0.22	1.3340	10 min.	0.27	1.3433	8 min.	0.3	1.3472
7	—	—	—	0.19	1.3341	10 min.	0.21	1.3412	10 min.	0.22	1.3468

quickly to the normal values. The chamber-liquid follows these changes more slowly, here probably abnormally slowly, because, as a second adrenalin-action, the blood-supply to the eyes had decreased very considerably for the time being. Thus, for example, the sugar-percentage in the bloodplasm rose in 30 minutes from 0.25 °/₀ to 0.38 °/₀, in the chamber-liquid (first in the right eye, then the left) it rose in the same time from 0.17 °/₀ to 0.24 °/₀; in another experiment the sugar-percentage in the bloodplasm had increased in 2 hours after the injection from 0.21 to more than 0.6 °/₀, in the chamber-liquid from 0.19 °/₀ to 4 °/₀; the rapidly regenerating secondary chamber-liquid then contained 0.63 °/₀ which corresponds strikingly to the bloodplasm at that moment.

When in the second period the sugar-percentage of the bloodplasm decreases, this decrease is followed more rapidly by the chamber-liquid than the increase which preceded. This is to be expected because in this period the blood-supply to the eyes and consequently the rapidity of diffusion has become greater. Yet we succeeded in establishing a moment when the decrease in blood-sugar-percentage outstripped the chamber-liquid, so that the relations were reversed: 5 hours after the injection the sugar-percentage of the plasm was 0.27°/₀, that of the chamber-liquid was 0.32°/₀.

It might also be possible to explain the great difference (0.6 and 0.4) which was established at the culminating-point of the hyperglycaemia, not by means of retarded diffusion (consequently: equilibrium not yet reached), but by a relative increase also of the combined sugar during the hyperglycaemia. The difference of 0.2°/₀ could then correspond to the quantity of combined sugar and the 0.4°/₀ sugar in the chamber-liquid would indicate the moment of the equilibrium of the diffusion. But this supposition is no longer valid, for during a separate experiment we have, (during the maximum of hyperglycaemia) taken off a slightly larger quantity of blood and we have determined the sugar-percentage of this together with the sugar-percentage of the ultra-filtrate, obtained from it; the same had been done before with the normal blood-plasm. In the beginning the sugar-percentage of the chamber-liquid was 0.24, of the blood-plasm it was 0.26 and of the ultra-filtrate of the plasm it was 0.16; the difference between the last two is therefore 0.09. This difference now remained equal during the adrenalin-hyperglycaemia (0.63°/₀ and 0.54°/₀), while then the sugar-percentage in the chamber-liquid was much lower (0·44°/₀) than that in the plasm. *Hence the quantity of combined sugar does not increase during adrenalin-hyperglycaemia.*

It will strike that in vitro in this experiment we find a quantity

of combined sugar of 0.09 % whereas the mean difference between eye-chamber liquid and bloodplasm amounted to only 0.044 %. This figure is really low; for as an average in eight experiments we found a quantity of combined sugar of 0.075 % in ultra-filtrates (in vitro) of serum of our test-rabbits.

Now this difference between the processes in vitro and in vivo is not yet such that on the strength of this we should no longer regard the chamber-liquid as a kind of ultra-filtrate, but moreover we are inclined to think that this difference is not essential. For in comparing the eye-chamber liquid and the plasm of the blood from an ear-vein we found an average difference of 0.044 %. But the blood which interacts with the eye-chamber liquid, will in no case be venous blood, but it will agree more with the composition of arterial blood. Now, as a consequence of the sugar-consumption of the various organs, the sugar-percentage of venous-blood will be lower than that of arterial blood; the magnitude of this difference will depend on the intensity of the sugar-metabolism of the particular organ. We may take for granted that this metabolism will be very slight in the case of the tissues (cornea, crystalline-lens) etc., which surround the eye-chamber, and also that the venous blood flowing from it, would differ very little from arterial blood, supposing we could investigate the former separately. This difference exists very distinctly when we compare blood taken simultaneously from the a. carotis and from the v. facialis posterior, the latter of which practically corresponds to the blood from an ear-vein. In three experiments we found here differences of 0.09, 0.03 and 0.02, on the average therefore over 0.04 %. If, therefore, we increase the sugar-percentage of the plasm from the ear-vein with this amount, the sugar-percentage of the chamber-liquid will correspond very well to what we should expect of an ultra-filtrate.

As regards the second liquid investigated by us, the *cerebro-spinal fluid*, we can dispose only of a much smaller number of experiments. The statements in the literature of the subject made it probable that here also we should find a sugar-percentage, lower and even considerably lower than that in the blood-plasm. Thus for example FINE and MYERS[1]) state that with a number of patients the sugar-percentage of the cerebro-spinal liquid amounted to only 57 % of that of the total blood. A similar statement we find in WESTON[2]).

For the reasons given before, this difference would become more

[1]) Proceedings Soc. Exp. Biol. **13**, p. 126, 1916.
[2]) Journ. of Med. Research. **35**, p. 199.

striking still, when compared with bloodplasm instead of the total blood.

As regards the technique to obtain cerebro-spinal liquid from rabbits, we obtained it by puncturing (with a glass capillary tube); the ligament connecting occiput and atlas after this had first been exposed by cutting the skin and preparing the muscles of the neck under local anaesthesia (without adrenalin); consequently obtaining the liquid of the fourth ventricle. The animals can bear this quite well.

We have embodied the results in tables III and IV.

In examining these tables we must bear in mind that a comparison of cerebro-spinal liquid and bloodplasm under physiological conditions is much more troublesome than in the case of chamber-liquid; the operation generally lasts half an hour, and, when the liquid can be obtained, distinct hyperglycaemia has occurred in the blood in the mean time (this appears from the tables). Hence in the values found for cerebro-spinal liquid, which in themselves are not abnormally low (average 0.18 °/₀ in table III) there is already a certain amount owing to the increase of the quantity of blood-sugar. The value of this quantity cannot be given however, as we do not know the rapidity of diffusion here. The physiological difference with the bloodsugar-percentage is, therefore, fairly certainly smaller than that which we find if we compare with the plasm, taken simultaneously (column 3). But it is most certainly larger than would appear from a comparison with the plasm at the beginning of the experiment (column 1). By means of a larger number of experiments and by causing the operation to last as short as possible we may probably obtain more accurate data here. We shall, besides, obtain an insight into the rapidity of diffusion from an investigation of the speed with which adrenalin-hyperglycaemia manifests itself in the cerebro-spinal liquid and also of the degree of this manifestation. Onr next experiments will lie in that direction [1]).

On the strength of table IV we may accept as certain that the sugar-percentage in cerebro-spinal liquid is considerably lower than it is in the chamber-liquid which was investigated simultaneously (cf. columns 3 and 4).

So we see here two "ultra-filtrates" with diverging sugar-percentages. Are we to think here of an "active" stopping of glucose by the plexus chorioideus? It seems to us that we need not call in the aid of a similar force, but that the cause should rather be

[1]) The results of these have been mentioned in the more detailed publication in the "Biochemische Zeitschrift". **123**. 190. 1921.

TABLE III.

Number of Rabbit	1 Blood-plasm		Time passed between 1 and 2	2 Cerebro-spin. liquid		Time passed between 2 and 3	3 Blood-plasm	
	Sugar-percentage	Refraction		Sugar-percentage	Refraction		Sugar-percentage	Refraction
1	—	—	—	0.14	1.3341	15 min.	0.40	1.3451
2	—	—	—	0.18	1.3341	8 min.	0.48	1.3441
3	0.23	1.314	20 min.	0.20	1.3341	50 min.	0.40	1.3456
4	0.30	—	35 min.	0.20	1.3341	15 min.	0.36	1.3451

TABLE IV.

Number of Rabbit	1 Blood-plasm		Time between 1 and 2	2 Prim. aqueus humour right eye		Time between 2 and 3	3 Cerebro-spin. liquid		Time between 3 and 4	4 Prim. aqueus humour left eye		Time between 4 and 5	5 Blood-plasm	
	Sugar-percentage	Refraction		Sugar-percentage	Refraction		Sugar-percentage	Refraction		Sugar-percentage	Refraction		Sugar-percentage	Refraction
5	0.30	—	10 m.	0.24	1.3336	25 m.	0.20	1.3341	10 m.	0.28	1.3341	—	0.36	1.3451

looked for in the action of the entire surrounding tissues, which, in the case of the eye-chamber, will fairly certainly show a smaller metabolism than the cerebral tissues with which the cerebro-spinal liquid is as much in interaction as it is with the blood.

A somewhat rapid sugar-consumption in the cerebral-tissues will necessarily cause a continuous diffusion of glucose from the cerebro-spinal liquid to this place of lower sugar-concentration. The quantity of sugar in the cerebro-spinal liquid therefore remains constantly lower than its value would be if it had interacted only with the blood; lower also than is found in the chamber-liquid. Thus we may expect for the same reasons that the blood itself will yield more sugar in the brain than round the eye-chamber and that, therefore the blood in a cerebral vein will show a greater decrease in sugar-percentage than the blood flowing from the eye-chamber and its surroundings.

A comparison of the ultra-filtrates of the blood from the a. carotis and the v. facialis posterior taught us that the difference in sugar-percentage between arterial and venous blood is almost completely due to the free sugar, while the quantity of "combined sugar" suffers hardly any modification in passing the capillary tubes.

It looks tempting to suppose that the combined sugar plays a part in the consumption of the sugar in the tissues: here the sugar would continually be combined (adsorbed?) and be combusted in that condition; this would continually cause the fixing of fresh "free" sugar from the neighbourhood; this would cause the decrease of concentration of free sugar on that spot, followed by diffusion from the blood, etc. A correct opinion about this supposition can only be pronounced when it is settled that the reducing substance, which cannot be ultra-filtrated, is sugar and when it is further settled of what kind the substance is to which this sugar is "combined".

SUMMARY AND CONCLUSIONS.

We can sum up the results of our investigations as follows:

1. In the case of rabbits *the sugar-percentage of tissue-liquids containing practically no colloids* (*aqueus humour, cerebro-spinal fluid*) is as a rule *smaller than that of blood-plasm, investigated at the same time*. This phenomenon is in agreement with the lower value for the sugar-percentage which ultra-filtrates of serum in vitro show when compared with this serum.

2. As regards *the liquid of the eye-chamber*, this *difference of sugar-percentage*, compared with *plasm* of arterial blood, is *the same* as the

quantity of combined sugar in blood-serum which does not [illegible] time.

3. As to aqueous-humour [illegible] the results do not yet give us [illegible] certainty concerning the exact proportions of its sugar-percentage [illegible] and that of blood-serum [illegible] sugar-percentage is [illegible] than it is in the liquid of the eye-chamber. We wish to [illegible] the possibility of explaining the [illegible] difference by assuming a [illegible] [illegible] in the [illegible] tissues that occurs in the [illegible] during the eye-chamber.

4. From the fluctuations in the sugar-percentage of eye-chamber [illegible] liquid under normal conditions and with hyperglycaemia [illegible] [illegible] we must conclude that the equilibrium [illegible] [illegible] is here chiefly brought by diffusion and partly by the [illegible] of liquid, such corresponds to what is accepted about the speed [illegible] circulation of the eye-chamber liquid. On the other hand, the [illegible] [illegible] liquid of the eye-chamber [illegible] sugar-percentage perfectly equal to blood plasma at the same moment. This is accounted for as follows: what has entered is [illegible] [illegible] with a high percentage of [illegible] and a corresponding [illegible] of combined sugar.

5. In comparing arterial blood from the [illegible] and venous blood from the [illegible] it appeared that the difference of sugar-percentage between these two is to be ascribed to the free sugar which is therefore yielded to the tissues.

6. The hyperglycaemia caused by adrenalin-injection depends entirely on an increase of the free sugar.

Physiological Laboratory.

Groningen, March 1911.

Physiology. — "*Experiments on the Quick component Phase of Vestibular Nystagmus in the Rabbit*". By A. DE KLEYN. (Communicated by Prof. R. MAGNUS).

(Communicated at the meeting of March 26, 1921).

When cold water is allowed to run into the right external auditory canal, a nystagmus will appear whose immediate effect is a slow deviation of both eyes towards the syringed, ergo the right side.

This deviation is succeeded by a quick movement of the two eyes towards the non-syringed, i. e. the left side. Then again follows the slow deviation to the right. These alternations of slow-, and quick eye-movements will recur while the syringing continues, and for some time after.

With every vestibular nystagmus the primary deflection is such a slow deviation, the so-called slow component phase of the nystagmus; the rapid movement, the so-called quick component phase is secondary.

It would be reasonable therefore to determine the direction of a nystagmus by the direction of the slow component phase. However, the quick eye-movements strike the observer more particularly, so that what in the clinical and in the physiological literature is called a nystagmus to the right is almost exclusively a nystagmus in which the quick component phase moves to the right; whereas by a nystagmus to the left a nystagmus is meant in which the quick component phase moves to the left. Although in strictness this does not square with the theory, from a practical point of view it will be well to adhere to this conception.

All researchers agree that the slow component phase arises from a direct reflex from the labyrinth via the nucleus of the N. vestibularis to the nuclei of the eye-muscle and the eye-muscles. As regards the origin of the quick component phase, opinions differ widely. Hardly any experiments have been made, so that we possess only a large number of more or less probable theoretical speculations and are still left much in the dark.

We have no intention to discuss these several theories. Only one of them, viz. BARTELS' theory, we will test experimentally. BARTELS assumes that the source of the reflex that gives rise to the quick

quantity of *"combined sugar"* in bloodserum which does not ultrafiltrate.

3. As to *cerebro-spinal fluid,* the results do not yet give sufficient certainty concerning the exact proportions of its sugar-percentage and that of blood-plasm; *to all appearance it is considerably lower than it is in the liquid of the eye-chamber.* We wish to indicate the possibility of accounting for this difference by assuming a larger glucose-consumption in the cerebral tissues than occurs in the tissues lining the eye-chamber.

4. From the fluctuations in the sugar-percentage of eye-chamber liquid under normal conditions and with *hyperglycaemia after adrenalin-injection* we must conclude that *the equilibrium with the blood is here chiefly caused by diffusion* and hardly by the circulation of liquid; such corresponds to what is accepted about the speed of circulation of the eye-chamber liquid. On the other hand, the so-called *secondary liquid of the eye-chamber is, as regards its sugar-percentage, perfectly equal to blood plasm at the same moment.* This is accounted for as follows: what has entered is practically *blood-plasm with a high percentage of colloids and a corresponding quantity of combined sugar.*

5. In comparing *arterial* blood from the a. carotis and *venous* blood from the v. facialis posterior it appeared that the *difference of sugar-percentage* between these two is to be ascribed to the *free sugar* which is therefore yielded to the tissues.

6. The hyperglycaemia caused by adrenalin-injection depends entirely on an increase of the free sugar.

Physiological Laboratory.

Groningen, March 1921.

Physiology. — "*Experiments on the Quick component Phase of Vestibular Nystagmus in the Rabbit*". By A. DE KLEYN. (Communicated by Prof. R. MAGNUS).

(Communicated at the meeting of March 26, 1921).

When cold water is allowed to run into the right external auditory canal, a nystagmus will appear whose immediate effect is a slow deviation of both eyes towards the syringed, ergo the right side. This deviation is succeeded by a quick movement of the two eyes towards the non-syringed, i. e. the left side. Then again follows the slow deviation to the right. These alternations of slow-, and quick eye-movements will recur while the syringing continues, and for some time after.

With every vestibular nystagmus the primary deflection is such a slow deviation, the so-called slow component phase of the nystagmus; the rapid movement, the so-called quick component phase is secondary.

It would be reasonable therefore to determine the direction of a nystagmus by the direction of the slow component phase. However, the quick eye-movements strike the observer more particularly, so that what in the clinical and in the physiological literature is called a nystagmus to the right is almost exclusively a nystagmus in which the quick component phase moves to the right; whereas by a nystagmus to the left a nystagmus is meant in which the quick component phase moves to the left. Although in strictness this does not square with the theory, from a practical point of view it will be well to adhere to this conception.

All researchers agree that the slow component phase arises from a direct reflex from the labyrinth via the nucleus of the N. vestibularis to the nuclei of the eye-muscle and the eye-muscles. As regards the origin of the quick component phase, opinions differ widely. Hardly any experiments have been made, so that we possess only a large number of more or less probable theoretical speculations and are still left much in the dark.

We have no intention to discuss these several theories. Only one of them, viz. BARTELS' theory, we will test experimentally. BARTELS assumes that the source of the reflex that gives rise to the quick

phase must be looked for in the periphery. By stimulating the labyrinth certain eye-muscles gradually contract and their antagonists relax simultaneously (slow phase). This process is supposed to elicit a reflex which causes a rapid contraction of the originally relaxed eye-muscles and a rapid relaxation of the originally contracted eye-muscles (quick phase). He relinquished his primary view that, owing to the contractions of the eye-muscles, the terminal branches of the trigeminus are stimulated and the reflex for the quick phase has its origin here, and adopted the view that the proprioceptive nerve-ends in the eye-muscles themselves, demonstrated experimentally by TOZER and SHERRINGTON [1]), represent the beginning of the reflex arc for the quick phase of the nystagmus.

Now it seems easy at first sight to test experimentally the validity of BARTELS' conception. When isolating one of the eye-muscles and recording during a nystagmus the movements of this muscle according to BARTELS' method on a Kymograph we need only to paralyse the proprioceptive nerve-ends in this muscle and see whether the rapid phase disappears.

BARTELS [2]) himself had tried this previously by injecting cocain into one of the orbitae. This however engendered such a rapid paralysis of all the eye-nerves that the problem could evidently not be solved in this way.

However we now possess a means to paralyse the proprioceptive nerve-ends in the muscle. LILJESTRAND and MAGNUS [3]) have demonstrated that after injections of certain doses of novocain the muscles at the place where the nerves enter, the proprioceptive nerves are paralysed while the motor nerves retain their functions.

If it should appear, however, that in these experiments the rapid phase does not become extinct (or would with larger doses disappear only simultaneously with the slow phase at the moment when the entire eye-nerve has been paralysed) this could not be put forward as an argument against the theory of BARTELS.

When a nystagmus occurs, contractions and relaxations of various muscles of both eyes take place. If, therefore, one wishes to paralyse the proprioceptive nerve fibers, which come into play for the reflex of the rapid phase, it is not enough to inject novocain into the isolated muscle, but one has to eliminate *all* proprioceptive fibers of *all* eye-muscles.

---

[1]) Folia Neurobiologica. IV, p. 626.

[2]) Klin. Monatsblätter für Augenheilkunde 1914. Bd. LIII, p. 365.

[3]) Pflügers Archiv. Bd. 176, p. 168.

This was effected in the following way: Tracheotomy was performed in a rabbit under ether anaesthesia and the respiration was maintained artificially. The vagi were severed and the carotids were temporarily closed by a clamp. Subsequently the M. externus of one eye, say the left one, was prepared and at its extremity a thread was attached through which with the aid of a lever the contractions of this muscle could be registered on a kymograph. Then followed the extirpation of the eyeball.

Hereafter the skull-cap was removed, likewise the cerebrum and, after a frontal section through the brainstem a little before the corpora quadrigemina, part of the brainstem. Finally section of the two N.N. occulomotorii and the two N.N. trochleares at the base of the skull, and of the right N.N. abducens- and trigeminus at their entry into the orbita. So all the eyemuscle-nerves and consequently also *all* proprioceptive nerves of the eyemuscles were cut through with the exception of one N. abducens, whose associated M. externus was used for the registration of the nystagmus.

In some experiments also the two N.N. trigemini were severed at their exit from the medulla, with a view to ascertain once more whether a normal nystagmus could be evoked after severing both trigemini. Of this little is known in the literature. HÖGYES[1]) points out that in experiments conducted in his laboratory by KERTESZ and v. MARSCHALKO, vestibular eye-movements still occurred after section of the ganglion Gasseri; however, in his description he does not state clearly whether or no a quick phase came forward. Moreover these experiments were made only unilaterally.

KUBO[2]) only states in his well-known communication on the vestibular reflexes in rabbits: "Die Durchtrennung des N. vagus, glossopharyngeus und trigeminus in der Schädelhöhle bleibt ohne Einfluss auf die Reaktion nach der Einspritzung".

Fig. 1 shows the nystagmus in a rabbit whose M. externus had been isolated to the *left* and whose N.N. trigemini, N.N. oculomotorii, N.N. trochleares and *right* N. abducens had been cut through. With the contraction of the M. externus the curve ascends.

From this experiment it appears that deviation and nystagmus are still demonstrable after severing *both N.N. trigemini* and *all the eyemuscle-nerves with the exception of one N. abducens* by douching alternately the two meatus with cold water and registering the movements of the still innervated M. externus. When syringing the

[1]) Ueber den Nerven mechanismus der assoziierten Augenbewegungen, übersetzt von MARTIN SUGAR. Urban und Schwarzenberg 1913, p. 82.

[2]) Pflügers Archiv. 1906. Bd. 114, p. 151.

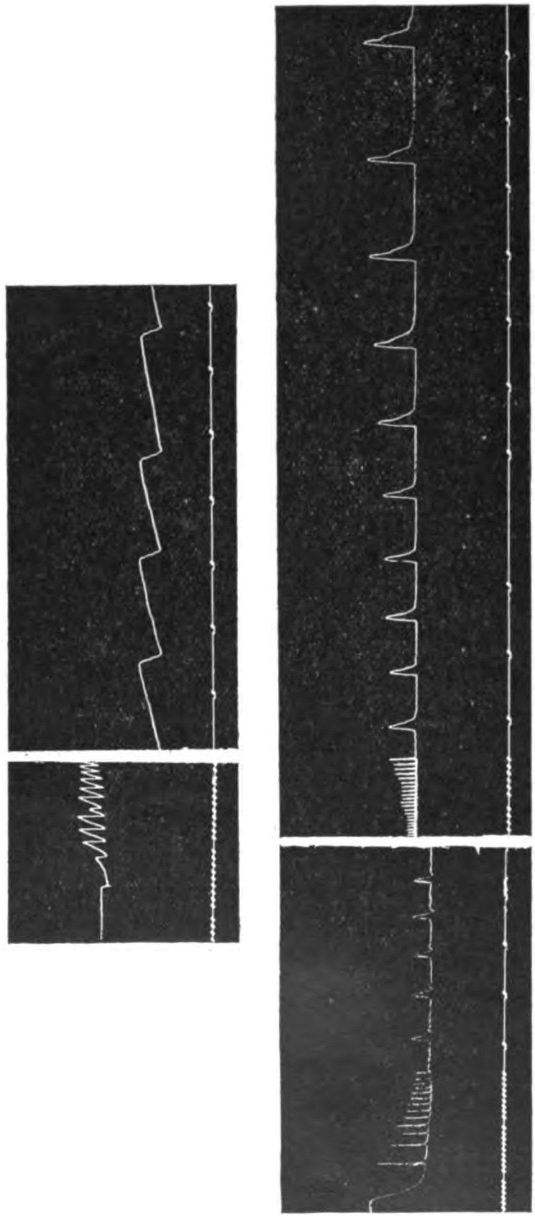


Fig. 1*a*: Syringing of the *left* auditory meatus with cold water. It will be seen that immediately after the douching a nystagmus appears, in which a slow contraction is invariably followed by a quick relaxation.

Fig. 1*b*: Illustrates the same with a quicker rotation of the kymograph. (The lever is shifted more towards the timeline, so that the contraction of the M. externus is not so small as it looks to be).

Fig. 1*c*: Syringing of the *right* meatus with cold water.

Here just the reverse takes place to Fig. 1*a*. Slow relaxations are followed by rapid contractions. The intense relaxation is very apparent just at the beginning of the douching.

Fig. 1*d*: Presents the same picture as Fig. 1*c* at a later douching of the left meatus, with which the alternating relaxations and contractions are still distinctly visible.

meatus at the prepared side a nystagmus is elicited in which slow contractions of the muscle are followed by quick relaxations and conversely, when syringing the other meatus a nystagmus occurs in which slow relaxations are followed by quick contractions.

This finding calls in question the soundness of BARTELS' theory. If stimulation of the proprioceptive nerve-fibers in the eyemuscles were to generate the reflex for the quick phase, these nerve-fibers would have to be stimulated by contraction as well as by relaxation of the muscle. Another theoretical, hitherto unrecorded objection is this that with the compensatory eye-positions, in which also occur considerable contractions and relaxations of the eyemuscles, only a large deviation but not a nystagmus is observed.

Our doubt proved to be well-grounded, as borne out by the following experiments in which novocain was injected into the isolated M. externus.

Technical difficulties caused a failure of some experiments. Six experiments succeeded to perfection and all yielded precisely the same results.

In every one of them the *left* M. rectus externus was isolated and the right N. trigeminus and all the eyemuscle-nerves were severed with the exception of the *left* N. abducens.

Registration of the nystagmus evoked by syringing the *left* meatus with cold water (slow contractions of the M. externus, followed by quick relaxations). Hereafter injection of 0.75—1 % of novocain into the isolated M. externus (Lower concentrations of novocain did not seem to distinctly influence the nystagmus). With the above-named concentrations not only the proprioceptive nerves were paralysed but also a paralysis of the motor eye-nerve manifested itself.

Also during the complete paralysis the syringing was continued; after some time the eye-nerve recovered itself slowly. In this manner, while syringing continually, we were also able to observe how this recovery occurred.

Now if the theory of BARTELS is correct we have after the novocain-injection to expect at the moment when the proprioceptive nerve-fibers are paralysed but the motor nerve-fibers are still functionating, a stage in which the quick phase of the nystagmus disappears, but the deviation still persists. Not before the moment when also the motor nerve-fibers are paralysed, will the deviation disappear also. Likewise during the recovery a stage of deviation is to be expected without a quick phase and only then, when the proprioceptive nerve-fibers begin to functionate, a normal nystagmus with a quick phase.

A similar process has already long been known during narcosis.

88*

Fig. 2.

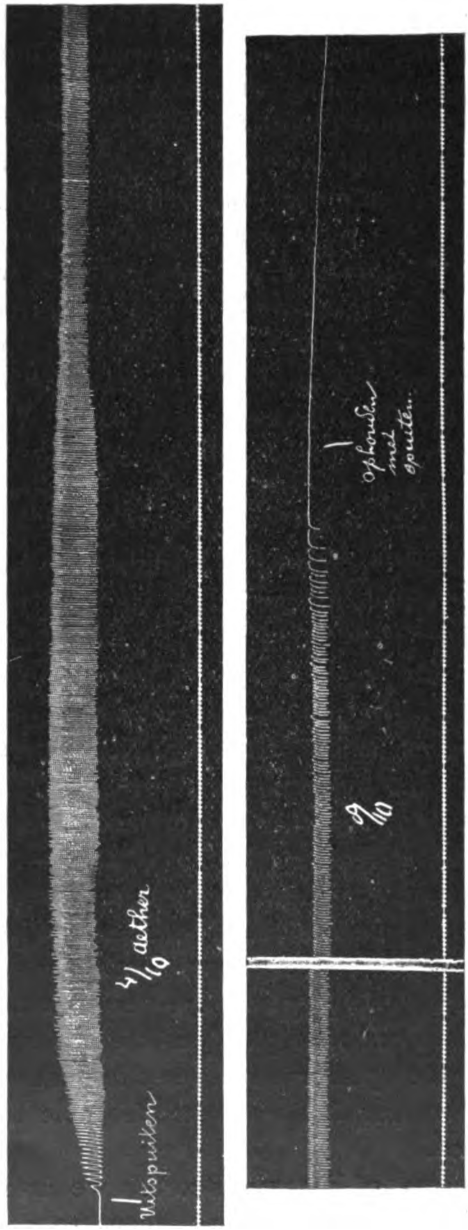


Fig. 2: Illustrates such an experiment. Directly after syringing the left meatus with cold water a deviation and a nystagmus is evoked, which soon attains a constant intensity.

It will be observed that during the etherization (at first ether 4 to 10, then rising slowly as high as 8 to 10), the muscle contracts more and more and the rapid beats of the nystagmus become smaller, until ultimately the latter disappear entirely and the isolated contracted muscle does not display any more relaxations. After the syringing is stopped the contraction decreases and gradually the slow phase of the nystagmus disappears.

The experiments with injections of novocain into the isolated muscle present just the opposite picture.

This is instanced in Fig. 3.

Fig. 3.

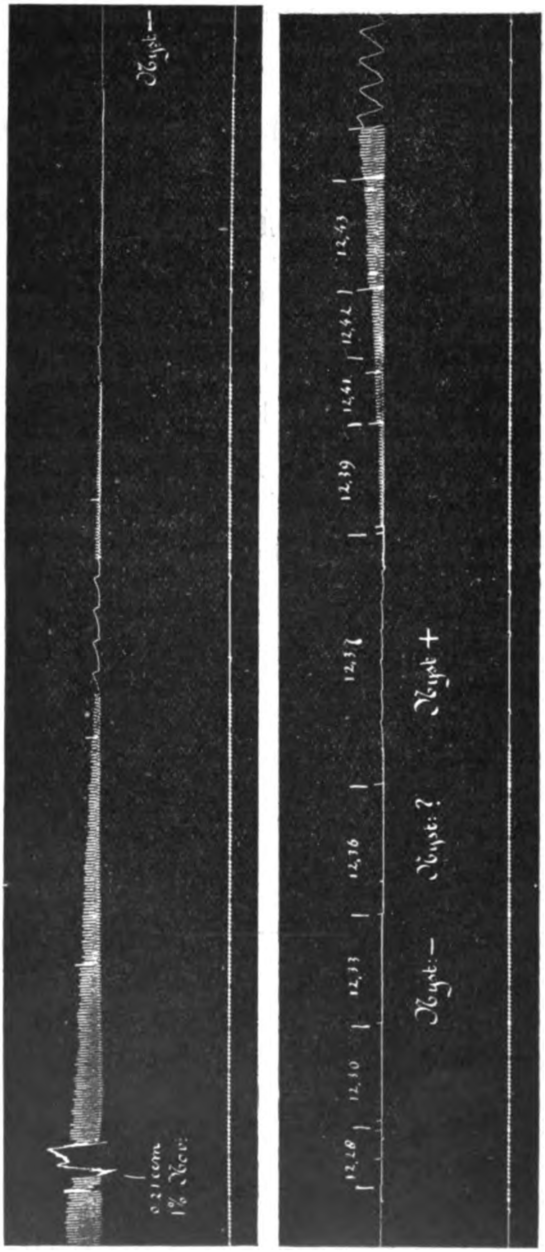


Fig. 3: Deviation and nystagmus evoked by syringing the left meatus with cold water.

After the injection with novocain the contractions of the muscle (slow phase) slacken, every slow contraction is however invariably followed by a rapid relaxation ***in the same measure as before the novocain-injection,*** **so that ultimately** ***minimal***

contractions and relaxations alternate, or in other words the quick phase of the nystagmus does not disappear before the entire muscle is paralysed.

After the total disappearance of the nystagmus the kymograph is stopped (at 28 sec. past 12), and now after one or more minutes we ascertain whether recovery of the function is already discernible. As shown by the curve at 37 sec. past 12 new small minimal contractions recur. Every one of them is again followed directly by a quick relaxation.

As stated above, five other quite successful experiments yielded the same result.

CONCLUSIONS.

Bartels assumes in his theory that the rapid phase of the vestibular nystagmus is brought about by a reflex originating near the proprioceptive nerve-fibers and that the terminal branches of the trigeminus in the orbita play a part in this process.

As the facts brought out in the present investigations have proved this not to be the case, his theory cannot be accepted.

The place where the reflex for the quick phase of the nystagmus arises is therefore to be looked for more towards the centra in the brainstem.

Physics. — "*Research by means of Röntgen Rays on the Structure of the Crystals of Lithium and Some of its Compounds with Light Elements*". I. By J. M. BIJVOET and A. KARSSEN. (Communicated by Prof. P. ZEEMAN).

(Communicated at the meeting of March 26, 1921).

[For some ten years the researches at the Laboratory for general and inorganic chemistry of this University have for the greater part been directed to the solid substance. With a view to extend the methods of research also to those that make use of Röntgen rays, hoping in this way finally to get more knowledge about the finer internal states of equilibrium, steps were taken to get the required Röntgen apparatus.

Through the great kindness of the municipality and of the Amsterdam University Association we have now this institution at our disposal, and we feel obliged to express our hearty thanks to both bodies.

The task we have set to leads us to the typical allotropic substances, but first we wanted to examine some simple, but nevertheless very interesting cases, in which results could be expected which are of importance for the knowledge of the nature of the link. For these cases were chosen Li and LiH, with the result described below. A. SMITS].

I. LITHIUM.

1. *Introduction.* The investigation of Li and some of its compounds with light elements is perhaps more suitable than many of the compounds examined hitherto for drawing conclusions concerning structure and binding of the particles on account of the small number of electrons outside the nucleus of the component atoms[1]).

The notion e.g. that electrons and ions must be considered as equivalent elements of the space lattice, which view offers inter alia great advantages for the explanation of the structure of the

[1]) Thus DEBYE and SCHERRER could show for LiF, that there are no atoms, but ions in the lattice-points. Physik. Zeitschr. **19**, 474 (1918).

crystal, and for the relation between infra-red and ultra-violet frequencies, may be tested in a simple way in the case of lithium. In the case of an equally simple lattice set with atoms of a higher atomic number the problem will be more difficult to solve. Thus we hope to make out presently for lithium hydride, whether possibly negative hydrogen ions occur here [1]).

The simplicity of these substances in contrast with more intricate compounds gives scope for hope that the result will be unequivocal. Moreover the said lithium compounds have the advantage that in using the method of DEBYE and SCHERRER place and intensity of the lines are much less influenced by the absorption in the rod than they are with heavier compounds. The latter, when results of rays of different wave-lengths [K_{Cu}- and K_{Cr}-rays] are recorded, may give rise to small changes in intensity relations even as far as reversing them if intensity is slightly differing. So it lately appeared to us with sodium bromate.

2. *Apparatus.* A Rausch von Traubenberg-Debye tube with exchangeable anti-cathode was used, as described by BIJL and KOLKMEYER [2]). The required high voltage direct current was obtained by rectifying transformed alternate current with a Snook. The radius of the camera used was 5.0 cm., the dimensions of the diaphragms were the same as those described by BIJL and KOLKMEYER. The sample was rotated by means of a clockwork.

3. *Photograms.* After being cleaned under paraffin-oil and washed with dry ether a rod of lithium, 1,5 to 2 mm. thick was covered with a thin protecting layer of paraffin, and fastened by means of a glass foot in the axis of the camera. Even after days the surface remained bright metallic and shiny. A Cr-anticathode was used. The exposure lasted $\pm$ 12 hours with a mean current of $\pm$ 12 mA., parallel spark between plate and point 3 cm. Then a film was made of a glass rod covered with paraffin to be able to eliminate the interference lines caused by these substances.

4. *Observations and calculation.*

In column 1 of Table I are recorded the distances on the film from the middle of the image to the interference lines, expressed in mm., and

---

[1]) Compare MOERS, Z. f. anorg. u. allg. Chemie **113**, 179 (1920).

[2]) These Proc. Vol. 21, p. 405.

TABLE I.

Distance in mm. and estimated intensity	$10^3 \sin^2 \frac{\vartheta}{2}$	$Cr_{K\alpha}$-radiation			$Cr_{K\beta}$-radiation		
		Σh^2	$10^3 \sin^2 \frac{\vartheta}{2}$ calculated	$h_1h_2h_3$	Σh^2	$10^3 \sin^2 \frac{\vartheta}{2}$ calculated	$h_1h_2h_3$
1	2	3	4	5	6	7	8
43.1 z	179				2	177	110
47 5 zs	214	2	213	110			
70.4 m	428	4	427	200			
81.0 zz	535				6	530	211
91.6 s	640	6	640	211			
116.1 ms	852	8	854	220			
120. zz	879				10	884	310

the estimated intensities; in column 2 the values for $10^3 \sin^2 \frac{\vartheta}{2}$ calculated from this (on account of the slight absorption in the lithium correction for the thickness of the rod was here unnecessary). In the well-known way the values referring to β-lines have been separated by the aid of the ratio $\lambda_\beta : \lambda_\alpha = 2{,}079 \,.\, 10^{-8} : 2{,}284 \,.\, 10^{-8}$. In correspondence with the regular crystalline form a common factor for $10^3 \sin^2 \frac{\vartheta}{2}$ of the α-lines was found, great 213,4. In connection with density, atomic weight, value of AVOGADRO and wave-length (resp. 0,534, 6,94, $0{,}6062 \,.\, 10^{24}$ and $2{,}284 \,.\, 10^{-8}$) it follows that if the number of particles per cell is n, a value for the common factor A is calculated for $n = 2$, which corresponds with the observation and is equal to half the factor mentioned, while for $A = 106{,}7$ follows $n = 1{,}99$; hence per cell (lattice parameter $a = 3.50 \,.\, 10^{-8}$ cm.) there are two particles. In connection with the intensities of the diffraction lines, those of planes with odd Σh being absent, it is obvious that lithium crystallizes in centered cubes [1]).

Table II gives the observed and calculated intensities, in which only the factor of the number of planes, the factor of LORENTZ, and the structure factor (which in this case is the same for the planes with

[1]) HULL already studied lithium, but could not decide between cubes with two atoms per lattice-point and centered cubes. Phys. Rev. **10**, 661 (1917).

even Σh, and zero for those with odd Σh) have been taken into consideration. Hence absorption in the rod, temperature factor, and polarisation factor are not taken into account. We hope to ascertain in how far anything can be concluded with regard to the configu-

TABLE II.

Planes	Intensity	
	Observed	Calculated
100	—	—
110	zs	6.0
111	—	—
200	m	1.5
210	—	—
211	s	4.0
220	ms	1.5
221, 300	—	—

ration of the electrons outside the nucleus after photometry of the film. It can, indeed, already be stated that it is not possible to satisfy the crystallographical and intensity conditions by a model, in which by the side of the Li-ion, the valency-electron occupies a definite place in the space-lattice. We tested the arrangement which may be obtained in the following way, and which is the only one that deserves consideration:

Draw the system of non-intersecting trigonal axes in the cell with edge $a' = 2a$ and place the valency-electron on each axis in the middle between the ions. That the intensities calculated for this model are not in agreement with the observed intensities appears from table III (supposition B; the effect of the distance of the remaining electrons from the nucleus can be neglected in this case.)

On the other hand agreement with the observed intensities *is* found, when in this model the valency-electron is placed not on the trigonal axis, but revolving in circles normal to the trigonal axis in the midst of the Li-ions (Table III, supposition C); here r = radius of the path supposed circular. The weakening factor occur-

ring in this case for the electron [1]) makes the calculated intensities, for values of $\frac{r}{a'} > \pm 0{,}1$, i.e. $r > \pm \frac{1}{4}$ of the distance between two nuclei, differ but little from those found for case A (simply centred cubic lattice; the intensities of column A are not essentially affected by taking into account a weakening factor for the Li-atom.) The choice between the two models that are in agreement with the Röntgen investigation, viz. that with atoms in the lattice points and that with ions between which binding circles, will be postponed till after photometry of the film [2]).

TABLE III [3])

Planes	Intensity			
	Observed	Calculated		
		A	B	C $\left(\frac{r}{a'} = 0{,}1\right)$
211	—		16	8
220	zs	216	96	78
321	—		14	0
400	m	54	6	20
332	—		4	1
422	s	144	64	77
431, 510	—			2
521, 440	ms	54	60	22

It is, however, very questionable whether a decision can be made with sufficient probability along this line, i.a. on account of the un-

[1]) Cf. the analogous calculations on binding circles in diamond by COSTER. These Proc. Vol. 22, p. 536 and KOLKMEYER, Vol. 23, p. 120.

[2]) Possibly this choice will be still more difficult here than it is for diamond, as the number of valency-electrons is here only $^1/_3$ of the total number.

[3]) The planes for which the structure factor becomes zero independent of the value of $\frac{r}{a'}$ have not been included in the table.

certainty of some intensity factors, e.g. those referring to the thermal movement of the valency electrons.

For the value of the diameter of BRAGG's *atomic domain* [1]) follows from the given structure $3.04.10^{-8}$ which is in very good agreement with the value given by BRAGG ($3.00.10^{-8}$).

5. *Summary.* Lithium crystallizes in centered cubes, lattice-parameter $a = 3{,}50.10^{-8}$ cm.; no lattice of stationary valency electrons. Possibly binding circles normal to the trigonal axes.

In conclusion we express our great indebtedness to Prof. SMITS for his assistance and the great interest he has taken in this work.

Laboratory of Physical and Inorganic
Chemistry of the University.

Amsterdam, March 14th. 1921.

Whilst this paper was being printed our attention was directed by a paper of THIRRING (Z. f. Phys. 4, 1, 1921), to the appendix of a paper by HABER (Sitz. Ber. der Preuss. Ak. d. Wiss. 51, 990, 1919), from which appears that at a meeting of the d. Chem. Ges. DEBYE already communicated the result of still unpublished investigation of Li, which agrees with our conclusion as to the arrangement of the Li-particles, and the rejection of the lattice of stationary electrons.

1) Phil. Mag. August. (1920), p. 169.

Experimental Psychology. — *"On the Development of Attention from the 8th until and including the 12th year of life."* By F. ROELS and J. FELDBRUGGE. (Communicated by Prof. C. WINKLER).

(Communicated at the meeting of February 26, 1921).

I.

The literature records little about the development of attention in children, the writers having confined themselves only to the value which the experimental investigation of attention in adults, possesses for a proper notion of the development of attention in children. Their investigations regarded the range of attention for simultaneous and successive impressions, its intensity, the aptness for distraction or the power of resistance to disturbing influences, the degree of clearness of the several elements observed in one action. True, in the first part of his "Vorlesungen zur Einführung in die experimentelle Pädagogik und ihre psychologischen Grundlagen" [1]) MEUMANN describes, on the basis of his own investigations, the development of attention in young individuals in all these respects, but for want of space he had to forego the publication of his experimental results. Children were also experimented on a.o. by F. N. FREEMAN [2]), A. KOCH [3]), J. HABRICH [4]), D. KATZ [5]) and M. v. KUENBURG [6]); with exception of the first, all about attention and conscious isolating abstraction, i.e. fixating of uniform elements from simple pictures.

---

1) Leipzig 1916, p. 179 volg.

2) Untersuchungen über den Aufmerksamkeitsumfang und die Zahlauffassung bei Kindern und Erwachsenen. Pädagogische Arbeiten des Leipziger Lehrervereins I, 1910.

3) Experimentelle Untersuchungen über die Abstraktionsfähigkeit von Volksschulkindern. Zeitschrift für angewandte Psychologie 7, p. 332.

4) Experimentelle Untersuchungen über die Abstraktionsfähigkeit von Schülerinnen, Ibid. 9, p. 189.

5) Studiën zur Kinderpsychologie. Wissenschaftliche Beiträge zur Pädagogik und Psychologie. Heft 4, 1913.

6) Ueber Abstraktionsfähigkeit und die Entstehung von Relationen beim vorschulpflichtigen Kinde. Ibid. 17, p. 270; See also K. BÜHLER: Die geistige Entwicklung des Kindes². Jena 1921, p. 162.

Experimental psychological investigation of children has been of still rarer occurrence with regard to the process of the concentration of attention in continued labour. Counterparts to the psychological investigations of children by VON VOSS [1]), RIVERS and KRAEPELIN [2]), AMBERG [3]), WEYGANDT [4]) and other experimentalists of the München school, do not exist. They would not, indeed, have satisfied us in the form given to them by these authors. For, although the fluctuations manifesting themselves in different places of their activity-curve, whether they are due to the start, to accommodation, to practice, to fatigue, to the final spurt etc., are chiefly put down to the degree of attention, generally speaking, the task imposed upon the subjects was too complicate to bring out the effect of attention as clearly as possible. For this reason we have not made use of KRAEPELIN's cipher-tests but of the cancellation-test.

We need not enter into an argumentation about the well-grounded validity of BOURDON and BINET's [5]) crossing-test, applied in the inquiry into the process of a prolonged concentration of attention. It has an advantage over KRAEPELIN's cipher-test in that, unlike this test, it does not put in requisition any other function except attention [6]).

In a previous investigation one of us established the advantages of this test notably with regard to the attention in children [7]). They consist above all in the fact that the test may be classed among the ordinary schooltasks, and that also young children take an interest in it provided they can read. Besides this, the experiment may be carried out, without any difficulty whatever, with several children simultaneously.

The text consisted of a printed paper of 36 lines, comprising 1768 letters distributed over 304 meaningless words. Some twenty

1) Ueber die Schwankungen der geistigen Arbeitsleistung. Psychologische Arbeiten 2, p. 399.

2) Ueber Ermüdung und Erholung. Ibid. 1, p. 627.

3) Ueber den Einfluss von Arbeitspausen auf die geistige Leistungsfähigkeit. Ibid. 1, p. 300.

4) Ueber den Einfluss des Arbeitswechsels auf fortlaufende geistige Arbeit. Ibid. 2, p. 118.

5) Attention et adaptation. L'année psychologique 6, blz. 364; B. BOURDON: Observations comparatives sur la reconnaissance, la discrimination et l'association. Revue philosophique 40, p. 209.

6) KRAEPELIN: Die Arbeitscurve. Philosophische Studien 19, blz. 459; vgl. ook E. MEUMANN: Vorlesungen zur Einführung in die experimentelle Pädagogik und ihre psychologischen Grundlagen III, Leipzig 1914, p. 390.

7) F. ROELS en JOH. WERKER: Proeven over opmerkzaamheid bij doove, slechthoorende en normale kinderen. Tijdschrift voor Zielkunde en Opvoedingsleer 10, p. 209.

teachers of elementary middle-class schools assisted us. A special procedure had been prescribed for them. The investigation was made with a whole class of from 20 to 30 children at the same time. The task was written on the blackboard and enjoined the pupils to cross as quickly and as well as possible all the *a*'s, the *e*'s and the *h*'s of which there were respectively 122, 331 and 59 in the piece. The trialpaper was put before the pupils immediately before the commencement of the experiment. As the letters to be crossed were written distinctly at the top of it, there was no necessity for the pupil to look at the blackboard in case he should waver.

Before starting the experiment proper the children had some preparatory practice. A piece of paper was laid before them on which were written similar meaningless words, in which they had to cross the *a*'s, the *e*'s and the *h*'s with a pencil. This afforded an opportunity to give several hints necessary for an undisturbed and uniform progress of the experiment. The children were enjoined to cross out the letters only once with a single light stroke, and not to return to words and lines that had already been read.

After the children had received all due information the real tests commenced. The teacher gave the sign for starting and marked the time. At intervals of a minute a signal sounded for the children to indicate with a mark the place they had reached. When a pupil was ready he had to turn over his paper, to put down his pencil and fold his arms till the last of them had finished his task. In all 1123 children from 8—12 years were investigated: 596 boys and 547 girls. Of either sex ± 100 children of the age of 8, 9, 10, 11 or 12 years were at our disposal. As we could make use only of the experimental results of those children who had passed the age-limit by no more than three months and as we also had to put aside some unreliable results, no more than 1073 protocols were at our diposal, distributed over the two sexes and the various ages as follows:

Boys 558.		Girls 513.	
8 years	104	8 years	98
9 „	111	9 „	101
10 „	112	10 „	107
11 „	111	11 „	105
12 „	120	12 „	10

In making up the protocols we determined: 1° the time taken up by the test, a fraction of a minute being eliminated; 2° the total number of mistakes. We considered as mistakes: *a* letters which should have been struck out and were skipped; *b* letters that were struck out by mistake. The mistakes sub *a* and *b* were added up. Finally the mistakes made in every minute were cast up (those sub *a* and *b* also put together).

Table I shows the time required for the test by the children of various ages, boys and girls together and each sex separately. The first column gives the arithmetical means (A. M.); the second the mean deviations (M. D.) and the third the central values (C. V.).

TABLE I.

Age (years)	Boys			Girls			Children		
	A.M.	M.D.	C.V.	A.M.	M.D.	C V.	A.M.	M.D.	C.V.
8	13.5	2.3	13	16	2.2	15	14.7	2.4	14
9	12	2	12	14.5	2.1	14	13.3	2.3	13
10	11	2	11	13.3	2.1	13	12.2	2.2	12
11	10	1.3	10	12.5	2.3	12	11.2	2.2	11
12	10	1.3	9	11	1.9	11	10.5	1.6	10

From this it appears that for boys and girls together, as well as for the two sexes separately, the time required for the labour decreases with the increase of age, leaving aside one exception: the boys of 12 years. (The average decrease from 8—12 years is about 30 %). It should be observed that the working-times of the girls are invariably longer than those of the boys. As to the amount of work done in a certain time the boys are in advance of the girls by two years: the mean working time of the boys of 8 years is only little more than that of the girls of 10, while the mean time of the boys of 10 is exactly equal to that of the girls of 12. These data are substantiated by the mean deviation, which in all cases is at most $^1/_6$ of the arithmetic mean, and by the fact that the central values are invariably equal to or smaller than the arithmetic means. That the mean deviation lessens with the increase of years points to the fact that with the increase of years a decrease is observable of the individual differences regarding the working-speed in various types of life-time.

Table II contains the average number of mistakes, with the mean deviation and the central value, made by the children of different ages, by boys and girls collectively and by each sex separately.

TABLE II.

Age (years)	Boys			Girls			Children		
	A.M.	M.D.	C.V.	A.M.	M.D.	C.V.	A.M.	M.D.	C.V.
8	50.5	23.1	47	53	25.3	44	51.7	24.1	45.5
9	40	17.5	36	36.5	17.5	35	38.2	17.2	36
10	40.3	21.2	33.5	33.4	12.2	31	36.7	16.9	32
11	30.5	13.8	27	33	14.8	31	31.7	14.3	30
12	23.5	11.8	22	27	14.8	25	25.2	13.3	23

Also the number of mistakes, as well as the time-values decreases rather regularly, both for the boys and the girls collectively and for each sex separately, with the increase of years. The boys of 10 years form the only exception. While the average decrease of the number of mistakes for the boys of 8—12 years is 46 %, that for the girls is much larger, viz. 51 %. Since the working-speed in the same space of time increases for both categories only with 30 %, the years seem to exert a greater influence upon accuracy than upon speed. Boys of 8, 9 and 10 years generally make more mistakes than girls of the same age. When considering the arithmetic mean the boys of 8 years seem to form an exception to this rule, but the rule holds good also for them when we consider the central value. From the 11th year, however, the boys have the better of the girls; however, the differences are generally inconsiderable. For either sex and all lifetimes the central values are invariably smaller than the arithmetical means. The mean deviations, on the other hand, are rather considerable, as they amount to $^1/_2$—$^1/_3$ of the arithmetical mean; here also they lessen with the increase of age. As for accuracy the individual differences become less significant as compared with the typical regularities, characterizing a certain age.

Times and mistakes per se do not throw much light upon the nature and the quantity of the work done. To realize both we first have to reduce the quantitative data of times and mistakes to one experimental value, in which either the time values have been reduced to mistake-values or the reverse. We followed the first method and obtained our experimental value by adding to the average

number of mistakes, made by children of a certain age and the same sex, the average number of mistakes made during the minutes, by which the working-time of the group of children under consideration exceeded the minimum time required by a group of one and the same sex for the performance of the task. The same method was applied to the data of boys and girls taken together. Only, in this case we added to the average number of mistakes, made by children of a certain age, the average number of mistakes made by them in the minutes they worked longer than the children of the group that required the shortest time for the performance of the task.

The values thus obtained are collected in

TABLE III.

Age (years)	Boys	Girls	Children
8	61.4	62	61.7
9	45	47.5	46.25
10	39.5	40.6	40.05
11	29.7	38.8	34.35
12	22	29.6	25.8

The quality and the quantity of the work achieved being in inverse proportion to the experimental value, it is evident from the table that boys work better than girls of the same age. At 8-, 9-, and 10 years the difference is not so great, but at 11-, and 12 years it manifests itself distinctly. The proficiency exhibited with the advance of years is best illustrated by a comparison of the experimental values for boys and girls together: from 8—12 years it amounts to as much as 58 %.

In Table IV we have given in the first column in percentages of the experimental values the superiority of the achievements of the boys over those of the girls of the same age. In the next three columns we have tabulated the progress made by the boys and girls together, and the two sexes separately, from 8—9, from 9—10, from 10—11, and from 11— 12 years. The increase of the achievement with every transition is expressed in percentages of the experimental values.

When examining tables III and IV more closely we see, that the achievements at 12 of boys and girls separately and of the two sexes collectively surpass those at 8 resp. 2.8, 2.2, and 2.4 times.

Broadly speaking the work done by boys of 11 years is equal to the work done by girls of 12 years. Whereas with boys a slowing

TABLE IV.

Age (years)	Advantage gained by the boys	Age	Progress of the boys	Progress of the girls	Progress of the children
8	1	8—9	27	23	25
9	5	9—10	13	15	14
10	2	10—11	25	5	15
11	24	11—12	26	24	25
12	26				

of the regular progress is observed between 9 and 10 years, with girls it is not noticeable before the 10th year. Before and after that time the yearly progress is much more considerable both with boys and with girls.

Table V divides the children in quick, fairly quick and slow workers. The time required for the task varying from 6 to 18 minutes, these three categories corresponded respectively with the three time-groups 6—9, 10—13, and 14—18 minutes [1]).

For every age, for boys and girls separately, and for the two sexes collectively, we give the percentages of the frequency, with which quick, fairly quick and slow workers occur.

TABLE V.

Age (years)	Boys			Girls			Children		
	Quick	Fairly quick	Slow	Quick	Fairly quick	Slow	Quick	Fairly quick	Slow
8	6	43	51	0	22	78	3	32.5	64.5
9	14	57	29	0	34	66	7	45.5	47.5
10	24	56	20	6	49	45	15	52.5	32.5
11	45	50	5	20	43	37	32.5	46.5	21
12	50	49	1	32	56	12	41	52.5	6.5

[1]) Vgl. F. Roels en Joh. Werker: Proeven over opmerkzaamheid bij doove, slechthoorende en normale kinderen. l.c., p. 212 and 213.

The percentage of quick workers increases regularly with age for the boys and girls separately and for the two sexes collectively, that of the slow workers lessens regularly. As to the fairly quick workers the change in their percentage is not regular, neither for the boys nor for the girls. The moving up from slow to fairly quick workers is among the boys, with the exception of boys of from 8 to 9 years, invariably less considerable than from fairly quick workers to quick ones. Among the girls, on the other hand, the transition from the fairly-quick to the quick workers is more marked only from 10—11 and from 11 to 12 years of age. The difference in this respect between boys and girls manifest itself still more distinctly by the fact, that of the 50 % decrease in the percentage of slow workers among the boys from 8—12 years, 44 % falls to the benefit of quick workers and 6 % to the fairly quick-ones; for the girls these values are respectively 32 % and 34 %.

This has been tabulated in

TABLE VI.

Age (years)	Boys			Girls			Children		
	Quick	Fairly quick	Slow	Quick	Fairly quick	Slow	Quick	Fairly quick	Slow
8—9	+ 8	+14	—22	—	+12	—12	+ 4	+13	—17
9—10	+10	— 1	— 9	+ 6	+15	—21	+ 8	+ 7	—15
10—11	+21	— 6	—15	+14	— 6	— 8	+17.5	— 6	—11.5
11—12	+ 5	— 1	— 4	+12	+13	—25	+ 8.5	+ 6	—14.5
Increase in five years	+44	+ 6	—50	+32	+34	—66	+38	+20	—58

It illustrates how the decrease in the number of slow workers for the boys and girls separately and for the two sexes collectively benefits the groups of quick and fairly quick workers. The values were obtained by calculating the difference, positive and negative, of the percentage of the quick, the fairly quick, and the slow workers in two successive years. For each of the three several groups the algebraic sum of these differences expresses the increase or the decrease of the percentage of the quick, and the slow workers in the five years.

When reverting to Table V we see that more than half the number of boys and rather more than 3/4 of that of the girls of 8 years are slow workers. Among the boys of 8 and 9 years there are respectively 6 and 14 % quick workers; among the girls of the same age there are none. Since among the girls of 10 years there is as large a percentage of quick workers as among the boys of 8, the latter have the start of the former by two years. Later on this advantage lessens: then the boys between 9 and 10 years are on a level with the girls of 11; boys between 10 and 11 years on a level with girls of 12.

Conversely, the percentage of slow workers among the girls is invariably greater than that of the boys of the same age. In course of time it also decreases far less rapidly among the girls. So also here the boys are in advance of the girls. Among the boys of 9

TABLE VII.

Age (years)		Boys			Girls			Children		
		Quick	Fairly quick	Slow	Quick	Fairly quick	Slow	Quick	Fairly quick	Slow
	A.M.	72	53	46	—	79	46	(72)	66	46
8	M.D.	30	23	17	—	42	19	(30)	31	20
	C.V.	61	49	40	—	59	39	(61)	51	40
	A.M.	50	40	34	—	41	34	(50)	40	34
9	M.D.	23	17	16	—	17	15	(23)	17	15
	C.V.	44	35	28	—	43	42	(44)	39	32
	A.M.	48	39	29	35	35	30	42	37	29
10	M.D.	26	20	13	7	14	11	24	17	12
	C.V.	40	34	27	33	35	27	39	34	27
	A.M.	32	28	24	48	33	27	40	30	25
11	M.D.	14	14	8	14	14	12	16	14	12
	C.V.	32	25	22	46	31	24	35	26	23
	A.M.	25.5	21	8	34	25	19	30	23	13
12	M.D.	13	10	—	13	15.5	8	13	13	8
	C.V.	23	20	—	33	17.5	16.5	28	20	18

years e.g., there are fewer slow workers than among the girls of 11 years. Boys between 10 and 11 years are about on a par with girls of 12.

For the boys and girls separately and for the two sexes collectively we have calculated the mean number of mistakes, made by the quick, the fairly quick, and the slow workers of each lifetime. The arithmetical mean, by the side of it the mean deviation and central value is given for each of the three groups and for each age in the columns of Table VII. Some figures in the column of quick workers under the heading "children" have been put in brackets to indicate that they are the results of calculation of the data for boys, since among the girls of 8 or 9 years there are no quick workers. (See Table VII p. 1379).

The average number of mistakes invariably increases at each age for the boys and girls separately and for both sexes together with the speed of working. There is only one exception; for the girls of 10 years the mean number of mistakes of the quick workers is as great as that of the fairly quick workers. The mean deviation is rather high: $^1/_3$—$^1/_2$ of the arithmetical mean; however the central value is generally lower than the average.

In discussing the data of table IV we have already pointed out a slowing in the regular progress of the boys between 9 and 10 years. Among the girls this occurs only between the age of 10 and 11. This phenomenon is corroborated by the data of table VII. In all categories the decrease of the number of mistakes is least for boys of 9—10 and for girls of 10—11. As to the quick workers among the girls the mean number of mistakes made by girls of 11 is even greater than that made by girls of 10.

The above is elucidated by table VIII which shows for boys and for girls the decrease of the number of mistakes from year to year in percentages for quick, fairly quick, and slow workers. We also added the percentages of the total decrease of the number of mistakes in the five years[1]). The slowing in the decrease of the mistakes of boys of 9—10 and girls of 10—11 years, irrespective of their being quick, fairly quick or slow workers, is conspicuous. The abrupt fall among the boys of 8—9 years is striking, among the girls it does not appear before a twelvemonth later.

The decrease of the number of mistakes in the five years is

[1]) Data concerning the total decrease of the number of mistakes in the five years for slow workers among the boys and quick workers among the girls have not been tabulated here, seeing that there were not enough slow workers among the boys of 12 and no quick workers at all among the girls from 8—9 years.

TABLE VIII.

Age (years)	Boys			Girls		
	Quick	Fairly quick	Slow	Quick	Fairly quick	Slow
8—9	28	29	30	—	28	+ 8
9—10	9	3	4	—	19	36
10—11	20	27	19	+ 28	12	11
11—12	28	20	—	26	44	31
Decrease in the five years	62 %	59 %	—	—	70 %	58 %

largest among the girls; it progresses with the speed of working. For the quick, and the fairly quick workers among the boys and for the fairly quick, and the slow workers among the girls it amounts resp. to 62 %, 59 %, 70 % and 58 %.

In Tables IX and X the data of the other tables are specified. Table IX contains for the boys and the girls separately and for the two sexes collectively the percentage of children of each of the three categories, that actually made from 0—10 %, from 11—20 %, from 21—30 %, from 31—40 % and from 41—50 % of all the mistakes that could be made.

In arranging the data for this table we considered as mistakes the letters which should have been struck out and were skipped; letters that were crossed mistakenly were left out of consideration. We could readily do so because the latter are by far fewer in number than the former. In fact they might have been left out of consideration in all our calculations without interfering with the accuracy of the experimental data. The scarcity of percentages in the columns 31—40 % and 41—50 % is due to the fact that the groups concerned did not yield enough cases of the percentages under consideration to be worked out mathematically.

Table X gives for boys and girls separately in percentages the increase of the percentage of children that made from 0—10 % and from 11—20 % of the possible number of mistakes. As mistakes were considered, just as in table IX letters that should have been struck out but were skipped. No calculations were made from the

TABLE IX.

Age (years)		0—10	11—20	21—30	31—40	41—50
8	B.	58	37	2	2	1
	G.	60	30	6	4	—
	C.	59	33.5	4	3	0.5
9	B.	76	22	2	—	—
	G.	75	24	1	—	—
	C.	75.5	23	1.5	—	—
10	B.	72	20	8	—	—
	G.	88	12	—	—	—
	C.	80	16	4	—	—
11	B.	86	14	—	—	—
	G.	86	13	1	—	—
	C.	86	13.5	0.5	—	—
12	B.	95	5	—	—	—
	G.	90	9	1	—	—
	C.	92.5	7	0.5	—	—

TABLE X.

Age (years)	Boys		Girls	
	Increase of mistakes 0-10%	Increase of mistakes 11-20%	Increase of mistakes 0-10%	Increase of mistakes 11-20%
8—9	+ 18	— 15	+ 15	- 6
9—10	— 4	— 2	+ 13	- 12
10—11	+ 14	— 6	— 2	+ 1
11—12	+ 9	— 9	+ 4	— 4
Increase in five years	+ 37	— 32	+ 30	-- 21

data of the columns 31—40 %, and 41—50 %, their number being too small. Lastly we annexed the percentages of the total increase of the number of mistakes in the five years.

The data of tables IX and X confirm in every respect the results hitherto obtained. For boys and girls a constant increase is to be observed of the number of cases in which from 0 to 10 % of the total number of possible mistakes was made; on the contrary a constant decrease of the number of cases with 11—20 % of the total number of possible mistakes. Here also the downward progress among the boys from 9—10 years and the girls from 10—11 years is remarkable. The increase and the decrease for boys is most marked from 10—11 years; whereas the girls present the best results before the transition from 9—10 years, certainly with respect to the decrease but to some extent also as regards the increase. Let it finally be observed that, taking the five years together, the increase and the decrease are invariably greater among the boys than among the girls. This experience, of course, does not conflict with the fact shown in table II, that the number of mistakes for the boys from 8—12 years diminishes on an average with 45 %, for the girls, however with 57 %.

In that table we recorded the absolute decrease of the mistakes; here, however, the decrease of a certain group — the cases of 11—20 % of the total of the possible number of mistakes — to the benefit of another group, viz. that of cases of 1—10 % of the total of the possible mistakes.

CONCLUSIONS.

1. The time, required for the work, decreases on an average with 30 % from the 8th to the 12th year.

2. The working times of the girls are invariably longer than those of the boys. As regards speed of working the boys are generally in advance of the girls by 2 years.

3. The accuracy of working increases from the 8th to the 12th year for the boys with 46 %, for the girls with 51 %. Boys of 8, 9 and 10 years generally make more mistakes than girls of the same age; after the eleventh year, however, the boys surpass the girls.

4. By adding up to the mean number of mistakes, made by children of a certain age and a definite sex, the mean number of mistakes made in the minutes, by which the working time of the category of children under consideration surpassed the smallest

number of minutes, required by a group of the same sex for the achievement of the same sex for the achievement of the task, we obtained an experimental value, comprising the quantitative data regarding time and mistakes. From these experimental values it appears that boys always work better than girls. At 8, 9 and 10 years of age the difference is not so great. It is very conspicuous however for the 11-, and 12-year old children. The improvement in the work of 8—12 year old children is 58 %.

5. Among boys from 9—10 years a slowing is observed in the regular progress; among the girls it appears a year later.

6. The percentage of quick workers increases regularly as they grow older for the boys and girls separately and for the two sexes collectively; that of the slow workers decreases. The advance from slow to fairly quick workers is among the boys (with the exception of those from 8—9 years) invariably less than the advance from the fairly quick-, to the quick workers. Among the girls, on the contrary, the advance from the fairly quick-, to the quick workers is most marked only from 10—11 and from 11—12 years.

7. As regards the number quick workers among boys of 8 years, the boys have the better of the girls. This superiority lessens later on; then the boys from 9—10 years are on a par with girls of 12. The same applies to the number of slow workers. Among the boys of 9 years e.g. there are fewer slow workers than among the girls of 11. Boys between 10—11 years are on a level with girls of about 12 years.

8. With one exception the accuracy of working decreases at each age for the boys and girls separately and for the two sexes collectively. The increase of accuracy in the five years is largest among the girls; it augments with the speed of working.

9. For the quick-, the fairly quick, and the slow workers among the boys a slowing of the regularly increasing accuracy is observed from 9—10 years; for those among the girls a year later.

II.

In the Proceedings of the Meeting of Feb. 26. 1921 [1]) we published under the same title the results of an inquiry into the phenomena of attention, appearing during persistent labour. We now present a sequel to it in a number of data concerning the types of workers that acted as experimental subjects. For the technique and the arrangement of the investigations I refer to our previous publication.

[1]) Verslagen van de Kon. Akad. v. Wet. Wis- en Natk. Afd. Dl. XXIX, blz. 1077.

In Table I we have grouped for every one of the three several categories of quick, fairly quick, and slow workers among the boys and girls separately and among the two sexes collectively, the number of minutes required for the task in three groups of consecutive minutes. If the number of minutes was not divisible by 3, the first and the last group were made equal so that the middle one was one minute larger or smaller [1]). Then the average number of letters

TABLE I.

Age (years)	Periods	Boys			Girls			Children		
		Quick	Fairly quick	Slow	Quick	Fairly quick	Slow	Quick	Fairly quick	Slow
8	I	153	118	99	—	118	79	(153)	118	90
	II	152	157	97	—	125	110	(152)	142	106
	III	198	127	103	—	107	126	(198)	119	171
9	I	172	127	105	—	126	87	(172)	126	98
	II	179	158	101	—	135	120	(179)	146	113
	III	203	137	115	—	117	139	(203)	129	127
10	I	175	138	116	163	135	93	168	137	104
	II	176	170	108	161	143	137	171	156	121
	III	201	147	120	198	125	128	200	135	123
11	I	191	151	127	174	142	105	181	146	115
	II	192	185	117	172	159	134	181	171	125
	III	219	161	131	212	135	143	215	147	136
12	I	191	152	128	176	153	110	182	152	116
	II	193	185	117	176	157	152	183	170	134
	III	220	163	130	219	137	132	217	147	130

[1]) Cf. F. Roels and Joh. Werker: Proeven over opmerkzaamheid bij doove, slechthoorende en normale kinderen, l.c., p. 212 and 213.

F. Roels: Vergelijkend onderzoek van eenige met behulp der natuurlijke en experimenteele leerwijze bij de studie van het geheugen verkregen resultaten. Verslagen Kon. Akademie van Wet. 1917, deel XXV, blz. 1315 en 1316.

J. Dauber: Zur Entwicklung der psychischen Leistungsfähigkeit. Fortschritte der Psychologie und ihrer Anwendungen, 5, blz. 86, 108, 117 en 130.

read in a minute was calculated for every group. By comparing the values for every group we can conceive the changes in the working-rate of the quick, the fairly quick and the slow-workers as the labour proceeds.

The quick workers among the boys and the girls work invariably harder in the third period than in the first and second; in the last two their working rate is about equal. In course of time the difference in working-rate with this category of boys and girls between the first and the third period first decreases (from 8 to 10), then remains constant (the differences between the third and the first periods are resp. 29, 18, 15, 15 and 15 %). For the rapid workers among the girls the difference in working-rate between the third, and the first period decreases a little from the 10th to the 12th year, although not much; the differences are resp. 21, 22 and 24 %.

The fairly quick boys always work hardest in the 2d period, and in the third harder than in the first. As to the fairly quick girls we also find that they work hardest in the second period but, contrary to the boys, they work quicker in the first period than in the third. Since the changes in the working-rate in the three periods with the fairly quick and the slow workers among the boys and girls are not so simple as with the quick workers, we have tabulated below for the quick, the fairly quick and the slow workers among the boys and girls the percentage with which the working rate increased from the first to the second and decreased from the second

TABLE II.

Age (years)	Boys						Girls					
	Quick		Fairly quick		Slow		Quick		Fairly quick		Slow	
	I—II	II—III	I—II	II—III	I—II	II—III	I—II	II—III	I—II	II—III	I—II	II—III
8	—1	+30	+33	—24	— 2	+ 5	—	—	+ 6	—17	+39	+14
9	+4	+13	+24	—15	— 5	+14	—	—	+ 7	—15	+38	+16
10	+0.5	+14	+23	—16	— 7	+11	— 1	+23	+ 6	—14	+47	— 7
11	+0.5	+14	+22	—15	— 8	+12	— 1	+23	+12	—18	+28	+ 7
12	+1	+14	+22	—13	— 9	+11	0	+24	+ 3	—15	+38	—15

to the third period. The percentages express the increase or decrease of working rate in the second, resp. the third period as compared with the preceding period.

It now appears, that as regards the fairly-quick boys of 8 years, there is a large increase from I—II, on the contrary a less considerable decrease from II to III. For the other ages the increase from I to II is also larger than the decrease from II to III.

On the one side however the percentage of the increase in the working-rate from I—II, and on the other the percentage of decrease from II to III remains approximately constant (resp. + 24, + 23, + 22, + 22 %, and — 15, — 16, — 15, — 13 %). The percentages with which the working-rate among the fairly-quick girls of different ages increases from I to II and decreases from II to III do not require special discussion.

With a few exceptions the slow boys and girls (the girls from 11—12 years) work hardest in the third period. But whereas the boys work harder in the first period than in the second, the reverse occurs with the girls. Just as in the case of the fairly-quick girls, the percentages of increase and decrease in the working-rate of the slow boys and girls of different ages from I to II and from II to III, do not give rise to any further discussion. For the boys, with the exception of those of 8 years, these percentages are approximately constant. This constancy is however not noticeable among the slow girls.

It would be too bold to draw any conclusions from the changes in the working-rates in connection with the influence exerted by the various factors that come into play with persistently continued labour, such as the start, the adaptation, practice, fatigue, abrupt instinctive actions of the will, voluntary concentrations of longer duration, the finish etc. [1]) The task our experimental subjects were directed to perform, was too simple and too uniform for such conclusions. Similarly our inquiry does not afford reliable evidence relative to the problem of the working-types. Moreover, the factors governing the working-process and to whose interference the different types owe their existence, lacked scope to display the irinfluence to the full during the comparatively short period required for our experiments. Nevertheless some of the above regularities may be classed with one of the types distinguished by MEUMANN from a quantitative point of view with persistently continued labour MEU-

[1]) Cf. RIVERS und KRAEPELIN: Ueber Ermüdung und Erholung, Psychologische Arbeiten 1, blz. 636 en 639; LINDLEY: Ueber Arbeit und Ruhe. Ibidem 3, blz. 513; v. VOSS: Ueber die Schwankungen der geistigen Arbeitsleistung. Ibidem 2. blz. 399.

MANN observes: "Vielleicht können wir drei Hauptformen des Arbeitsverlaufs unterscheiden, indem bei einigen Individuen die Arbeit mit einer relativ grossen Leistung einsetzt und dann mit mancherlei Schwankungen allmählich abnimmt, bei einer zweiten Gruppe von Menschen erreicht die Arbeit erst nach längerer Zeit ihr Maximum, um dann allmählig abzunehmen, bei einer dritten tritt das Arbeitsmaximum erst gegen das Ende einer längeren Arbeit ein.[1]) Considering that not any of the groups of boys and girls worked quickest in the first period, we could not verify the first type. The fairly-quick children, with the highest working-rate in the second period, are no doubt to be classified with the second type. Nor can it be doubted that MEUMANN'S third type is represented by the quick workers among the boys and the girls that work hardest in the third period and about equally hard in the first and second, as well as by the slow girls that work quicker and quicker and by the slow boys that, in spite of a fair start, begin to slacken a little in the second period, but still reach their maximum in the third.

With regard to the types of workers we wish to say another word: With the exception of the eight-year-old boys and girls among the three categories and all the female subjects among the slow workers, all the other groups, whichever may be the age of the children, exhibit a striking constancy in the percentages of increase or decrease of rate resp. from the first to the second and from the second to the third. This appeared from a closer inspection of the data in table II. The conclusion therefore seems justifiable that the types are formed after the 8th year and maintain themselves at the very least until the 12th year. New experiments with older children will have to show whether or no and if so when the type changes again. We cannot find out to what cause the exception of the slow workers among the girls is to be ascribed.

Lastly we have given in Table III a survey of the mistakes made in each category in the three periods established for Table I. A comparison of the values for every group enables us to form an idea of the changes in the accuracy of working among the three categories of workers as the work progresses.

The quick workers among the boys and girls of different ages most often make most mistakes in the third period; only the boys from 9 to 12 and the girls of 12 years of age make most mistakes in the second period; for the boys of 9 and the girls of 12 the

[1]) Vorlesungen zur Einführung in die experimentelle Pädagogik und ihre psychologischen Grundlagen. II, Leipzig 1913, blz. 389; vgl. ook III, blz. 51.

TABLE III.

Age (years)	Periods	Boys			Girls			Children		
		Quick	Fairly quick	Slow	Quick	Fairly quick	Slow	Quick	Fairly quick	Slow
	I	25	28	31	—	23	29	(25)	27	31
8	II	28	38	32	—	41	35	(28)	38	33
	III	47	34	37	—	36	36	(47)	35	36
	I	32	26	29	—	32	29	(32)	28	29
9	II	35	42	39	--	38	34	(35)	41	35
	III	33	32	32	—	30	37	(33)	31	36
	I	24	27	28	28	24	31	25	26	34
10	II	37	38	34	28	42	34	35	40	30
	III	39	35	38	44	34	35	40	34	36
	I	22	26	31	29	23	28	25	24	29
11	II	38	44	45	33	37	39	36	38	42
	III	40	30	24	38	40	33	39	38	29
	I	29	28	—	32	25	27	31	26	29
12	II	40	35	—	35	42	37	37	39	36
	III	31	37	—	33	33	36	32	35	35

difference in the percentage of the mistakes is however very small in the second and the third periods. The smallest number of mistakes by the quick workers among the boys and girls of all ages is made in the first period. This phenomenon, which recurs with a single exception also among the fairly quick and the slow workers of both sexes, finds an explanation in the fact that the children are still fresh in the first period and consequently the influence of fatigue does not yet inhibit the favourable action of the start, the adaptation and may be also that of practice. That most mistakes are made in the third period is not surprising when we consider that, as appeared in our discussion of the data of table I and II, the working rate of the quick workers is just greatest in the third period and that, as we stated before, the number of mistakes increases with the working-rate.

The fairly-quick workers of either sex make the largest number of mistakes in the second period — with the exception of the boys of 12 and the girls of 11. Fewest mistakes were ever made in the first period (one exception: the girls of 9). Here again the difference in the percentage of the mistakes in the third and the second period is very small, so that we find also for the fairly-quick workers that the general rule (the number of mistakes increases with the working-rate) not only holds for the working-time and the total number of mistakes, but also for the groups into which we have split them up. The slow workers among the boys and girls form an exception to this almost general rule. Although they work hardest in the third period, it is only the boys of 8 and 10, and the girls of 8, 9, and 10 that make most mistakes in this period. The boys of 9 and 11 and the girls of 11 and 12, on the contrary, make most mistakes in the second period. And whereas the difference in the percentage of mistakes is mostly small for the second and third period in the exceptional cases of the other categories, it is here rather considerable except for the girls of 12.

The reasons for this deviation among the slow workers we were not able to detect. Finally it is worth recording that also the slow workers make fewest mistakes in the first period. Boys of 11 years form the only exception in the 9 cases of slow workers.

CONCLUSIONS.

1. As regards the quick workers among the boys and girls more work is done in the last period than in the first and the second; in the latter two the rate is equal. With years this difference in the working-rate, which occurs in this category of boys between the third and the first period, first decreases (from 8 to 10); then it remains constant. For the quick workers among the girls this difference increases a little, though not much. from 10—12.

2. The fairly-quick boys always work hardest in the second period and in the third harder than in the first. The fairly-quick girls also work hardest in the second period, but in the first harder than in the third.

3. The slow boys and girls work quickest in the third period, with the exception of a few. But whereas the boys work quicker in the first than in the second, just the reverse is the case with the girls.

4. We agree with Meumann in distinguishing three principal forms: "Indem bei einigen Individuen die Arbeit mit einer relativ grossen

Leistung einsetzt und dann mit mancherlei Schwankungen allmählig abnimmt, bei einer zweiten Gruppe von Menschen erreicht die Arbeit erst nach längerer zeit ihr Maximum, um dann allmählig abzunehmen, bei einer dritten tritt das Arbeitsmaximum erst gegen das Ende einer längeren zeit ein." As not a single group of boys or girls worked quickest in the first period, we could not verify experimentally the occurrence of the first type. To the second type belong no doubt the fairly-quick children with the greatest working-rate in the second period, which ends more slowly with the boys than the girls, but begins stronger with the latter than with the former. No more can it be doubted but that MEUMANN's third type is represented by the quick workers among the boys and the girls, that work hardest in the third period and equally hard in the first and the third, just as by the slow girls, that work quicker and quicker and by the slow boys who, in spite of a fair start, begin to slacken a little in the second period, but nevertheless attain their maximum in the third.

5. After the 8th year the working-type is developed to maintain itself at the very least till the 12th year. New inquiries with older children will have to be made to ascertain whether or no, and if so when, changes will occur still later.

6. The general rule that accuracy decreases with the working-rate does not only hold good for the working-time and the total number of mistakes, but also for the groups of the first, the second and the third period into which we have split them up.

Physiology. — "*Concerning Sulphaemoglobinaemia.*" By Prof. A. A. HIJMANS VAN DEN BERGH.

(Communicated at the meeting of March 26, 1921).

The blue colour of the skin and the mucous membranes, called *cyanosis* is almost exclusively due to a lowered oxygen content of the blood. Occasionally it is brought about by a slight modification of the blood-pigment into the so-called methaemoglobin, consequent on poisoning with some substances, such as nitrites, nitrobenzol, anilin-derivatives and the like. We are greatly indebted to STOKVIS and to TALMA[1]) for pointing out to us the fact that in some cases of intestinal disease poisonous substances formed in the intestinal canal are resorbed in the blood, and likewise evoke methaemoglobinaemia with cyanosis. My own researches made in 1905[2]) went to show that in these forms of enterogenous methaemoglobinaemia nitrites are answerable for the formation of methaemoglobin from the bloodpigment. In addition, on further investigation I came across people exhibiting such a cyanosis, which, however, appeared not to be due to the presence of methaemoglobin but to a substance that possessed qualities which proved it to be sulphaemoglobin i.e. the compound formed when allowing small quanta of sulphuretted hydrogen to act upon haemoglobin. In all these cases the serum was free from dissolved pigment, so that there was no haemolysis. Afterwards we met with another form of methaemo- and sulphaemoglobinaemia[3]), this time attended with marked haemolysis and most likely caused by anaerobe bacteria. So we know already three forms of this cyanosis: the septic sulphaemoglobinaemia with haemolysis; the intraglobular methaemoglobinaemia; and the intraglobular sulphaemoglobinaemia. We now intend to record some new researches concerning the intraglobular sulphaemoglobinaemia.

Our own observations together with those of various English physici-

[1]) STOKVIS, Festschr. f. Leyden, en Ned. Tijdschr. v. Geneesk. 1902, II, 678. TALMA, Ned. Tijdschr. v. Geneesk., 1902, II, 721.

[2]) HIJMANS V. D. BERGH, Deutsch. Arch. f. klin. Mediz. 1905, LXXXIII, 86. HIJMANS V. D. BERGH en GRUTTERINK, Berlin. klin. Woch. 1906, I.

[3]) HIJMANS V. D. BERGH, Ned Tijdschr. v. Geneesk. 1918, I. 1774.

ans, notably the contributions of WOOD CLARKE and HURTLEY [1]) and of MACKENZIE WALLIS [2]) seem to indicate that as a rule this sulphaemoglobinaemia is presumably owing to a stasis in the large intestine. That we had indeed to do with the sulphuretted hydrogen-compound of haemoglobin was evident from the spectrum of the blood in these conditions, which is quite similar to that of sulphaemoglobin: besides the two familiar bands of the haemoglobin, a band is also discernible in the red near λ 617. None of the other substances examined by us displayed such a spectrum. It appeared moreover that reducing substances, acting on the blood of these patients, did not take this band away, whereas the band of the methaemoglobin, which resembles most the sulphaemoglobin, disappears directly after the addition of ammonium sulphid or of STOKES' reagent. Chemically however we could not demonstrate sulphuretted hydrogen in the bloodserum. This was all the more surprising because according to ERICH MEYER's investigations the chemical methods to demonstrate H_2S are much more sensitive than the spectroscopic. This tallies with the fact that in cadaveric blood drawn twice 24 hours after death, sulphuretted hydrogen is chemically demonstrable in the serum, while spectroscopically nothing might be seen of a sulphaemoglobin-absorption band.

To my colleague Prof. LAMERIS I am especially indebted for the observation of a young patient, who was suffering from an enterogenous cyanosis due to sulphaemoglobinaemia, and who enabled us to inspect more narrowly the above-named problem. This boy suffers from the so-called HIRSCHSPRUNG's disease i. e. a marked dilatation of the colon, existing from birth, and in large measure obstructed defecation. The investigations, which I purpose to record here have for the major portion been conducted by Dr. ENGELKES. In co-operation with him I have tried to set at rest the above question. To begin with it appeared that the boy's blood presented a marked sulfo-band. We could confirm the phenomenon detected by WEST and CLARKE [3]) that, in sulphaemoglobinaemia, on passing pure carbon monoxid in a solution of sulphaemoglobin, the band in the red is shifted 5 wavelengths to the right. In addition we could superadd a new reaction on sulphaemoglobin to the previous one. If namely a drop of a 1 °/₀ sol. of potassium cyanid is added to a sol. of SHb, the sulfo-band will persist at room-temperature for a long time; the MHb-band disappears directly. Addition of a very

[1]) WOOD CLARKE and HURTLEY, Journ. Physiol. 1907, XXXVI, 62.
[2]) MACKENZIE WALLIS, the Quarterly Journal of Medicine, 1913, VII, 73.
[3]) WEST and WOOD CLARKE, Lancet, 1907, I, 272.

large quantity of a concentrated potassium-cyanid solution also makes the sulfo-band disappear, but then only after some minutes. If to an SHb-solution an equal volume of a 1 % solution of CNK is added, at a temperature of 37° the S-band disappears only after three hours.

By passing a current of CO_2 through a little serum from the boy and collecting this in a solution of lead-acetate or even in the sensitive reagent of CARO-FISCHER we again failed to demonstrate sulphuretted hydrogen in the serum.

This induced us to discontinue the chemical tests with sulphuretted hydrogen in the serum, and try another method. With due caution to ensure sterility we allowed serum to act at 37° on a normal haemoglobin solution. It then appeared that in the majority of the tests a little sulphaemoglobin had been formed from the normal haemoglobin after a sojourn of 2×24 hours in the incubator. Now the idea suggested itself whether the serum might contain bacteria to which the formation of sulphaemoglobin could be ascribed. This idea had already occurred to GIBSON [1]) regarding methaemoglobinaemia and, indeed, this author achieved a positive result. Later experimenters, however, could not confirm it. A bacteriological examination of our patient's blood, carried out by Mr. SCHAAP also showed it to be sterile.

We now allowed *very small* quanta of hydrogen sulphid to act upon normal haemoglobin-solutions; this yielded the sulphoband in the incubator at 37° only after a good deal of time (24 hours), the same result as had generally been obtained in the action of serum from our patient on normal haemoglobin-solution. We were justified in concluding from this fact, that small amounts of sulphuretted hydrogen must be contained in the serum. Supposing that by passing a current of carbon dioxid, too much of the small amounts of hydrogen sulphid would get lost to recover the rest, we resolved to react directly on the serum. Lead-salts did not answer our purpose as they precipitate protein, which tampered with our result. Various technical circumstances prevented us for the time from applying the most sensitive reagent of CARO-FISCHER. We then had recourse to the reagent of KRAL, consisting of a weak nitro-prussid-sodium solution in a soda-alkaline or ammoniacal environment. When adding such a solution to a solution containing a trace of hydrogen sulphid, a beautiful red, violet-tinged coloration comes forth. This reaction is believed to be less sensitive than the two others, nevertheless we obtained pronounced results, when we applied it as a

[1]) GIBSON and DOUGLAS, Lancet, 1906, II. 72.

ring-test on the patient's serum. In that case the reaction was always positive, so that in this way the presence of hydrogen sulphid in the serum was established chemically. To make assurance double sure possible sources of error had to be precluded. For acetone and kreatinin also give a coloration with nitroprussid in ammoniacal solution. We detected, however, that, as regards acetone, this reaction turns out negative for a multiple of the largest possible quantities occurring in the bloodserum. A negative result was likewise obtained for two patients with pronounced acetonuria (diabetes). Similarly kreatinin yields with this reagent a positive result only in a multiple concentration of the largest possible quantities occurring in the serum. Moreover the colour of the kreatinin-ring differs widely from that of the hydrogen sulphid ring. Finally we have examined in the same way the serum of a certain number of normal persons. The result was negative.

In order to get more certainty that it was sulphaemoglobin we had detected, we proceeded as follows. A little of the patient's blood was collected in a physiological common-salt solution and washed out with it repeatedly, so that all the serum was removed. The spectrum of the red bloodcorpuscles appeared to have retained the sulpho-band, while the pipetted liquid did not yield a reaction with Kral's reagent.

We now added to a mixture of red bloodcorpuscles and physiological saltsolution a little of a 2% potassium cyanid solution (neutralized and in physiological NaCl solution to prevent haemolysis). The mixture was placed some time in the incubator. After some hours the HCN had expelled the H_2S; the SHb-spectrum had made room for that of CyHb and in the supernatant fluid we obtained a positive reaction with nitroprussid [1]).

It would seem, then, that hereby the presence of H_2S, be it only in small quantities, in the serum as well as in the blood-pigment of our patient, had been established.

Earlier experiments of Claude Bernard have demonstrated that hydrogen sulphid injected intravenously into animals, is exhaled through the lungs. Although, comparatively speaking the quantities of H_2S in the blood of our patient are not inconsiderable, we did not succeed in demonstrating in this simple manner H_2S in the exhaled air. We did get a positive result, however, when the boy had been breathing for ± one hour in a specially contrived

[1]) When prosecuting our investigation we found this phenomenon to be of a more complex nature than can be anticipated from the description given in this paper.

apparatus. Of course we endeavoured to preclude errors by control-experiments with normal subjects and by taking due measures.

It proved possible to determine the amount of Hb which is converted to SHb. This determination is based on the now generally received conception that haemoglobin is a well-defined substance that contains to each molecule one atom of iron.

In a closed vessel supplied with pure oxygen, haemoglobin takes up a constant amount of oxygen. The amount of iron in a given quantity of haemoglobin and its loosely combined oxygen in an environment of pure oxygen is constant: 2 atoms of O to 1 atom of Fe, which is expressed in volumina: 401 cc of oxygen to one gram of iron. The derivatives of the red blood-pigment, the methhb., the sulphhb., the cyanhb., the haematin are assumed not to take up any oxygen from a gas-mixture. If, therefore, blood that contains besides oxyhb., also SHb is brought into contact with an atmosphere of pure oxygen and the amount of loosely combined oxygen and of iron is determined, the amount of the converted Hb may be calculated from the difference between the known ratio of Fe : O and of a pure OHb solution under the same conditions (401 : 1).

Now, BARCROFT's method affords a rather simple way to perform an accurate gas-analysis of the blood, while the titanium method is quite adapted for the iron-determination in this liquid. Dr. ENGELKES used them in investigating the blood of our patient. It became evident that the quantity of converted Hb varied at different times, as had already been made out spectroscopically.

Once we found a quantity of converted Hb of 19 %, another time of 12.5 %.

In these inquiries the clinician meets with an impediment in that, as a matter of course, he can work only with minimal quantities and cannot often repeat an experiment. The field of research is widened considerably when experimenting with animals. This proved possible. At the outset of our investigation when we had not yet succeeded in demonstrating chemically the presence of H_2S in the serum, we have injected intravenously some of that serum into rabbits. To our great surprise we found already after an hour in the blood of one of these rabbits a rather large quantity of SHb. It was not necessary to look for the cause of this surprisingly rapid action of such small amounts of the serum upon rabbit's blood, for on closer inspection we were still more surprised at detecting in the blood of some perfectly healthy, fresh rabbits a physiological amount of SHb which was distinctly demonstrable by the spectroscope. We have subsequently examined about 26 rabbits.

In 4 of them we found comperatively much SHb., in 13 a small quantity and in 9 none at all. A quantitative determination by gas-analysis and a Fe-determination in a rabbit with marked S-band yielded the result:

aorta-blood . . . . 10.5 °/₀.
vena-porta blood 12.5 °/₀.

Now let us revert to the original question, why in the serum from our patients (and the same will be the case with rabbits) so little H_2S is found with a more or less marked S-band., whereas in cadaveric blood the opposite ratio reveals itself. In my opinion this should be interpreted as follows; the H_2S that is taken up from the colon by the blood, is first distributed over corpuscles and serum. In the corpuscles it forms a solid compound, SHb, which does not dissociate, consequently it does not give off H_2S in the lungs. It is most likely destroyed slowly and removed from the blood, presumably together with the remaining Hb-molecule. It is quite different with the H_2S in the plasma, which is there combined with alkali in solution. Soon an equilibrium will be established between the gases dissolved in the serum and those of the alveolar air, and since the alveolar air is constantly refreshed by respiration and the atmospherical air does practically not contain H_2S, the H_2S dissolved in the plasma will escape from it. In serum or tissues part of it will be destroyed by oxydation. Thus, in consequence of this process and of the respiration the serum will, with the exception of a few traces only, be liberated from H_2S. In cadaveric blood more and more of H_2S is taken up by the blood as putrefaction progresses. Here again H_2S is distributed over corpuscles and serum. But respiration and oxydation are absent, the blood is locked up in the vessels, so the dissolved H_2S is not withdrawn from the liquid. The quantities of H_2S dissolved in the serum are large enough to be demonstrated by the sensitive chemical reaction. At the cadaveric temperature the quantity does not suffice to convert Hb into SHb within a given lapse of time. With progressing putrefaction and longer duration of the action the Shb-spectrum will reveal itself.

In the course of our inquiry we saw three more patients, in whom we noted marked Shb-aemie, consequent on a slowed passage of the contents of the colon. The fact that sulphaemoglobinaemia proved to be of more frequent occurrence than we had originally supposed, and especially the other fact that this blood anomaly to a certain percentage, occurs in otherwise healthy rabbits, enhance the significance of the results recorded in this paper.

For it is really surprising that, with men in pathological and with animals even in physiological condition, a substance is taken up by the blood from the intestinal canal to the amount of 20 %, which is poisonous for the central nervous system, and which, by combining with the respiratory pigment renders part of it worthless.

The investigations described above do not yield conclusive evidence for the Shb-aemia in patients with slowed progress of the contents of the colon. We have been impressed with the idea that in patients in whom there was an impediment in the advance of the intestinal content, cyanosis was evolved in a comparatively short time — almost rather abruptly. It still, looks as if another hitherto unknown factor besides the resorption of H_2S from the gut, is instrumental to the origin of Shb-aemia. Whether this factor be the presence of reducing substances, as MACKENZIE-WALLIS concludes from his interesting inquiry, will have to be made out by further experiments. We were not so fortunate as to demonstrate such reducing substances in the serum. Likewise the question whether SHb-aemia occurs perhaps more frequently than spectrosopic examination could make out up to the present day, must be left for subsequent investigation. In our judgment this spectroscopic result cannot be achieved before about 10 % of OHb have been converted to SHb.

Mathematics. — *"Bestimmung der Klassenzahl aller Unterkörper des Kreiskörpers der m-ten Einheitswurzeln."* (Verbesserung.) By N. G. W. H. BEEGER. (Communicated by Prof. W. KAPTEYN.)

(Communicated at the meeting of November 26, 1921).

Anstatt von 4. § 4 meines Beitrages in diesen "Proceedings" (Vol. XXII, S. 331 und 395) lese man Folgendes:

4. Der Grad der gemeinschaftliche Untergruppe, von g und der Zerlegungsgruppe von l_1, ist gleich $d_1 d'_1$ wenn d_1 der gröszte gemeinschaftliche Teiler aller Zahlen b_{1n} bedeutet und d'_1 der gröszte gemeinschaftliche Teiler von f_1 mit den Summen:

$$\left\} \tfrac{1}{2} \varphi\, a_0\, b_{0n} + 2\varphi\left(\frac{m}{2^{h_*}}\right) a_*\, b_{*n} + \varphi\left(\frac{m}{l_2^{\;h_2}}\right) a_2\, b_{2n} + \ldots . \right\{ : e_1\, \varphi_1$$

wo n nur die Werte durchlauft für welche $b_{1n} = 0$ ist. Die Exponenten a_i sind bestimmt durch die Congruenz

$$l_1 + n\frac{m}{l_2^{\;h_1}} \equiv A_0^{\;a_0} A^{\;a_*} A_2^{\;a_2} \ldots . \;(mod\; m)$$

während die Zahl im linken Gliede in Satz 5 definirt ist.

Beweis: Die Substitutionen der Zerlegungsgruppe entstehen durch Multiplizieren der Trägheitsgruppe (Satz 4) mit der zyklischen Gruppe aus Satz 5. Aus Satz 4 ergibt sich dasz die Substitutionen der Trägheitsgruppe gebildet werden von den Resten der Potenzen $A_1{}^y$ $(mod\; m)$ worin y gewisze Zahlenwerte hat; die Substitutionen der Zerlegungsgruppe werden gebildet von den f_1 ersten Potenzen der linken Gliedes obenstehender Congruenz. Die Substitution der Zerlegungsgruppe

$$A_0^{\;xa_0} A_*^{\;xa_*} A_1^{\;y} A_2^{\;xa_2} \ldots . \;(mod\; m)$$

wird also dann und nur dann eine Substitution der Gruppe g sein, wenn für jedes System der b die Congruenz

$$\tfrac{1}{2} \varphi\, a_0\, x\, b_{0n} + 2\varphi\left(\frac{m}{2^{h_*}}\right) a_*\, x b_{*n} + \varphi\left(\frac{m}{l_1^{\;h_1}}\right) y\, b_{1n} + \varphi\left(\frac{m}{l_2^{\;h_2}}\right) a_2\, x\, b_{2n} + \ldots .$$
$$\equiv 0 \;(mod\; \varphi).$$

gilt. Wir müszen also die Anzahl der Zahlensysteme x, y bestimmen, welche diesen Congruenzen genüge leisten.

Es folgt aus der Definition der Zahl f_1 und der Exponenten a_i:

$$f_1\, a_0 \equiv 0 \;(mod\; 2)\,; \qquad f_1\, a_* \equiv 0 \;(mod\; \tfrac{1}{2}\varphi_*)\,; \qquad f_1\, a_2 \equiv 0 \;(mod\; \varphi_2)\,; \ldots .$$

Die Summen

$$f_1\left(\tfrac{1}{2}\varphi\, a_0\, b_{0n} + 2\varphi\left(\frac{m}{2^{h_*}}\right)a_*\, b_{*n} + \varphi\left(\frac{m}{l_2^{\,h_2}}\right)a_2\, b_{2n} + \ldots.\right)$$

worin n alle Zahlenwerte hat für welchen $b_{1n}=0$ ist, sind daher teilbar durch φ, und da $e_1 f_1 = \dfrac{\varphi}{\varphi_1}$ ist, werden die Summen

$$\tfrac{1}{2}\varphi\, a_0\, b_{0n} + 2\varphi\left(\frac{m}{2^{h_*}}\right)a_*\, b_{*n} + \varphi\left(\frac{m}{l_2^{\,h_2}}\right)a_2\, b_{2n} + \ldots.$$

durch $e_1\varphi_1$ teilbar sein.

Die Congruenzen nehmen daher die folgende Form an:

$$x\varphi_1\,\frac{\tfrac{1}{2}\varphi_0 a_0 b_{0n} + 2\varphi\left(\frac{m}{2^{h_*}}\right)a_* b_{*n} + \varphi\left(\frac{m}{l_2^{\,h_2}}\right)a_2 b_{2n} + \ldots}{e_1\varphi_1} + f_1 y b_{1n} \equiv 0 \,(mod\, f_1\varphi_1). \quad (1)$$

Es ist $0 \leqq x < f_1$ und $0 \leqq y < \varphi_1$ und es ist erlaubt die ganze Zahlen welche durch die gebrochene Form dargestellt werden, durch ihre Reste (mod f_1) zu ersetzen. Der Modul der Congruenzen (1) ist das Product der Anzahlen der Wertevorräte f_1 für x und φ_1 für y; die Coefficienten der Unbekannten x und y, in diesen Congruenzen, bilden eine Gruppe im additieven Sinne. Die Congruenzen sind daher derselben Art wie (3) Seite 337 aber jetzt mit nur zwei Unbekannten. Nach der Bemerkung am Ende des § 3 ist die Anzahl der Lösungen (x, y) also gleich das Quotient der Zahl $f_1\varphi_1$ und der Zahl der Coefficientensysteme. Wir haben nun diese letzte Zahl zu bestimmen. Die Anzahl der verschiedenen Werte von b_{1n} ist $\dfrac{\varphi_1}{d_1}$

Zu den Zahlen $b_{1n}=0$ gehören $\dfrac{f_1}{d_1}$ verschiedenen Werte der Bruchform von (1) Der Wert $b_{1n}=0$ musz daher gleichviel Mal vorkommen und darum werden die übrige $\dfrac{\varphi_1}{d_1}-1$ andere Werte der b_{1n} auch soviel Mal auftreten. Es gibt also $\dfrac{\varphi_1}{d_1}.\dfrac{f_1}{d_1}$, Coefficientensysteme. Die Anzahl der Lösungen von (1) ist daher

$$f_1\,\varphi_1 : \frac{\varphi_1}{d_1}\frac{f_1}{d'_1} = d_1\, d'_1.$$

Weiter füge man Folgendes an dem Beweise des Satzes **10** auf Seite 396 zu, und lasse die letzten 12 Zeilen dieser Seite fort.

Auszerdem genügen die gesuchten Stellen der b noch den Con-

gruenzen, welche man bekommt wenn man in (8) die Systeme der a durch ihre Vielfache ersetzt. Man wird aber dann auch Systeme der a bekommen, die schon unter die Vorigen vorkommen. Der kleinste Wert von x für welchen

$$\tfrac{1}{2}\varphi\left(\frac{m}{l_1^{h_1}}\right) x\, a_0 b_{0n} + 2\varphi\left(\frac{m}{2^{h_*} l_1^{h_1}}\right) a_* x\, b_{*n} + \varphi\left(\frac{m}{l_1^{h_1} l_2^{h_2}}\right) x\, a_2 b_{2n} + \ldots \equiv 0 \left(\mathrm{mod}\ \frac{\varphi}{\varphi_1}\right)$$

ist, ist $\frac{f_1}{d'_1}$. Nimmt man also für x die Zahlen $0, 1, 2, \ldots . \frac{f_1}{d'_1} - 1$ so sind die Systeme xa_0, xa_*, xa_2, ... alle von den vorigen verschieden, weil diese letzten eben diejenige sind welche den Congruenzen

$$\tfrac{1}{2}\varphi\, a_0 b_{0n} + 2\varphi\left(\frac{m}{2^{h_*}}\right) a_* b_{*n} + \varphi\left(\frac{m}{l_2^{h_2}}\right) a_2 b_{2n} + \ldots . \equiv 0\ (\mathrm{mod}\ \varphi)$$

genügen wenn n nur die Zahlenwerte hat für welche $b_{1n} = 0$ ist. Die Totalanzahl der Systeme der a die man in (8) berücksichtigen musz, ist also

$$\frac{r}{d_1} \cdot \frac{f_1}{d'_1} = \frac{rf_1}{d_1 d'_1},$$

und daher die Anzahl der gesuchten Systeme der b:

$$\frac{\varphi}{\varphi_1} : \frac{rf_1}{d_1 d'_1} = \frac{e_1 d_1 d'_1}{r}.$$

Weiter folge man den Beweis auf Seite 397 dritte Zeile.

Physics. — "*The Propagation of Light in Moving, Transparent, Solid Substances*". III. *Measurements on the Fizeau-Effect in Flint Glass.* By Prof. P. ZEEMAN, W. DE GROOT, Miss A. SNETHLAGE and G. C. DIBBETZ.

(Communicated at the meeting of April 23, 1920.)

More accurate than the results obtained with quartz, an account of which was given in communication II [1]); are those of the Fizeau-effect in moving flint glass.

Six cylindric rods of a length of 20 cm. and a circular cross-section of a diameter of 25 mm. were made for us by the firm ZEISS at Jena. The kind of glass is the ordinary silicate flint glass of the type 0.103 of the firm SCHOTT und GENOSSEN. The endplanes are plane-parallel in close approximation. The clearness of the interference fringes appeared to be excellent with stationary glass column, while, when the necessary precautions were taken, also when the column was in rapid motion, the fringes remained still very good. The photos taken were much better than those that had been obtained before (II) with quartz. This is partly owing to the excellent material [2]), to the greater cross-sections of the rods (now 25 mm. as against 15 mm. before for quartz), and to the smaller number of internal reflections [2]). It appeared finally possible to observe also the Fizeau-effect for moving flint glass directly in a telescope, as clearly as it is possible for moving water, and we had the privilege to demonstrate the effect before several physicists.

The perfect sureness with which the rather complicated apparatus worked at last, was not obtained until some improvements had been made in the arrangement as it had been used for quartz. We will discuss the principal of them.

2. Through different causes the interference fringes can take an oblique position during the movement of the column of glass cylinders. It is, however, necessary that the fringes remain parallel to

1) These Proc. Vol. XXII, N°. 6, p. 512.

2) Compare II, 2.

the horizontal or vertical cross-lines. Else no photos are obtained on which measurements can be made.

Already in the experiments with quartz a *compensator* was inserted in one of the interfering beams of light, consisting of a plane-parallel circular glass plate of a thickness of 5 mm. and a diameter of 25 mm., to which every desired position could be given. The inclination of the interference fringes can be modified by rotation round a horizontal axis; a simple arrangement was, therefore, applied through which the observer, sitting at the eye-piece of the telescope; could bring about the desired rotation. Besides a plane-parallel plate was placed before the object glass of the telescope in such a way that an image of the interference fringes could be observed in a small telescope placed on one side, while at the same time after removal of the eye-glass a photo of the fringes was made with the large telescope. Thus the observer at the small telescope could at once observe an error in the position of the fringes, and if necessary, redress it during the photographing. This proved to be but rarely necessary when an experiment had been properly prepared.

3. As was set forth before (I, 4), it was necessary to superpose 20 to 30 photographs of the interference fringes, each with an exposure of a hundredth second, because otherwise the photographic image was too faint. This number could be greatly reduced by working without filters, hence directly with the white arc-light.

Diminution of the number of exposures increases the sharpness of the photos, and renders it possible to take more in succession, before the disturbances through fluctuations of the temperature in the glass-rods, which inevitably occur in consequence of the movement of the apparatus, become troublesome.

For the interpretation of the photo obtained it is then necessary to know what is the effective wave-length λ of the white arc-light, with which the fringes have been photographed.

The accuracy in the determination of λ need not be very great, as will appear presently (see 5).

4. *Determination of the effective wave-length of the light used.*

The effective wave-length of the operative light, had to be measured *after* it had left the last mirror of the interferometer, and of course for that kind of plates that was used in the experiments.

The beam from the interferometer was focussed with a cylinder lens on the slit of the collimator of a Hilger spectroscope with

constant deviation, from which the prism had been removed, and replaced by a small totally reflecting prism.

By placing a grating replica before the object glass of the photographic camera, a spectrum of the source of light could be photographed.

The most active part in this spectrum could be made directly visible by putting a wedge of smoked glass with the side horizontal before the slit of the collimator (KENNETH MEES method).

The prismatic action of the wedge was counteracted by a second wedge of clear glass.

The result for the effective wave-length λ was 4750 A°, with an uncertainty of ± 25 A°.

5. This accuracy is, however, sufficient. This can be verified by numerical calculation, or by the following consideration.

We know (II, 9) that the optical effect is given by the formula:

$$\Delta = \frac{4\,ho}{\lambda c}\left(\mu - 1 - \lambda \frac{d\mu}{d\lambda}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

By deriving from this the value of $\frac{1}{\Delta}\frac{d\Delta}{d\lambda}$, we can see how great the influence is of an error in the determination of the effective wave-length on the calculated effect.

Instead of λ we introduce the frequency ν, which brings about a slight simplication, and we put $\varphi = \mu - 1 + \nu \frac{d\mu}{d\nu}$. Then:

$$\frac{1}{\Delta}\frac{d\Delta}{d\lambda} = \frac{1}{\varphi \,.\, \nu}\frac{d\varphi \,.\, \nu}{d\nu} \times -\frac{1}{\lambda^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Now:

$$\frac{d\varphi}{d\nu} = 2\frac{d\mu}{d\nu} + \nu\frac{d^2\mu}{d\nu^2},$$

for which we may write in approximation:

$$\frac{d\varphi}{d\nu} = 2\frac{d\mu}{d\nu},$$

because μ depends almost linearly on ν.

Equation (2) then becomes:

$$\frac{1}{\Delta}\frac{d\Delta}{d\lambda} = \nu\left(\frac{2\nu\frac{d\mu}{d\nu}}{\varphi} + 1\right),$$

$\nu \dfrac{d\mu}{d\nu} : \varphi$ becomes about $\frac{1}{8}$ for the ordinary flint glass, so that:

$$\frac{1}{\Delta}\frac{d\Delta}{d\lambda} = \frac{1}{\lambda} \times \frac{5}{4}$$

For $\lambda = 5000$ A°, $d\lambda = 25$ A° becomes $\dfrac{d\Delta}{\Delta} = {}^{3}/_{4}\ {}^{0}/_{0}$.

6. *Ribbon-shutter.* The shutter which acts periodically and is worked electromagnetically, described in I, 4, repeatedly gave cause for disappointment, because it was never certain that the light was transmitted at the very moment that the cradle passes a chosen point of the path.

This is perfectly certain with the ribbon-shutter, which is diagrammatically represented in fig. 1.

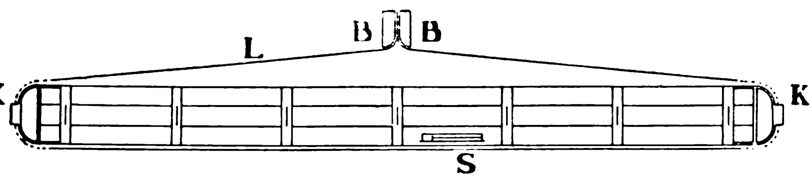


Fig. 1.

A band of ribbon L of lancaster linen is clasped between two blocks B, which are firmly fastened at a chosen place of the bed of the apparatus. It is passed round the beam with the glass column. The beam can execute its usual movement to and fro without being hindered by the ribbon, for this can easily slide over the copper pieces K, the length of the ribbon remaining constant. There have been made two openings in the ribbon of 10 or 15 cm., which at a certain position of the beam, but only then, allow the light to pass through circular holes in the pieces K, and during the time that corresponds with the length of the openings in the ribbon. By displacing BB along the bed, the moment at which the light is transmitted, may be chosen. The edges of the ribbon are provided with a hem to obtain greater firmness, and prevent fraying.

The copper pieces K are smoothly polished, and the friction of the ribbon is therefore very slight. Sometimes it was still diminished by some talc-powder.

The electric shutter, which was used in the experiments described in II, was now used after a small modification to admit the light only in one of the movements to and fro of the beam. For this purpose the movable arm is placed before the arc-lamp. The phase

is adjusted so that the arm turns when the beam has just passed the middle of its course.

7. *Improvement of the Velocity Measurement.* In the experiments with quartz rods the velocity was measured directly according to a method which has been described in II, § 11. We have made this method simpler and more delicate, and also arranged it so that the velocity could be immediately read in every experiment.

In main lines the arrangement is still the same as represented in II § 11. The screen with two slits S_1 and S_2 used formerly was however, replaced by a screen S (cf. figure of the preceding §), the construction of which will be further explained by referring to figure 2.

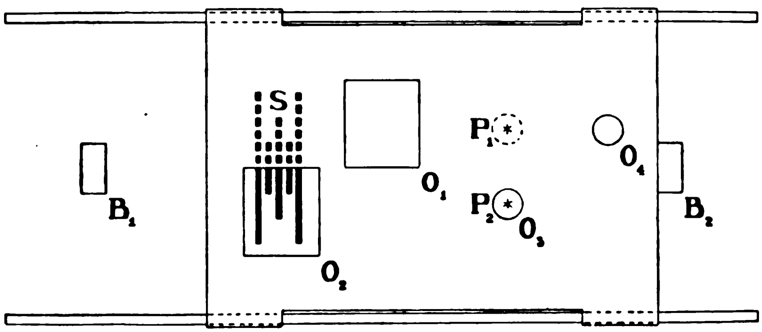


Fig. 2.

Our earlier slit S_1 is replaced by a glass scale, slit S_2 by a small aperture.

The graduated glass scale was obtained by covering a glass plate with a soot-layer, and by drawing by the aid of a chisel of the width of exactly 1 mm., five lines in the soot-layer at mutual distances of exactly 1 mm.

Then the soot-layer was fixed with a drop of varnish, and the first line was covered with red glass, the others with blue glass. All this was cemented on the beam. In fig. 2 the alternate long and short lines are indicated under S; the colours render errors of front and back in the observation through lenses impossible. In the figure the lines are partially dotted, as they are half covered by a screen, which can slide to and fro. During the movement of the beam in one sense the lower, during the movement in opposite sense the upper half of the scale is automatically covered through the inertia of the screen. The blocks B_1 and B_2 define the extreme positions of the screen.

As we said, our former slit S_1 has been replaced by the scale with the coloured lines; instead of the slit S_2 there are two fine

apertures P_1 and P_2 on either side of a horizontal line through the middle of the scale. In the position drawn in the figure the light can leave through P_2, because the opening O_2 in the moving screen allows this. When the screen rests against B_1, P_2 is covered, and O_2 comes in the position, in which it is possible that light is emitted through P_1.

As was explained in II § 11 an image of the slit arrangement is projected on the rotating disc R, which is provided with radial slits. With a weakly magnifying telescope the image projected on the rotating disc is observed with intervals of 0,001 sec. In this time the beam moves about 1 cm. at the velocity used in the neighbourhood of 10 $^M/_{sec}$. The observer sees the coloured scale S in the field of vision, and then the "star" P_2, or rather the after-image of the light emitted by this star at a former transmission. The place of the star on the scale can be read accurately down to ½ mm., and the distance from P_2 to S being about 50 mm., the velocity can be determined certainly accurately down to $1\,^0/_0$. All the changes in the velocity of the beam are immediately visible, and the velocity corresponding to every photo taken can at once be noted down.

It is necessary that the "star" moves at the same level at which the axis of the rotating disc has been placed, for else a small correction must still be applied to the velocity.

Thanks are due to Mr. W. M. Kok, assistant at the Physical Laboratory, for his valuable help in the execution of the arrangement for the velocity measurement.

8. *Results.* The extreme values of the velocity which were directly measured in our experiments, were 918 and 994 cm/sec.

There were made two series of measurements, which were distinguished by the way in which the velocity was found. In the first series, A, the method of communication II, in the second, B, that described above in § 7, was followed. All the results for the effect were reduced to a velocity of 1000 cm./sec.

Series A.

When the measurements of the 34 separate photos obtained on 11 plates are combined, the effect is found to be 0,247 ± 0,006.

When first the observations on each plate are combined, and the mean is taken of the results of 11 plates, the effect is found to be 0,247 ± 0,009.

Series B.

gives for the effect derived from 49 observations divided over 13 plates 0,238 ± 0,006, for the effect derived from the mean of the results of the 13 plates 0,240 ± 0,008.

Finally all the 83 observations may be combined; then 0,242 ± 0,004 is found.

When a result is calculated for each of the 24 plates separately, and then the mean is taken, the effect is found to be 0,243 ± 0,006.

The number after the ± sign indicates the mean error, and as was stated before all the results have been reduced to the velocity 1000 cm./sec.

Theoretical value of the effect for flint-glass.

The firm SCHOTT und GENOSSEN, Jena, gives the following indices of refraction for the flint-glass 0.103 used:

$$
\begin{array}{ll}
\mu_A' = 1.6099 & \lambda = 7677\ A^\circ \\
\mu_C' = 1.6153 & \quad 6563 \\
\mu_D' = 1.6202 & \quad 5893 \\
\mu_F' = 1.6324 & \quad 4862 \\
\mu_G' = \quad 6428 & \quad 4341
\end{array}
$$

The effect can be calculated according to the formula derived before (II, 9)

$$\Delta = \frac{4\, lw}{\lambda c}\left(\mu - 1 - \lambda \frac{d\mu}{d\lambda}\right).$$

By the aid of the given data, values for μ and $\frac{d\mu}{d\lambda}$ are derived for the effective wave-length 4750. The final result is:

$$\Delta = \frac{4 \,.\, 120 \,.\, 1000}{4750 \times 10^{-8} \times 3.10^{10}} (1,634 - 1 + 0,084) = 0,242.$$

This value is almost in perfect agreement with the value yielded by our experiments.

It is of interest to note that the dispersion term contributes to the value of the effect $= 0,242$ by an amount of 0,028.

If the dispersion had not been taken into account, 0,214 would have been found for the effect, which is incompatible with the experiments.

APPENDIX.

1. The publication of the above communication, which was already laid before the meeting of the Academy of April 23rd 1920, has been delayed through particular circumstances.

This affords me an opportunity to add a few remarks to the paper.

Our collaborator, Mr. W. DE GROOT, phil. docts. orally informed me of another derivation of the formula for the optical effect (II, 9),

and independent of this at about the same time Prof. F. ZERNIKE at Groningen did the same in a letter dated November 11th 1919. The short, elementary derivation, which in the two communications is founded on the same idea, will follow here.

In principle the experiment with the moving glass rod can be classed under the following scheme (fig. 3). From the source of

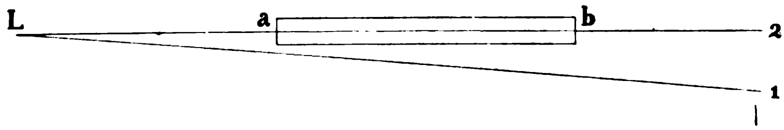


Fig. 3.

light L issue two beams of light, one through the glass ab, the other through the air. With the aid of the necessary devices the phases 1 and 2 are compared in two cases, first when the glass ab is at rest, secondly when it moves e.g. to the left with the velocity w. According to the principle of relativity the glass may as well be at rest, and the room with the other parts may be made to move to the right with the velocity w. Whether 1 and 2 are received with a moving or a stationary apparatus, makes no difference in the relative phase of the beams of light. Therefore only L is made to move to the right with the relative velocity w, and approaches the glass rod. This only gives a DOPPLER effect equal for the two beams, in which the wave-length varies from λ to $\lambda - \lambda \frac{w}{c}$. When everything is at rest, the phase-difference between 1 and 2 is $\frac{(\mu - 1)}{\lambda} l$, when l is the length of the glass rod ab.

Hence the change due to the movement is:

$$-\frac{(\mu-1)\,l}{\lambda^2}\,d\lambda + \frac{l}{\lambda}\frac{d\mu}{d\lambda}\,d\lambda = \frac{wl}{\lambda c}\left(\mu - 1 - \lambda\frac{d\mu}{d\lambda}\right)$$

To get the total effect the formula should be multiplied by 4 — i.e. a factor 2 for the movement to and fro of the rays, and a factor 2 on account of the reversal of the direction of the movement — so that the formula given before, appears.

Also FIZEAU's experiment with the moving water and stationary glass end-plates may be treated by the method sketched above, but then the calculation is not so simple.

Mr. ZERNIKE still points out that an actual experiment might be taken with the two beams of light running in opposite directions and stationary glass rod, as is supposed in the calculation. It would

then only be required in the experiment described before to make the glass prism slide to and fro, and keep the glass rod at rest. For different optical and mechanical reasons the execution seems to me attended with greater difficulties than the experiment made.

2. In a letter of October 22nd 1919 Prof. M. VON LAUE had the kindness to draw my attention to a thesis for the doctorate of P. HARRESS of 1912 [1]), which he sent me, and in which a subject is treated closely related to our investigations. In HARRESS's experiment the light runs to the right and to the left in a cycle of glass prisms, which as a whole is in rotatory motion.

The comparison of the observed displacement of the interference-fringes and the theory elaborated by HARRESS gave a very unsatisfactory agreement. This is chiefly owing to the theory, in which the absolute and the relative velocity of the light are mixed up. VON LAUE has redressed this error, which greatly improves the agreement between theory and observation. HARRESS's experiment closely resembles SAGNAC's experiment of 1913, of which, very remarkably, VON LAUE gave the relativistic theory already in 1911 [2]).

In SAGNAC's and HARRESS's experiments the displacement Δ of the interference fringes expressed as fraction of their distances is:

$$\Delta = \frac{2\omega}{c\lambda} \Sigma\, rl,$$

in which l is the length of the path passed over in the rotating apparatus, r the distance of this to the axis of rotation, ω the angular velocity, the sum extended to all the different paths.

Index of refraction and dispersion do not occur in this formula, which in itself is already a difference with our experiments.

It seems unnecessary to enter into a fuller discussion of HARRESS's work, as in an interesting paper by VON LAUE [3]) the experiments by FIZEAU, SAGNAC, HARRESS, and those made by us are discussed and compared, and as with exclusion of the influence of the dispersion the two last-mentioned experiments have also already been treated in the fourth edition of VON LAUE's Relativitätstheorie [4]).

3. I pointed out on an earlier occasion that it might be interesting to examine substances in which $\frac{d\mu}{d\lambda}$ is great.

---

[1]) Re-edited in O. KNOPF, Ann. d. Phys. 62, 389, 1920.

[2]) Münchener Sitz. Ber. 1911, 404.

[3]) VON LAUE, Ann. d. Phys. 62, 448, 1920.

[4]) Cf., p. 23, 25, 185—189.

In particular substances with strong absorption-bands or lines deserve attention, such as *didymium* compounds, which can be obtained as solid solutions in glass, and the vapour of *sodium*.

When these substances are chosen to work with, horizontal interference lines must be thrown on the slit of the spectroscope. Horizontal lines are observed in the telescope, which diminish in distance from red to violet. Through the FIZEAU-effect the lines would move up and down in case of rapid motion to and fro of selectively absorbing substances, and at those places in the spectrum where $\frac{d\mu}{d\lambda}$ assumes large values the amplitude of the movement might become considerable.

From preliminary experiments on the dispersion of didymium glass at the ordinary temperature and at that of liquid air it appeared that the value of $\frac{d\mu}{d\lambda}$ assumes nowhere great values in the visible spectrum. Though rods of didymium glass of excellent quality are to be had, I am yet of opinion that it would not be worth the trouble to make experiments with them on the FIZEAU-effect.

Nor can results on the FIZEAU-effect be expected with sodium vapour. Close to the absorption D-lines $\frac{d\mu}{d\lambda}$ can, indeed, become very great, but at the most interesting place near the D-lines the absorption too becomes very great. On continuation of the experiments it might perhaps have deserved recommendation to work with a stationary tube with sodium vapour and moving prism (see § 1 of this appendix). It appeared however, clearly enough from some experiments that the observation of the FIZEAU-effect with sodium vapour was out of the question.

Though these experiments did not yield the result for which they were undertaken, they gave occasion to the observation of an interesting interference phenomenon in sodium vapour, about which a separate communication will shortly follow [1]).

P. Z.

[1]) This communication has been published already. These Proceedings Vol. 24, p. 206, 1922.

Physics. — "*Motion relativated by means of a hypothesis of* A. FÖPPL". By H. ZANSTRA. (Communicated by Prof. P. EHRENFEST).

(Communicated at the meeting of March 26, 1921).

§ 1. *The fixation of "inertial systems" in classical mechanics without applying the principle of absolute motion.*

It is well known that in the equations of motion of classical mechanics for a system of n material points:

$$m_\nu \ddot{x}_\nu = X_\nu \qquad m_\nu \ddot{y}_\nu = Y_\nu \qquad m_\nu \ddot{z}_\nu = Z_\nu \quad . \; . \; . \quad (1)$$
$$(\nu = 1 \ldots . n.)$$

the position of those points is referred to a rectangular system of co-ordinates being at rest or moving uniformly in absolute space. But when we refer this position to another system of axes that does not move uniformly or rotates with respect to the above mentioned systems, the differential equation of motion assumes a more complicated form. E.g. if the system of axes has an absolute rotation, it is well known that we get on the left side of equation (1) terms of the type of centrifugal and Coriolis forces.

Those systems of co-ordinates in which the equation of motion assumes the simplest form (1), the so-called "inertial systems" [1]), are consequently defined by means of the idea of absolute space. This idea, introduced by NEWTON, was at first retained in the later elaboration of NEWTON's mechanics. NEWTON's contemporary BERKELEY however already gave a criticism of this principle of absolute motion (motion in "absolute space"). The purport of his demonstration is that motion of bodies must be referred to other bodies and not to an absolute space [2]).

In more recent times (± 1870) this question was taken up again, especially by C. NEUMANN, LANGE and MACH [3]). While NEUMANN still

[1]) The idea of "inertial systems" was introduced by LANGE he calls them "Inertialsysteme".

[2]) G. BERKELEY. The Principles of Human Knowledge, section 111. e.v.

[3]) E. MACH. Die Mechanik in ihrer Entwickelung. As for LANGE and NEUMANN see § 5. Literature is mentioned in the Enzyklopedie der Math. Wiss. VI 1, p. 30. See moreover H. SEELIGER. Ueber die sogenannte absolute Bewegung. Sitzungsber. der Math. Phys. Klasse der Bayr. Ac. der Wiss. 1906. Bd. 36, p. 85.

supported the idea of absolute motion, later criticism led to the general conviction that motion is relative. The latter being assumed, we have the following two problems:

1. How can inertial systems be fixed without the aid of an "absolute space".

2. Which are, according to this, the equations of motion of mechanics, if we require that absolute co-ordinates do no more occur in them, but exclusively relative co-ordinates (that fix the place of the material points with respect to each other) and their derivatives.

Föppl gave a solution of the first problem, which shall be treated in the next § [1]). In connection with this we shall give in § 4 a solution of the second problem. Here already can be remarked that this solution is quite different from the one given by the theory of Einstein, in § 5 we return to this.

2. *The hypothesis of* A. Föppl [2]), *by which special inertial systems are fixed.*

From here we will deal with motion in plane instead of space, for the sake of simplicity [3]).

With Föppl we assume that the total matter in space consists of a finite number n of material points. For the co-ordinates x', y' in an inertial system we then have the equations:

$$m_\nu \ddot{x}'_\nu = X_\nu. \qquad m \ddot{y}'_\nu = Y_\nu. \quad \ldots\ldots \quad (2)$$
$$(\nu = 1 \ldots \ldots n)$$

We suppose further that for the quantities on the right side (components of force) the law of reciprocal action holds:

$$\Sigma X = 0 \quad \Sigma Y = 0 \quad \ldots\ldots\ldots \quad (3a)$$
$$\Sigma (x' Y - y' X) = 0, \quad \ldots\ldots\ldots \quad (3b)$$

The sign Σ includes all n points. This is the case e.g. if the points apply forces upon each other in the direction of the lines that join them.

From (2), (3a) and (3b) follows:

1) Some short critical remarks on this and some other solutions shall be given in § 5.

2) A. Föppl. Vorlesungen über technische Mechanik. VI. Erster Abschnitt. Die relative Bewegung.

3) For space we have a quite analogic reasoning. Then the use of vectors can be recommended.

$$\Sigma m \ddot{x}' = 0 \quad \Sigma m \ddot{y}' = 0 \quad . \quad . \quad . \quad . \quad . \qquad (4a)$$

$$\Sigma m (x' \ddot{y}' - y' \ddot{x}') = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4b)$$

and after integration:

$$\Sigma m \dot{x}' = T_1 \quad \Sigma m \dot{y}' = T_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5a)$$

$$\Sigma m (x' \dot{y}' - y' \dot{x}') = R, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5b)$$

the 3 constants T_1, T_2 and R still being quite arbitrary. In analogic way we get in space 6 constants T_1, T_2, T_3, R_1, R_2, R_3. It is always possible to choose an inertial system in this way that 3 of these 6 quantities vanish, more of them cannot vanish without a special hypothesis.

The *hypothesis of* Föppl is:

There are inertial systems, for which all six constants T_1, T_2, T_3, R_1, R_2, R_3 *vanish together.*

Consequently for the plane:

$$T_1 = 0 \quad T_2 = 0 \quad R = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

or also, after (5a) and (5b):

$$\Sigma m \dot{x}' = 0 \quad \Sigma m \dot{y}' = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7a)$$

$$\Sigma m (x' \dot{y}' - y' \dot{x}') = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7b)$$

To these special inertial systems belong also those for which moreover $\Sigma mx' = 0$ and $\Sigma my' = 0$, the origin thus coinciding permanently with the centre of gravity of the system of points, such a system is called by Föppl a *principal system of reference* (Hauptbezugssystem) [1]).

Such a principal system $X_\beta Y_\beta$ can be constructed as follows: Take a system of axes $X_\alpha Y_\alpha$ of which the origin coincides permanently with the centre of gravity of the system of points, the axis of X passing permanently through one of the material points. Calculate $\Sigma mr^2 \dot{\theta}$ in $X_\alpha Y_\alpha$. Take a second system of co-ordinates $X_\beta Y_\beta$ with its origin also permanently in the centre of gravity and give it in $X_\alpha Y_\alpha$ a velocity of rotation $\omega = -\dfrac{\Sigma mr^2 \dot{\theta}}{\Sigma mr^2}$. In this way a principal system, consequently an inertial system, has been fixed without the aid of an "absolute space".

[1]) The hypothesis of Föppl in its original form is: for an inertial system with origin in the centre of gravity the total moment of momentum vanishes permanently.

3. *Motion relativated without the aid of the hypothesis of* FÖPPL.

First we will give a solution of the second problem of § 1, *without applying the hypothesis of* FÖPPL.

We assume the same as in § 2, so we start from the differential equation (2), for which hold the relations (4*a*) and (4*b*), we assume moreover that the forces, of which X_ν and Y_ν are the components only depend on the relative position of the points (e.g. Newton force).

Take now a new system of axes XY with its origin permanently in an arbitrary material point, which we will call point 1, and with its axis of X permanently through a second arbitrary-point 2 of the system of points. The new co-ordinates x and y are now relative co-ordinates as meant in § 1. After transformation of the former co-ordinates to the new system of axes [1]), the equations of motion (2) pass, considering (4*a*) and (4*b*), into:

$$\left.\begin{aligned} \ddot{x}_\nu - \dot{w}\, y_\nu - w^2 x_\nu - 2 w \dot{y}_\nu &= \frac{X_\nu}{m_\nu} - \frac{X_1}{m_1} \\ \ddot{y}_\nu + \dot{w}\, x_\nu - w^2 y_\nu + 2 w \dot{x}_\nu &= \frac{Y_\nu}{m_\nu} - \frac{Y_1}{m_1} \end{aligned}\right\} \quad \ldots \quad (8a)$$

in which w is given by:

$$a \dot{w} + 2 b w + c = 0 \quad \ldots \ldots \quad (8b)$$

a, b and c are functions of the x, y, $\dot{x}$, $\dot{y}$ and $\ddot{x}$, $\ddot{y}$ of the n points

$$\left.\begin{aligned} a &= \Sigma m \, \Sigma m (x^2 + y^2) - (\Sigma mx)^2 - (\Sigma my)^2. \\ b &= \Sigma m \, \Sigma m (x\dot{x} + y\dot{y}) - \Sigma mx \, \Sigma m\dot{x} - \Sigma my \, \Sigma m\dot{y}. \\ c &= \Sigma m \, \Sigma m (x\ddot{y} - y\ddot{x}) - (\Sigma mx \, \Sigma m\ddot{y} - \Sigma my \, \Sigma m\ddot{x}). \end{aligned}\right\} \quad (8c)$$

The sign Σ includes all n points.

There are $2n-3$ co-ordinates x_ν, y_ν and according to this $2n-3$ equations (8*a*) of the second order, the auxiliary quantity w occurs in one equation (8*b*) of the first order in w. So (8*a*) and (8*b*) form together a system of order $2(2n-3) + 1 = 4n-5$. After elimination of w there remain $2n-3$ co-ordinates, so that e.g. the system can be reduced to $2n-4$ *equations of the second and one of the third order*: In these equations only the relative co-ordinates, their derivatives and the quantities X_ν Y_ν occur. As we supposed the components $X_\nu Y_\nu$ dependent on the relative position of the material points only, the problem is solved.

---

[1]) See § 5. g.

4. *Motion relativated with the aid of the hypothesis of* Föppl.

In this § we shall give a much simpler solution of the second problem of § 1 by making the same suppositions as in the preceding §, *moreover making use of the hypothesis of* Föppl. Instead of (4*a*) and (4*b*) we take the equations (7*a*) and (7*b*), but further we proceed in exactly the same way.

After transformation to the new, relative co-ordinates [1]), the equations of motion (2) pass, considering (7*a*) and (7*b*), into:

$$\left.\begin{aligned} \ddot{x}_\nu - \dot{w}y_\nu - w^2 x_\nu - 2\,w\dot{y}_\nu &= \frac{X_\nu}{m_\nu} - \frac{X_1}{m_1}. \\ \ddot{y}_\nu + \dot{w}x_\nu - w^2 y_\nu + 2\,w\dot{x}_\nu &= \frac{Y_\nu}{m_\nu} - \frac{Y_1}{m_1}. \end{aligned}\right\} \quad . \ . \ . \ . \quad (9a)$$

in which

$$w = -\frac{\Sigma m \,\Sigma m\,(x\dot{y} - y\dot{x}) - (\Sigma mx\,\Sigma m\dot{y} - \Sigma my\,\Sigma m\dot{x})}{\Sigma m\,\Sigma m\,(x^2 + y^2) - (\Sigma mx)^2 - (\Sigma my)^2} \ . \quad (9b)$$

The sign Σ includes all n points.

The equations (9*a*) are the desired equations of motion in relative co-ordinates, if for w we substitute the value (9*b*). After this substitution we get $2n$—3 *independent differential equations of the second order.*

Without introducing Föppl's hypothesis we came in § 3 to $2n$—4 equations of the second and one of the third order. Owing to the hypothesis of Föppl we have found in this § $2n$—3 equations of the second order. For this reason the equations (9*a*) and (9*b*), we found here, are preferable to the equations (8*a*), (8*b*) and (8*c*).

5. *Remarks.*

a. Lange's method of trial bodies [2]) gives an experimental way of finding inertial systems. He does not discuss however their connection with the total of matter in space.

b. Neumann [3]) and afterwards Boltzmann [4]) try to do this by referring the place of the material points to the principal axes of inertia of the total system. They do not give differential equations. A further consideration of this question will bring the conviction,

[1]) See § 5, *g*.

[2]) L. Lange. Geschichtliche Entwickelung des Bewegungsbegriffs.

[3]) C. Neumann. Ueber die Prinzipien der Galilei-Newtonschen Mechanik.

[4]) Boltzmann. Vorlesungen über die Prinzipien der Mechanik. Leipzig 1904. II, p. 333.

that the differential equations holding in such a system must be of a different form from NEWTON's. On the other hand the system of FÖPPL does not require any alteration of NEWTON's differential equations, but only a definite value of some integration constants (§ 2).

c. The problems studied by KOLKMEIJER [1]) are in many respects more extensive. So he eliminates also absolute time. In this connection it is sufficient to state, that he does not find equations of the type of § 4, because he does not make use of FÖPPL's hypothesis.

d. EINSTEIN gives in his theory of relativity a way of relativating, founded on quite different principles from those held in this essay. This is connected with the fact that he wants to relativate not only mechanical but also electromagnetical and optical phenomena.

e. In connection with an experiment, made by FÖPPL [2]) and a remark by FREUNDLICH [3]), the following thought-experiment may be discussed: At the north-pole of the earth the pendulum-experiment of FOUCAULT is made. Under the pendulum a heavy flywheel with vertical axis of rotation has been mounted. Problem: Does the pendulum's motion alter, when we revolve the flywheel? FÖPPL's hypothesis and our equations give us the following answers: With respect to the principal system remains: 1°. the rotation of the pendulum's plane permanently $=0$; 2°. the sum of moments of momentum of flywheel and earth $=$ a constant, also this sum for the rest of the bodies of the universe, because the total sum has to remain $=0$. Consequently the rotation of the pendulum's plane with respect to the rest of the bodies (the fixed stars) does not alter (it does alter with respect to the earth, the rotation of the latter having undergone some change).

f. From the point of view of NEWTON's mechanics can be said: 1°. If only two celestial bodies were in universe, it were possible that they moved round each other at a constant distance. 2°. A liquid mass, supposed to be the only body in space, can be flattened by centrifugal forces, though no relative motion of its particles is observed. With the view taken here, which is based on FÖPPL's hypothesis, this is impossible.

g. For the transformations of § 4 we can start from a principal

[1]) N. H. KOLKMEYER. Eliminatie van de begrippen assenstelsel, lengte en tijd uit de vergelijkingen voor de planetenbewegingen. Dissertation Amsterdam 1915.

[2]) A. FÖPPL. Sitzungsber. der math.-phys. Klasse der Bayr. Ac. der Wiss. 1904. Bd. 34, p. 3.

[3]) ERWIN FREUNDLICH. Die Grundlagen der EINSTEINschen Gravitationstheorie. Berlin 1916, p. 27.

system, the origin of which is consequently situated in the centre of gravity of the system of points, while the total moment of momentum continuously vanishes (§ 2).

$$m_\nu \ddot{x}'_\nu = X_\nu \qquad m_\nu \ddot{y}'_\nu = Y_\nu \quad \ldots \ldots \ldots \quad (2)$$

$$\Sigma\, m x' = 0 \quad \Sigma\, m y' = 0 \quad \ldots \ldots \ldots \quad (10a)$$

$$\Sigma\, m (x'\dot{y}' - y'\dot{x}') = 0 \quad \ldots \ldots \ldots \quad (10b)$$

First we pass to a second system of axes, with its origin permanently in point 1 of the system of points and which is continuously parallel to the principal system, for this second system holds:

$$\ddot{x}_\nu = \frac{X_\nu}{m_\nu} - \frac{X_1}{m_1} \qquad \ddot{y}_\nu = \frac{Y_\nu}{m_\nu} - \frac{Y_1}{m_1} \quad \ldots \ldots \quad (11)$$

$$\Sigma\, m\, \Sigma\, m (x\dot{y} - y\dot{x}) - (\Sigma\, mx\, \Sigma\, m\dot{y} - \Sigma\, my\, \Sigma\, m\dot{x}) = 0 \ . \quad (12)$$

(10*b*), becomes (12) because according to (10*a*) $x'_1\, \Sigma m = -\Sigma mx$, $y'_1 \Sigma m = -\, \Sigma\, my$.

If we take a third system of axes with its origin also in point 1 and its axis of X permanently through point 2, and indicate the velocity of rotation of the third system of axes with respect to the second by one w, the equations (11) and (12) pass after transformation into the final equations (9*a*) and (*b*). Now w can be considered as an auxiliary quantity, that can be substituted from (9*b*) into (9*a*).

For the transformations of § 3 the first system of axes can also be chosen with its origin in the centre of gravity of the system of points, the calculations are quite analogous, however more complicated.

Physics. — "*Space-time symmetry.* I. *General Considerations*". By N. H. KOLKMEIJER. Communication N°. 7*a* from the Laboratory of Physics and Physical Chemistry of the Veterinary College at Utrecht. (Communicated on behalf of Prof. W. H. KEESOM, Director of the Laboratory, by Prof. H. KAMERLINGH ONNES).

(Communicated at the meeting of December 18, 1920).

§ 1. *Introduction.* In the literature of later years some propositions are found for changes in the original atom-model of RUTHERFORD-BOHR. These propositions are founded on different considerations. So LEWIS [1]) and LANGMUIR [2]) were led by considerations on the chemical structure. SOMMERFELD [3]) was brought to his combinations of ellipses by investigations of the cause of the fact that the defect of charge of the nucleus is not a whole number for the L-series. BORN, LANDÉ and MADELUNG [4]) again studied the absolute dimensions of the elementary cells of crystals, the transformations of energy, and especially the compressibility. By these considerations they were led (partly in cooperation) to the invention and nearer inspection of cubical models analogous to those of LEWIS and LANGMUIR.

Evidently BORN, LANDÉ and MADELUNG especially felt the necessity of a change in the considerations on symmetry that were valid until now, because they considered moving systems. On the same ground I felt necessitated to introduce some new symmetry-elements, in which time also plays a role [5]). It now seemed desirable to consider the space-time-symmetry more systematically than could be done in Communication N°. 4. In this paper the way to attack this problem will be indicated.

§ 2. *Restriction to a definite kind of operations.* A symmetry operation will further on be denoted by Δ, a complex symmetry

1) G. N. LEWIS, Journ. of the Amer. Chem. Soc. 38 (1916) p. 762.
2) I. LANGMUIR, Journ. of the Amer. Chem. Soc. 41 (1919) p. 868.
3) A. SOMMERFELD, Physik. ZS. 19 (1918) p. 297.
4) M. BORN and A. LANDÉ, Verh. d. D. Phys. Ges. 20 (1918) p. 210.
M. BORN, Verh. d. D. Phys. Ges. 20 (1918) p. 230.
A. LANDÉ, Sitz.-Ber. d. Berl. Akad. 1919 p. 101.
Verh. d. D. Phys. Ges. 21 (1919) p. 2, 644, 653.
ZS. f. Phys. 2 (1920) p. 83.
E. MADELUNG and A. LANDÉ, ZS. f. Phys. 2 (1920) p. 230.
5) N. H. KOLKMEIJER, Comm. N°. 4, These Proceedings 23 (1920) p. 120.

operation [1]) by $\Delta\Delta$. SCHOENFLIES [2]) and his predecessors only consider such Δ's that an application of them to a point A produces a point B, the coordinates of which are found from those of A by a linear orthogonal substitution. By these operations the distance between two points therefore does not change.

It seems natural to introduce for space-time symmetry-operations too the restriction that an application of them to two four-dimensional *xyz-ict*-points (or rather $x^1x^2x^3x^4$-points) A and B does not change the four-dimensional distance $A\,B$.

In the first place we thus limit our considerations to those Δ's the algebraic representation of which is a linear orthogonal four dimensional substitution and to corresponding symmetry-elements.

Secondly, (as was also done by SCHOENFLIES and his predecessors in an analogous sense) we exclude those Δ's, the repeated application of which to a point A gives an infinite number of points at the same time within a finite space or within a finite time-interval at the same place.

Thirdly it will prove desirable to introduce still one restriction, which has not its analogue in the three-dimensional problem. From the algebraic substitution mentioned we see, that x^1, x^2, x^3 of the new point B depend on x^4 of the original point A. Thus, application of the Δ's in question to A gives a point B that is displaced in the course of time to an infinite distance even when A remains on the spot. This fact is an objection against the consideration of such a Δ; an objection however that may be avoided by considering only the final result of subsequent applications of more than one Δ of the kind mentioned to a point A, of a $\Delta\Delta$ therefore.

So we limit ourselves to the consideration of such Δ's, the application of which to a point A gives a point B with a world-line parallel with the x^4-axis, when the world-line of A has that direction.

In the next §§ we shall see to which kind of Δ's we are led by this restriction.

§ 3. *Geometrical meaning and analytical indication of one of the kinds of operations considered.* In a R_4 with coordinates x^1, x^2, x^3 and $x^4 = ict$, R_3 be an arbitrary linear space of three dimensions. We shall call R_3 a s y m m e t r y - s p a c e [3]) (symbol r) when the corresponding operation (symbol $\mathfrak{R}$, name space-time-reflection) changes

[1]) In the same sense as f.i. a rotatory-reflection is a $\Delta\Delta$.

[2]) A. SCHOENFLIES. Krystallsysteme und Krystallstructur, Leipzig 1891.

[3]) This name has already been used by P. H. SCHOUTE, Verh. Kon. Ak. Amst. Eerste Sectie II 7 (1894) p. 16.

a point $A(x_1^1, x_1^2, x_1^3, x_1^4)$ into a point $B(x_2^1, x_2^2, x_2^3, x_2^4)$ that is geometrically to be found in the following way.

Draw through A a perpendicular to $\mathfrak{r}$ and measure the length of that perpendicular on the other side of its point of intersection with $\mathfrak{r}$. The end-point of this stretch is the point B.

When $\mathfrak{R}$ is given by the length l_1 of the perpendicular from the origin of R_4 on $\mathfrak{r}$ and its direction cosines φ_1^1, φ_1^2, φ_1^3 and φ_1^4, where

$$\varphi_1^{1^2} + \varphi_1^{2^2} + \varphi_1^{3^2} + \varphi_1^{4^2} = 1, \quad . \; . \; . \; . \; . \quad (1)$$

then we find by substitution of one of the four indices 1, 2, 3 and 4 for n

$$x_2^n = x_1^n + 2\,\varphi_1^n\,(l_1 - \varphi_1^1\,x_1^1 - \varphi_1^2\,x_1^2 - \varphi_1^3\,x_1^3 - \varphi_1^4\,x_1^4) \quad . \; . \quad (2)$$

When in a three-dimensional $x^1x^2x^3$-system we consider a plane V through the origin, the direction cosines of its normal being in the ratio φ_1^1, φ_1^2 and φ_1^3, then the points A' and B', corresponding in this system to A and B in the four-dimensional system, are lying on the same perpendicular to V, $x_2^n - x_1^n$ being proportional to φ_1^n for the values $n = 1, 2, 3$.

When the distances from A' and B' to V are denoted by $-y_m$ with $m = 1$ and 2 resp., then we have

$$y_m = \frac{\varphi_1^1\,x_m^1 + \varphi_1^2\,x_m^2 + \varphi_1^3\,x_m^3}{\sqrt{1-\varphi_1^{4^2}}} \quad . \; . \; . \; . \; . \quad (3)$$

and therefore

$$\left.\begin{aligned} y_2 &= y_1 + 2\sqrt{1-\varphi_1^{4^2}}\,(l_1 - \sqrt{1-\varphi_1^{4^2}}\;y_1 - \varphi_1^4\,x_1^4) \\ x_2^4 &= x_1^4 + 2\,\varphi_1^4\,(l_1 - \sqrt{1-\varphi_1^{4^2}}\;y_1 - \varphi_1^4\,x_1^4) \end{aligned}\right\} \quad . \; . \quad (4)$$

Thus the distance from B' to V and the new value of *ict* are evidently found from the values of these quantities for the point-incident A by drawing in a two-dimensional yx^4-system a line in such a way, that the perpendicular to it from the origin has a length l_1 and forms with the x^4 axis an angle with a cosine $= \varphi_1^4$ and by reflecting the point A'' with the coordinates x_1^4 and y_1 in that line. The coordinates of the point B'' thus found give the new value of the time and the new distance to V in the three-dimensional figure.

§ 4. *Each space-time-symmetry-operation of the considered kind may be regarded as a complex symmetry-operation of space-time-reflections.*

The Δ's treated by Schoenflies and predecessors, reflection in a plane, inversion about a centre, translation, rotation through $\frac{2\pi}{2}$, $\frac{2\pi}{3}$, $\frac{2\pi}{4}$ and $\frac{2\pi}{6}$, 2-, 3-, 4- and 6-al screws, 2-, 3-, 4- and 6-al rotatory-reflections and gliding reflection may all be regarded as

complex reflections in one or more planes[1]). This is a consequence of the linear-orthogonal character of the substitution by which only congruent and symmetrical figures are possible.

For the same reason we must also assume that each imaginable space-time-Δ of the considered kind may be considered as a complex reflection in one or more symmetry-spaces. This shows at once the way in which each space-time-symmetry-element can be found.

§ 5. *General formula for the coordinates of the point B, found from the point A by application of an arbitrary complex space-time-symmetry-operation of the above considered kind.*

$$x_m^n = x_0^n + 2 \sum_{k=1}^{k=m} \left\{ (l_k - \varphi_k^1 x_0^1 - \varphi_k^2 x_0^2 - \varphi_k^3 x_0^3 - \varphi_k^4 x_0^4) \left[\varphi_k^n + \right.\right.$$

$$\left.\left. + \sum_{l=k+1}^{l=m} \varphi_l^n \Big((l,k) + \Sigma\, (l,p)\,(p,k) + \Sigma\, (l,p)\,(p,q)\,(q,k) + \ldots \Big) \right] \right\} \quad . \; . \; (5)$$

where (p,q) has been written for

$$-2\,(\varphi_p^1 \varphi_q^1 + \varphi_p^2 \varphi_q^2 + \varphi_p^3 \varphi_q^3 + \varphi_p^4 \varphi_q^4),$$

while we must take $l > p > q > k$ etc.

§ 6. *We can derive all space-time-symmetry-operations by simply combining all space-symmetry-operations that have been mentioned without the limiting to 2-, 3-, 4- or 6-al axes with time-symmetry-operations.* The linear orthogonal substitution, expressed by formula (5) will have a scheme of coefficients:

$$\left.\begin{matrix} {}_1a_1 & {}_1a_2 & {}_1a_3 & {}_1a_4 & {}_1a_5 \\ {}_2a_1 & {}_2a_2 & {}_2a_3 & {}_2a_4 & {}_2a_5 \\ {}_3a_1 & {}_3a_2 & {}_3a_3 & {}_3a_4 & {}_3a_5 \\ {}_4a_1 & {}_4a_2 & {}_4a_3 & {}_4a_4 & {}_4a_5 \end{matrix}\right\}$$

while these coefficients are connected by the following relations:

$$\left.\begin{array}{l} {}_1a_1^2 + {}_2a_1^2 + {}_3a_1^2 + {}_4a_1^2 = 1 \\ {}_1a_2^2 + {}_2a_2^2 + {}_3a_2^2 + {}_4a_2^2 = 1 \\ {}_1a_3^2 + {}_2a_3^2 + {}_3a_3^2 + {}_4a_3^2 = 1 \\ {}_1a_4^2 + {}_2a_4^2 + {}_3a_4^2 + {}_4a_4^2 = 1 \end{array}\right\} (6) \text{ and } \left.\begin{array}{l} {}_1a_1\,{}_1a_2 + {}_2a_1\,{}_2a_2 + {}_3a_1\,{}_3a_2 + {}_4a_1\,{}_4a_2 = 0 \\ {}_1a_1\,{}_1a_3 + {}_2a_1\,{}_2a_3 + {}_3a_1\,{}_3a_3 + {}_4a_1\,{}_4a_3 = 0 \\ {}_1a_1\,{}_1a_4 + {}_2a_1\,{}_2a_4 + {}_3a_1\,{}_3a_4 + {}_4a_1\,{}_4a_4 = 0 \\ {}_1a_2\,{}_1a_3 + {}_2a_2\,{}_2a_3 + {}_3a_2\,{}_3a_3 + {}_4a_2\,{}_4a_3 = 0 \\ {}_1a_2\,{}_1a_4 + {}_2a_2\,{}_2a_4 + {}_3a_2\,{}_3a_4 + {}_4a_2\,{}_4a_4 = 0 \\ {}_1a_3\,{}_1a_4 + {}_2a_3\,{}_2a_4 + {}_3a_3\,{}_3a_4 + {}_4a_3\,{}_4a_4 = 0 \end{array}\right\} (7)$$

---

[1]) (Note added during translation). See f.i. C. VIOLA N. Jahrb. f. Miner., Geol. und Pal. Beil. Bd. 10 p. 495 1896. G. WULFF Zs f. Kryst. u. Miner. 27 p. 556 1896.

Because of the third restriction introduced in § 2 we must suppose ${}_1a_4$, ${}_2a_4$ and ${}_3a_4$ to equal zero. Substituting this in (6) and (7) we find:

$$ {}_4a_4^2 = 1 \quad {}_4a_1 = 0 \quad {}_4a_2 = 0 \quad {}_4a_3 = 0 \quad . \quad . \quad . \quad . \quad . \quad (8) $$

while (6) and (7) are then reduced to:

$$ \left. \begin{array}{l} {}_1a_1^2 + {}_2a_1^2 + {}_3a_1^2 = 1 \\ {}_1a_2^2 + {}_2a_2^2 + {}_3a_2^2 = 1 \\ {}_1a_3^2 + {}_2a_3^2 + {}_3a_3^2 = 1 \end{array} \right\} \ . \ (9) \text{ and } \left. \begin{array}{l} {}_1a_1\,{}_1a_2 + {}_2a_1\,{}_2a_2 + {}_3a_1\,{}_3a_2 = 0 \\ {}_1a_1\,{}_1a_3 + {}_2a_1\,{}_2a_3 + {}_3a_1\,{}_3a_3 = 0 \\ {}_1a_2\,{}_1a_3 + {}_2a_2\,{}_2a_3 + {}_3a_2\,{}_3a_3 = 0 \end{array} \right\} \ . \ (10) $$

Equations (8) say, that the transformed time depends on the time only, and that the transformed space-coordinates are dependent of the original ones only. Equations (9) and (10) show, that this last transformation is linear orthogonal. By this we have proved the proposition stated at the beginning of this §. We need therefore only apply equation (5) for values of $\varphi_m{}^4 = 0$ viz. for a pure space-transformation and of $\varphi_m{}^4 = 1$ viz. for a pure time-transformation.

§ 7. *Meaning of the cases* $\varphi_m{}^4 = 0$ *and* $\varphi_m{}^4 = 1$. A $\mathfrak{R}$ with $\varphi_m{}^4 = 0$ is nothing else than a reflection. (Symbol $\mathfrak{S}$, symbol of the symmetry-plane $\mathfrak{s}$).

It might seem interesting to derive all imaginable space-Δ's by investigating which combinations of $\mathfrak{S}$'s when considered as complex Δ, are compatible with the restrictions 1 and 2 of § 2 [1]). A point of consideration could be whether the order of application of the reflections in the ΔΔ should be chosen arbitrarily or not. In the first case [2]), we find, that each ΔΔ may be regarded as a combination of those already used by SCHOENFLIES and predecessors, but we might say just as well, that the Δ's used by SCHOENFLIES are but combinations of $\mathfrak{S}$'s and that there exist combinations, which were not treated by him. Proceeding in the indicated way, we find some Δ's that are aequivalent with point- and space-*groups* of SCHOENFLIES. After this we might investigate which space-groups can be formed from those Δ's.

As however the result of such an investigation has already been obtained by SCHOENFLIES we shall do better to combine each of his

[1]) (Note added during translation). C. VIOLA and G. WULFF partly executed such a plan (l.c.).

[2]) This seems natural by analogy with Δ's that were known before and is also demanded by the principle that around each particle the configuration of the other particles is the same. An exception to this last demand is formed by the definition of the sense of rotation and translation resp. dilation for a screw resp. time-rotation (see further on).

Δ's without restriction to 2-, 3-, 4- or 6-al axes with the possible time-Δ's [1]). Which are these?

One single $\mathfrak{R}$ with $\varphi^4 = 1$ will be called a "retroduction" [2]), the corresponding symmetry-element a "symmetry-moment". In a moving system of particles there exists a $\mathfrak{M}$ (symbol for a retroduction), when each point P, where at the moment t a particle A is present, is also occupied by a particle B at the moment $2\mathfrak{m}-t$, where $\mathfrak{m}$ is the symbol for a symmetry-moment and so the value of its t too. When then at the moment $t+\Delta t$ A is at Q, there must also be present a particle at Q at the moment $2\mathfrak{m}-t-\Delta t$. Because of the second restriction of § 2 we conclude that this last particle must be particle B. The velocities of A and B at P are therefore equal and opposite. At the moment $\mathfrak{m}$ there would thus be at the same place two particles with opposite velocities. This would be in conflict with the impermeability of matter (which we shall assume to hold for the electrons too), unless the two particles are identical [3]). Let us therefore suppose this to be the case. Then each particle must have come to rest at the moment $\mathfrak{m}$ and hence describe its path in the opposite direction.

When we have however a $\Delta\Delta$ of a $\mathfrak{M}$ and a $\mathfrak{S}$ we must change the above "at the same place" into "at the image of the place in $\mathfrak{s}$". Then the difficulty of two particles with different velocities being at the same moment (at the moment $\mathfrak{m}$) at the same place, would be avoided, unless at the moment $\mathfrak{m}$ the particles were lying in the $\mathfrak{s}$. In this case the velocity at that moment would not necessarily be 0, when only the two particles were supposed to be identical. Having passed the $\mathfrak{s}$ the particle then describes the symmetrical path and when moreover the $\mathfrak{s}$ was intersected perpendicularly by the path there would not be any discontinuity in the motion. In Comm. n°. 4 l.c. the symmetry-element of such a Δ (symbol $\mathfrak{M}\mathfrak{S}$) has been called "reversal-symmetry-plane". Further on we shall call it reversal-plane and the operation reversal-reflexion.

Other $\Delta\Delta$'s of time- and space-Δ's may be investigated in the indicated way.

---

[1]) After this we have still to form groups with the Δ's used by SCHOENFLIES and with the newly introduced ones.

[2]) This name (from retro = back and duco = I lead) and the name dilation (from differo = I postpone) introduced later on have been chosen in consultation with Prof. DAMSTÉ of Utrecht.

[3]) We exclude therefore cases as imaginéd by LANDÉ l.c., in which after a collision two electrons suddenly get each others velocities in direction and magnitude. LANDÉ himself designs these cases as improbable.

§ 8. *The ΔΔ's of two and more* 𝔐's. Regarding time-Δ's as special cases of ℜ's, we see from the end of § 4, that we can find all kinds of time-Δ's by only studying ΔΔ's of 𝔐's.

The applicability of the complex symmetry-operation of $\mathfrak{M}_1$ and $\mathfrak{M}_2$ (symbol 𝔓, name "dilation", symbol and name of the symmetry-element 𝔭 and "period") to a system of particles, means that when at a moment t a particle A is at the point P there are also particles B resp. C at P at the moments $2\mathfrak{m}_1—2\mathfrak{m}_2+t$ and $2\mathfrak{m}_2—2\mathfrak{m}_1+t$ respectively. In this case P is every time occupied by a particle after a lapse of time, $2(\mathfrak{m}_1—\mathfrak{m}_2)$. When the number of particles at our disposition is not infinite, the same particle A must necessarily at the end arrive at P again. Moreover, each particle, when arriving at P must have the same velocity and the same direction of motion, which will become evident, when we consider the state at moments $2(\mathfrak{m}_1—\mathfrak{m}_2)+t+\Delta t$. All particles are thus distributed in unequal numbers over differently shaped closed paths in which they circulate with phase-differences, that are the same for the different paths and also for the different particles in one and the same path. The times of revolution in two paths are proportional to the numbers of the circulating particles.

A ΔΔ of a 𝔓 and a rotation through $\frac{2\pi}{n}$ about a n-al symmetry-axis was already used in Comm. n°. 4 l.c. We shall call its symmetry-element n-al time-axis, the symmetry-operation time-rotation [1]) (symbol 𝔓𝔄).

The complex operation of $\mathfrak{M}_1$, $\mathfrak{M}_2$ and $\mathfrak{M}_3$ (symbol 𝔔, name reversal-dilation) is a symmetry-operation of a system of particles, when it fulfills this condition: When at the moment t the point P is occupied by a particle A, we shall find there particles B, C etc. at the moments:

$—2\mathfrak{m}_1+2\mathfrak{m}_2+2\mathfrak{m}_3—t$, $2\mathfrak{m}_1—2\mathfrak{m}_2+2\mathfrak{m}_3—t$ and $2\mathfrak{m}_1+2\mathfrak{m}_2—2\mathfrak{m}_3—t$.

In the first place we have therefore three symmetry moments. At those moments all particles must therefore return in their paths. As this must happen at more than one moment each particle oscillates in a different path of arbitrary form, while the moments of returning are the same for all paths. It is evident that in each path one particle only can circulate now. Secondly there evidently exists a period. To find it the following considerations will be of use: When to a moment t we apply the order $\mathfrak{M}_3\mathfrak{M}_2\mathfrak{M}_1$ and to the

[1]) The distinction we must make here between the two possible combinations of sense of rotation and sense of dilation is analogous to that which SCHOENFLIES and his predecessors made between left- and right-handed screws.

result of this operation the order $\mathfrak{M}_2 \mathfrak{M}_3 \mathfrak{M}_1$, the influences of $\mathfrak{M}_1$ neutralize each other, so that in fact we have only applied the double dilation $\mathfrak{M}_2 \mathfrak{M}_3 \mathfrak{M}_2 \mathfrak{M}_3$ [1]). Besides the intervals of time $4\,(\mathfrak{m}_2 - \mathfrak{m}_3)$ between the passages of particles by P, we find the intervals $4\,(\mathfrak{m}_3 - \mathfrak{m}_1)$ and $4\,(\mathfrak{m}_1 - \mathfrak{m}_2)$ too. Now we come into conflict with the second restriction of § 2, unless the quantities $\mathfrak{m}_2 - \mathfrak{m}_3$ and $\mathfrak{m}_3 - \mathfrak{m}_1$ have a greatest common measure. This is the time of oscillation. We can easily prove that then all demands of § 2 are satisfied.

By the investigation of $\Delta\Delta$'s of $\mathfrak{Q}$'s and space-Δ's we shall find i.a. that the paths may be closed in the same way as has been found for $\mathfrak{P}$, but that then half of the paths (chosen in a definite way) is described in the opposite direction.

§ 9. *There are no other time Δ's than $\mathfrak{M}$, $\mathfrak{P}$ and $\mathfrak{Q}$.* For all $\Delta\Delta$'s of even numbers of $\mathfrak{M}$'s the same considerations hold as the following for four $\mathfrak{M}$'s. When at a moment t the particle A is at P, and when we have to do with a $\Delta\Delta$ of four $\mathfrak{M}$'s we must find at P also particles B, C etc. at the moments $\pm 2\mathfrak{m}_1 \pm 2\mathfrak{m}_2 \pm 2\mathfrak{m}_3 \pm 2\mathfrak{m}_4 + t$, where the sign $+$ has to be chosen for half of the $\pm$-signs, the sign — for the other half. This gives therefore more than one dilation, which together yield however (comp. the considerations on $\mathfrak{Q}$) only a dilation equal to their greatest common measure, which case is already comprised in $\mathfrak{P}$.

For all $\Delta\Delta$'s of uneven numbers of $\mathfrak{M}$'s we can follow the reasoning on the case of $\mathfrak{Q}$. Thus this neither gives something new.

Combinations of time-Δ's yield nothing that has not yet been treated.

§ 10. *Symbols for the new symmetry-operations and symmetry elements.* For shortness sake we shall give names and symbols to the *s.-t.*-Δ's and symmetry-elements. As a preliminary system we propose the following:

With a small change now and then we retain the names and symbols of SCHOENFLIES. When now a Δ of SCHOENFLIES is combined with a retroduction the name of the first Δ might be changed by joining to it the prefix reversal. The same may be done with the names of the symmetry-elements. Before the symbols of Δ's and symmetry-elements we add $\mathfrak{M}$ and $\mathfrak{m}$ resp. When the change relates to a dilation the prefix is "time", for the symbols this becomes

[1]) In the here indicated way the treatment of $\Delta\Delta$'s of $\mathfrak{R}$'s (and therefore of $\mathfrak{M}$'s and $\mathfrak{S}$'s) is much simplified. By applying one of the Δ's thus found to the symmetry-elements of another one we can see whether this brings us into conflict with the restrictions of § 2. A $\Delta\Delta$ found in this way evidently forms a *group* of Δ's.

$\mathfrak{P}$ and $\mathfrak{p}$. When an operation is combined with a reversal-dilation we add the prefix reversal-time and for the symbols $\mathfrak{Q}$ and $\mathfrak{q}$. Sometimes the name obtained in this way is still somewhat shortened. In the following table we find these provisionally fixed names together with the symbols.

Without time △			*With* $\mathfrak{M}$		
Identity	1		Retroduction (symm.-moment)	$\mathfrak{M}$	$\mathfrak{m}$
Inversion (centre)	$\mathfrak{J}$	$\mathfrak{i}$	Reversal-inversion	$\mathfrak{M}\mathfrak{J}$	$\mathfrak{mi}$
Reflection (symmetry-plane)	$\mathfrak{S}$	$\mathfrak{s}$	Reversal-reflection (reversal-plane)	$\mathfrak{M}\mathfrak{S}$	$\mathfrak{ms}$
Rotation (n-al axis)	$\mathfrak{A}$	$\mathfrak{a}$	Reversal-rotation	$\mathfrak{M}\mathfrak{A}$	$\mathfrak{ma}$
Rotatory-reflection (n-al reflect.-axis)	$\overline{\mathfrak{A}}$		Reversal-rotatory reflection	$\mathfrak{M}\overline{\mathfrak{A}}$	
Translation (place-period)	$\mathfrak{T}$	$\mathfrak{t}$	Reversal-translation	$\mathfrak{M}\mathfrak{T}$	$\mathfrak{mt}$
Gliding-reflection (gliding plane)	$\overline{\mathfrak{T}}$		Reversal-gliding-reflection	$\mathfrak{M}\overline{\mathfrak{T}}$	
Screw (n-al screw axis)	$\mathfrak{X}$		Reversal screw	$\mathfrak{M}\mathfrak{X}$	
With $\mathfrak{P}$			*With* $\mathfrak{Q}$		
Dilation (period)	$\mathfrak{P}$	$\mathfrak{p}$	Reversal-dilation	$\mathfrak{Q}$	
Time-inversion	$\mathfrak{P}\mathfrak{J}$	$\mathfrak{pi}$	Reversal-time-inversion	$\mathfrak{Q}\mathfrak{J}$	$\mathfrak{qt}$
Time-reflection (time-plane)	$\mathfrak{P}\mathfrak{S}$	$\mathfrak{ps}$	Reversal-time-reflection	$\mathfrak{Q}\mathfrak{S}$	$\mathfrak{qs}$
Time-rotation	$\mathfrak{P}\mathfrak{A}$	$\mathfrak{pa}$	Reversal-time-rotation	$\mathfrak{Q}\mathfrak{A}$	$\mathfrak{qa}$
Time-rotatory-reflection	$\mathfrak{P}\overline{\mathfrak{A}}$		Reversal-time-rotatory-reflection	$\mathfrak{Q}\overline{\mathfrak{A}}$	
Time-translation	$\mathfrak{P}\mathfrak{T}$	$\mathfrak{pt}$	Reversal-time-translation	$\mathfrak{Q}\mathfrak{T}$	$\mathfrak{qt}$
Time-gliding-reflection	$\mathfrak{P}\overline{\mathfrak{T}}$		Reversal-time-gliding-reflection	$\mathfrak{Q}\overline{\mathfrak{T}}$	
Time-screw	$\mathfrak{P}\mathfrak{X}$		Reversal-time screw	$\mathfrak{Q}\mathfrak{X}$	

§ 11. *The way in which s.-t.-symmetry-operations may be combined into groups.* When the point groups of SCHOENFLIES are completed by those, which contain other than 2-, 3-, 4- and 6-al rotations etc. we can form from each of the thus found groups, *s.-t.*-groups by combining each of the non-aequivalent operations of a group with either no time-operation or with a $\mathfrak{M}$ or with a $\mathfrak{P}$, or with a $\mathfrak{Q}$. Each of the thus found groups must then still be examined to find out whether the time-operations added are perhaps in conflict with each other. Several of the groups obtained will also be found to be the same.

The same might be done with the translation-groups [1]), which are formed by SCHOENFLIES as a means to change point-groups into space-groups. After this, all obtained *s.-t.*-point-groups are multiplied by each of the *s.-t.*-translation-groups found. Examples of such groups will be given in a following paper (N°. 7*b*).

[1]) In the case of translation-groups we have no longer a ground for the assumption that a $\mathfrak{P}$ and a $\mathfrak{Q}$ cause the paths to be closed. The only thing we should have won by omitting this hypothesis however would be the allowance of a continuous translatory motion of the whole system of particles. It would not be desirable to include this motion in our considerations.

Physics. — *Space-time-symmetry.* II. *Discussion of a special case. The tetrahedrical atom-models of* LANDÉ". By N. H. KOLKMEIJER. Communicaton N°. 7*b* from the Laboratory of Physics and Physical Chemistry of the Veterinary College at Utrecht. (Communicated on behalf of Prof. W. H. KEESOM, Director of the Laboratory, by Prof. H. KAMERLINGH ONNES).

(Communicated at the meeting of January 29, 1921).

§ 1. *Introduction.* Recently, BORN, LANDÉ and MADELUNG [1]) (partially in cooperation) have studied atom models in which the electrons circulating about the nucleus are distributed over a number of shells over each of which they are spread symmetrically.

LANDÉ treats [2]) the problem of finding "orbits with polyhedrical symmetry" with the intention of reducing the p-bodies-problem of the p electrons (the nucleus is thought at rest) to a one-body-problem.

A survey of the possibilities arising in this problem and an insight in the symmetry-character of the models in question can easily be obtained by considering the latter from the point of view of space-time-symmetry (denoted further on by s-t-symmetry) treated in a former communication [3]). At the same time we may test in this way the usefulness of the considerations in question. We shall only consider the tetrahedrical models of LANDÉ.

§ 2. *The space group of* SCHOENFLIES, *on which the tetrahedrical atom models of* LANDÉ *are based.* In fig. 1 the 24 points are indicated, that arise from the symmetry of the tetrahedrical group of the second kind T_d (hemimorphous hemihedry of the cubic system) of SCHOENFLIES, when one of them is chosen arbitrarily [4]). SCHOENFLIES gives a summary of the non-aequivalent symmetry-operations of this group in the following way:

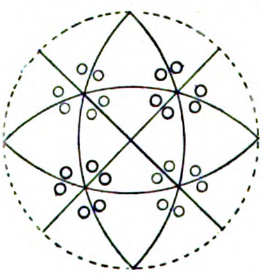


Fig. 1.

[1]) See the papers cited in note 4 p. 1419.

[2]) A. LANDÉ Verh. d. D. Phys. Ges. 21 (1919) p. 2, 644, 653, Zs. f. Physik 2 (1920) p. 83.

[3]) N. H. KOLKMEIJER Comm. N^0. 7*a*. These Proceedings p. 1419.

[4]) The points at the other side of the sphere are denoted by thinner small circles.

$$I \left\{ \begin{array}{llll} 1 & \mathfrak{U} & \mathfrak{V} & \mathfrak{W} \\ \mathfrak{A} & \mathfrak{A}' & \mathfrak{A}'' & \mathfrak{A}''' \\ \mathfrak{A}^2 & \mathfrak{A}'^2 & \mathfrak{A}''^2 & \mathfrak{A}'''^2 \end{array} \right. \qquad II \left\{ \begin{array}{llll} \mathfrak{S}_d & \mathfrak{U}\mathfrak{S}_d & \mathfrak{V}\mathfrak{S}_d & \mathfrak{W}\mathfrak{S}_d \\ \mathfrak{A}\mathfrak{S}_d & \mathfrak{A}'\mathfrak{S}_d & \mathfrak{A}''\mathfrak{S}_d & \mathfrak{A}'''\mathfrak{S}_d \\ \mathfrak{A}^2\mathfrak{S}_d & \mathfrak{A}'^2\mathfrak{S}_d & \mathfrak{A}''^2\mathfrak{S}_d & \mathfrak{A}'''^2\mathfrak{S}_d \end{array} \right.$$

where $\mathfrak{U}$, $\mathfrak{V}$ and $\mathfrak{W}$ denote rotations through $\frac{2\pi}{2}$ about the binary axes, $\mathfrak{A}$, $\mathfrak{A}'$, $\mathfrak{A}''$ and $\mathfrak{A}'''$ rotations through $\frac{2\pi}{3}$ about the ternary axes and $\mathfrak{S}_d$ a reflection by one of the planes of symmetry.

§ 3. *The model with* 24 *electrons.* When we think the nucleus coinciding with the centre of the sphere on which the points have been drawn, then an electron will collide with another one when in its motion it reaches the sides or the edges of the solid angle within which its motion takes place. As has been remarked in Comm. N°. 7*a* Landé sometimes supposes such a collision to take place; but he thinks it rather improbable. Here we shall consider such collisions as impossible. Each electron must therefore describe a path that remains inside the solid angle (elementary domain).

Now a s.-t.-symmetrical atom is continually changing its aspect and at the same time perhaps not all its properties but at least those depending on the aspect. It would be difficult, if not impossible to recognize such an atom by its properties when not approximately the same configuration of the electrons came back from time to time. This would be an indication of time-symmetry when at least the intervals between the moments of two equal configurations are approximately equal te each other.

Perhaps the returning configuration might take another position in space than the original one. As long however as we have not to do with the relation of an atom to a neighbouring one, it is allowed to choose a system of coordinates the origin of which moves with the nucleus, and which may rotate about its origin in such a way that the recurring configuration takes the same position with respect to the system as the original one.

In Communication Nr. 7*a* we have seen that an electron returning to the same place (in the system of coordinates) after the lapse of a certain time while the same relation holds for all electrons, is an indication of the existence of a "dilation" $\mathfrak{P}$ or a "reversal dilation" $\mathfrak{Q}$.

In the same way as in a space lattice an infinite group is formed by the translations, an infinite group 1, $\mathfrak{P}$, $\mathfrak{P}^2$ etc.[1]) is formed by

[1]) According to Schoenflies each symmetry operation of a group transforms a particle into another one (at the same moment) but for the identity. Therefore the

the dilations. Let us denote this group by Π. SCHOENFLIES denotes some space groups by placing between broken brackets the symbol of the translation group used for the formation of the group and separated from it by a comma, the symbol of the isomorphous point group. Now the group of symmetry operations of the above mentioned atom model with 24 electrons will evidently be represented by $\{T_d, \Pi\}$. The group contains therefore the symmetry operations I and II and moreover the same symmetry-operations, each multiplied by $\mathfrak{P}^m$, where for m must be taken each positive or negative whole number. Fig. 2 shows possible paths of 12 of the 24 electrons.

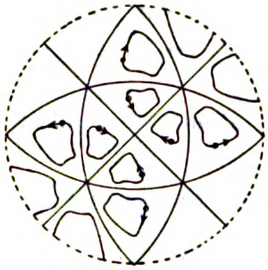


Fig. 2.

§ 4. *The model with* 12 *electrons.* Let us suppose now that only 12 electrons are circulating with as much tetrahedrical symmetry as is possible. Of the above 24 electrons each pair must then coincide. In this case an electron must be able to cross the boundary of its domain, but for the trivial case that it is always moving in a $\mathfrak{s}$. When only space symmetry existed this crossing would be impossible. At the moment of the crossing of a $\mathfrak{s}$ the two electrons coincide, but the velocity of one of them is symmetrical to that of the other one with respect to the plane. The collision caused by such velocities might be avoided when one electron reverted in its path. From comm. Nr. 7a this evidently happens, when the reflection in the $\mathfrak{s}$ is accompanied by a retroduction $\mathfrak{M}$ or a reversal dilation $\mathfrak{Q}$ viz. a retroduction $\mathfrak{M}$ multiplied by a dilation $\mathfrak{P}$.

For causing each two electrons to coincide we have only to choose for the symmetry moment of the retroduction that of the crossing of one of the six $\mathfrak{s}$'s. We may form the group of the symmetry operations for this case in the following way. Replace II from § 2 by:

$$III \left\{ \begin{array}{llll} \mathfrak{M}\mathfrak{S}_d & \mathfrak{M}\mathfrak{U}\mathfrak{S}_d & \mathfrak{M}\mathfrak{V}\mathfrak{S}_d & \mathfrak{M}\mathfrak{W}\mathfrak{S}_d \\ \mathfrak{M}\mathfrak{A}\mathfrak{S}_d & \mathfrak{M}\mathfrak{A}'\mathfrak{S}_d & \mathfrak{M}\mathfrak{A}''\mathfrak{S}_d & \mathfrak{M}\mathfrak{A}'''\mathfrak{S}_d \\ \mathfrak{M}\mathfrak{A}^2\mathfrak{S}_d & \mathfrak{M}\mathfrak{A}'^2\mathfrak{S}_d & \mathfrak{M}\mathfrak{A}''^2\mathfrak{S}_d & \mathfrak{M}\mathfrak{A}'''^2\mathfrak{S}_d \end{array} \right.$$

---

number of symmetry operations of a group equals that of the particles. In the group in question however each operation transforms a particle into the same particle at a different moment. By this the number of particles is no longer equal to that of the operations in the group. This last number is equal now to that of the positions of all the particles required for the definition of the model.

I and III then form a group which we shall call Γ_1. This group multiplied by Π gives the demanded group. Thus $\Gamma_1' = \{\Gamma_1, \Pi\}$.

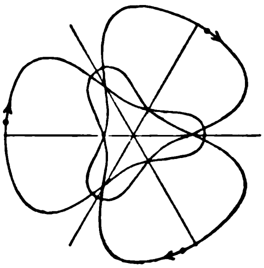


Fig. 3.

Here we may make the following remark. In the deduction of the above model we have quite followed LANDÉ's considerations spoken of in the introduction. According to these p electrons describe $n = pq$ paths in such a way, that they continually show the symmetry of a sub group, while each of the p electrons describes q definite paths successively, where pq denotes the number of operations in a group. The above model is therefore also a "tetrahedrical atom model". LANDÉ however does not consider this model with 12 electrons. In the model discussed above each of the 12 electrons describes a path (see fig. 3 where the projection of the paths of 3 electrons on a plane perpendicular to a ternary axis has been represented) symmetrical with respect to a $\mathfrak{s}$, while the two electrons that may be derived from a third one by application of $\mathfrak{A}$ and $\mathfrak{A}^2$, describe paths that may be derived from that of the third one in the same way. (The period $\mathfrak{p}$ of $\mathfrak{P}$ is just equal to the time of revolution in such an orbit). Without pointing expressively to the necessity of this, LANDÉ considers the case that the three paths in question coincide and form one path, symmetrical with respect to the three $\mathfrak{s}$'s, while the three electrons are circulating in it with difference in phase of $^1/_3$ (see fig. 4).

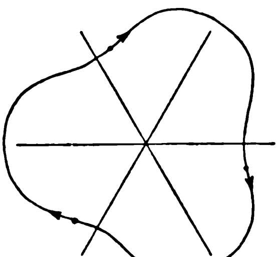


Fig. 4.

We may find the group of the symmetry operations of the model of LANDÉ by multiplying the whole group found above by $\mathfrak{M}'\mathfrak{S}_d'$, where $\mathfrak{S}_d'$ means a second $\mathfrak{S}$ while $\mathfrak{M}'$ refers to the moment when it is passed. $\mathfrak{M}\mathfrak{S}_d$ and $\mathfrak{M}'\mathfrak{S}_d'$ being at the same time symmetry operations, $\mathfrak{M}\mathfrak{M}'\mathfrak{S}_d\mathfrak{S}_d'$ is also a symmetry operation viz. a time-rotation with a rotation through $\frac{2\pi}{3}$ and a period $^1/_3$ of the time of revolution = twice the time between the moments in which the two $\mathfrak{s}$'s are crossed. When the group of the powers of dilations with period equal to $^1/_3$ of time of revolution is called Π', then the new group is $\Gamma_1'' = \{\Gamma_1, \Pi'\}$. The group Π discussed above is a sub-group of group Π'.

§ 5. *The model with* 4 *electrons.* LANDÉ also discusses the case of

4 electrons. This case is found from the foregoing model of LANDÉ when the two electrons that are derived from a third one by a single and a double rotation through $\frac{2\pi}{3}$ about a ternary axis, coincide with this third one and when this is the case for each of the ternary axes. Thereto we have only to change the ternary axes into ternary timeaxes, to choose the period equal to $^1/_3$ of the time of revolution of the electrons in their orbits and to take care to combine positive dilations with rotations about the axes in senses corresponding with the direction of revolution of the electrons. In that case however I and III, both changed in the indicated way, do not longer form a group. Let us consider however the group.

$$I \left\{ 1 \quad \mathfrak{U} \quad \mathfrak{B} \quad \mathfrak{W} \qquad II \left\{ \mathfrak{M}\mathfrak{S}_d \quad \mathfrak{M}\mathfrak{U}\mathfrak{S}_d \quad \mathfrak{M}\mathfrak{B}\mathfrak{S}_d \quad \mathfrak{M}\mathfrak{W}\mathfrak{S}_d \right.\right.$$

which forms namely a group differing in the same sense from the group V_d of SCHOENFLIES as the group Γ_1, from T_d. When we multiply this group which we shall call Γ_2 by the infinite group Π'' of the time-rotations 1, $\mathfrak{P}'\mathfrak{A}$, $\mathfrak{P}'^2\mathfrak{A}^2$ etc., where $\mathfrak{P}'$ has a $\mathfrak{p} = {}^1/_3$ time of revolution, then we obtain the required group $\Gamma_2' = \{\Gamma_2, \Pi''\}$ [1]). Π is a sub-group of Π'' too. This case too is represented in fig. 4 when two of the three indicated particles are cancelled.

§ 6. *The model of* MADELUNG *and* LANDÉ. MADELUNG and LANDÉ have treated still a model (l.c.), in which four electrons are moving in the lastly mentioned way. They are circulating with a uniform velocity in circles, while each electron is followed by another one in the same orbit at an angular distance of 75°, so that totally eight electrons are circulating. This model may easily be derived from the preceding one by choosing the moment of $\mathfrak{M}$ not equal to that of the crossing of one of the $\mathfrak{s}$'s. Then two electrons are namely circulating in each orbit, with a constant phase-difference when the orbits are circular and when the velocity is uniform. This last condition cannot be expressed by the used operations. The model in

1) Though it is not here our purpose to find out all possible cases, we may point at another way in which the electrons may be brought to coincide, viz. when the binary axes are replaced by time-axes. We then obtain six electrons each of which could move f.i. over a face of a cube. Perhaps LANDÉ did not consider this model because it has no binary axes. Still we must also consider this model as a case of "tetrahedrical" s.-t.-symmetry. Moreover it may also be treated as a special case of the model of LANDÉ of six electrons in rhombohedrical symmetry connection.

question thus belongs as a special case to the group $\Gamma_s' = \{\Gamma_s, \Pi''\}$, where Γ_s is:

$$I \left\{ 1 \quad \mathfrak{U} \quad \mathfrak{V} \quad \mathfrak{W} \right. \qquad II \left\{ \mathfrak{M}'\mathfrak{S}_d \quad \mathfrak{M}'\mathfrak{U}\mathfrak{S}_d \quad \mathfrak{M}'\mathfrak{V}\mathfrak{S}_d \quad \mathfrak{M}'\mathfrak{W}\mathfrak{S}_d \right.$$

§ 7. *Final remarks.* In the same way we can also study the cubic model with 48, 24 and 8 electrons respectively, and also the models with other numbers of electrons. We may still draw the attention to the problem of the s-t-symmetrical relation between the different shells of an atom with respect to the above said on the periodicity of the configurations of the electrons.

Finally the following may be remarked:

When a certain number of electrons is moving in such a way that s.-t.-symmetry exists, it is evident from the preceding that perhaps no pure space symmetry element exists. In the equations of motion given by LANDÉ for one of four electrons the remaining space symmetry of the four electrons was taken into consideration. These equations proved to possess the symmetry of the group T_d of SCHOENFLIES, so that the four-bodies-problem of the four electrons was reduced to a one-body-problem. From the preceding we now see, that this reduction could only be the consequence of the existence of the s.-t.-symmetry of the electron configuration in the equations; and that this symmetry would not suffice when the time-symmetry parts are omitted. It may be verified easily though that, in the cases discussed above no change is brought into the conclusion by this remark. We have seen moreover, that the appearance of s.-t.-symmetry instead of space-symmetry was always connected with the coincidence of two or more orbits; and algebraically this is expressed by boundary conditions, not by a property of the equations of motion.

Physics. — "*Space-time-symmetry.* III. *Remarks on the deduction of groups of space-time symmetry-operations.*" By N. H. KOLKMEIJER. Communication N°. 7*c* from the Laboratory of Physics and Physical Chemistry of the Veterinary College at Utrecht. (Communicated on behalf of Prof. W. H. KEESOM, Director of the Laboratory, by Prof. H. KAMERLINGH ONNES.)

(Communicated at the meeting of January 29, 1921).

§ 1. *Introduction.* In Comm. N°. 7*a*[1]) the s.-t.-symmetry operations have been derived, Comm. N°. 7*b*[2]) gave examples of the application of these on a couple of models. With the purpose to start the study of the s.-t.-symmetrical atom models I have spent some time with finding out all groups of s.-t.-symmetry operations that are connected with point groups. The results of this investigation would take too much space here. Therefore only a few remarks follow on the way of deduction of the groups in question.

As moving particles we shall choose one atom nucleus and a finite number of electrons, all without dimensions.

§ 2. *Possibility of the existence of the s.-t.-symmetrical atom models.* In the first place we may prove in a way analogous to that followed by LANDÉ[3]) in the discussion of the possibility of his models, that the s.-t.-symmetrical atom models are consistent with the action of the forces.

In the second place: Among the models found in the way here indicated there will be some that do not satisfy the demand, (Comm. N°. 7*b*[4])), for an atom model, that the configuration of nucleus and electrons returns periodically. This is f.i. the case with all models in which we have to do with one retroduction only. We might call them quasi atom models. For the sake of a systematic treatment we shall have to consider these too.

In the next § we shall see, that it is impossible that to the ope-

---

1) These Proceedings, p. 1419.

2) These Proceedings, p. 1428.

3) A. LANDÉ, Verh. d. D. Physik. Ges. 21, p. 2, 1919.

4) l.c.

rations in question there belong those that are combined with translations. Thus we have only to consider rotations, rotatory reflections (including reflections and inversions) and combinations of these two with retroductions, dilations and reversal dilations.

§ 3. *We need not consider other configurations of space symmetry elements than those considered in the theory of space-symmetry.* Evidently the configuration of the space symmetry elements (including the geometrical parts of the s.-t.-Δ) is a figure at rest, so that when one of the s.-t.-Δ, composed of a s.- and a t.-Δ, is applied to it, it will give a congruent or symmetrical configuration, though at a different moment, but also at the same moment. The s.-Δ alone being then a Δ of the configuration [1]), this latter must correspond with one of those obtained in the theory of space-symmetry [2]).

§ 4. *Introduction of the generative operations of a group.* That the method indicated in communication N°. 7*a* l.c. for the searching of all groups of s.-t.-symmetry-operations is valid has been proved in the preceding. This method however may be much simplified by making use of the generative operations of the point-groups [3]) viz. of operations chosen in such a way, that each operation of the group can be considered as a product of some of those generative operations (the order in which the factors are taken having influence). We shall indicate each group by placing the symbols of the generative operations, separated by comma's, between broken brackets.

When to each of the generative operations of the group $G = \{\mathfrak{A}, \mathfrak{B}, \mathfrak{C}\}$ we add a time operation (including the identity) we obtain the group $\Gamma = \{\mathfrak{S}\mathfrak{A}, \mathfrak{H}\mathfrak{B}, \mathfrak{K}\mathfrak{C}\}$. When moreover pure time operations are added as generative operations, so that we obtain $\Gamma_1 = \{\ldots \mathfrak{L}, \mathfrak{M}, \mathfrak{S}\mathfrak{A}, \mathfrak{H}\mathfrak{B}, \mathfrak{K}\mathfrak{C}\}$, we need only vary the operations $\mathfrak{S}, \mathfrak{H}, \mathfrak{K}, \mathfrak{L}, \mathfrak{M}\ldots$ in all possible ways to find all groups of s.-t.-symmetry operations, that can be derived from G. Now though it is possible that in a group Γ_2 which has been found in some other way and which

[1]) This holds also for translations. As we have confined the number of nuclei to 1, no Δ's composed by means of translations can occur in the groups sought for.

[2]) It might seem doubtful, whether for time rotations the restriction to those through $\frac{p}{q} 2\pi$ radians, remains valid. In fact in these considerations there is no objection to time rotations with infinitesimal rotation. This has a meaning only when besides the time period is infinitesimal. Then this symmetry element indicates, a uniform circular motion. But for the rest this does not change the following considerations.

[3]) A. SCHOENFLIES, Krystallsysteme und Krystallstructur, Leipzig 1891.

necessarily must have $\mathfrak{S}\mathfrak{A}$, $\mathfrak{H}\mathfrak{B}$ and $\mathfrak{K}\mathfrak{C}$ in common with one of the varied groups Γ (f.i. Γ'), we find another one of the non-aequivalent operations of G multiplied by the time operation $\mathfrak{R}$, while in Γ this added operation is $\mathfrak{P}$; we can however add to Γ as a generative operation a time operation $\mathfrak{Q}$ in such a way that $\mathfrak{Q}\mathfrak{P} = \mathfrak{R}$, and then we obtain a group Γ'_1 which does no longer show the indicated difference with Γ_2 and which doubtlessly belongs to the groups found by the variation of the group Γ_1.

§ 5. *The time operations added cannot be in contradiction.* From the preceding it is evident that in some one of the groups obtained in this way, one of the non-aequivalent operations of G could occur more than once, each time multiplied by another time operation. When these should be in conflict with each other, this would only mean, that all electrons are at rest. For a point at rest has each conceivable time operation as a symmetry-operation. Then the group is the same as G itself with the special condition that the particles do not move.

§ 6. *Restriction of the time operations that are to be added.* Never need reversal-dilations to be added to the generative operations of a group as G. A reversal dilation $\mathfrak{Q}$ namely means nothing else but a group of time operations among which f.i. $\mathfrak{M}_1\mathfrak{M}_2\mathfrak{M}_3$ and $\mathfrak{M}_2\mathfrak{M}_1\mathfrak{M}_3$. It is evident that the same effect is obtained by adding to $\mathfrak{A}$ the order $\mathfrak{M}_1\,\mathfrak{M}_2\,\mathfrak{M}_3$ only. For, when later on time operations are added as generative operations to the groups that have been formed already we necessarily add f. i. also the operation $\mathfrak{M}_2\mathfrak{M}_1\mathfrak{M}_2\mathfrak{M}_1$, which multiplied by $\mathfrak{M}_1\mathfrak{M}_2\mathfrak{M}_3\mathfrak{A}$ gives again $\mathfrak{M}_2\mathfrak{M}_1\mathfrak{M}_3\mathfrak{A}$. As further $\mathfrak{M}_1\mathfrak{M}_2\mathfrak{M}_3$ is a retroduction and as in the derivation of the groups we necessarily add each retroduction it has no sense to consider the form $\mathfrak{M}_1\mathfrak{M}_2\mathfrak{M}_3$ especially. Similar considerations show, that of the orders $\mathfrak{M}_1\mathfrak{M}_2$ and $\mathfrak{M}_2\mathfrak{M}_1$, that are included in a dilation we need only to add one. Thus the generative operations of group G are multiplied by $\mathfrak{M}$ or $\mathfrak{M}_1\mathfrak{M}_2$ only; for the same reason we add later on as generative operations only either $\mathfrak{M}$, or $\mathfrak{M}_1\mathfrak{M}_2$, or $\mathfrak{M}_1$ and $\mathfrak{M}_2$.

§ 7. *Choice of the generative operations.* Proceeding in the indicated way we are sure to omit none of the groups of s.-t.-symmetry operations. It will however very well be possible that we thus obtain the same group more than once, each time in a different form and thus that those different forms do not always show their aequivalence directly. We shall see how that aequivalence of two such forms can

be found out. For this purpose it will prove to be desirable to choose the generative operations for each group G in such a way, that a possibly low power of each of these operations is aequivalent to the identity. Thereto we shall even sacrifice sometimes the advantage of a possibly small number of generative operations.

$\{\mathfrak{A}_n\}$ [1]		
$\{\mathfrak{A}_2, \mathfrak{A}'_2\}$	$(\mathfrak{a}_2\, \mathfrak{a}'_2) = \frac{\pi}{n}$	
$\{\mathfrak{A}_3, \mathfrak{A}'_3\}$	$(\mathfrak{a}_3\, \mathfrak{a}'_3) = 2 \operatorname{arc\,tg} \sqrt{2}$	$\{\mathfrak{A}_2, \mathfrak{A}_3\}$
$\{\mathfrak{A}_4, \mathfrak{A}'_4\}$	$(\mathfrak{a}_4\, \mathfrak{a}'_4) = \frac{\pi}{2}$	$\{\mathfrak{A}_2, \mathfrak{A}'_2, \mathfrak{A}''_2\}$
$\{\mathfrak{A}_5, \mathfrak{A}'_5\}$	$(\mathfrak{a}_5\, \mathfrak{a}'_5) = 2 \operatorname{arc\,tg} \frac{-1+\sqrt{5}}{2}$	$\{\mathfrak{A}_2, \mathfrak{A}'_2, \mathfrak{A}''_2\}$
$\{\bar{\mathfrak{A}}_{2n}\}$ [1]		
$\{\mathfrak{S}, \mathfrak{S}'\}$	$(\mathfrak{s}\, \mathfrak{s}') = \frac{\pi}{n}$	
$\{\mathfrak{A}_n, \mathfrak{S}_h\}$	$(\mathfrak{a}_n\, \mathfrak{s}_h) = \frac{\pi}{2}$	
$\{\mathfrak{S}_v, \mathfrak{S}'_v, \mathfrak{S}_h\}$	$(\mathfrak{s}_v\, \mathfrak{s}'_v) = \frac{\pi}{n}, (\mathfrak{s}_v\, \mathfrak{s}_h) = (\mathfrak{s}'_v\, \mathfrak{s}_h) = \frac{\pi}{2}$	
$\{\mathfrak{A}_2, \mathfrak{S}_d\}$	$(\mathfrak{a}_2\, \mathfrak{s}_d) = \frac{\pi}{2n}$	
$\{\mathfrak{A}_3, \mathfrak{S}_h\}$	$(\mathfrak{a}_3\, \mathfrak{s}_h) = \operatorname{arc\,cos} \frac{1}{3}\sqrt{3}$	
$\{\mathfrak{A}_3, \mathfrak{S}_d\}$	$(\mathfrak{a}_3\, \mathfrak{s}_d) = \operatorname{arc\,sin} \frac{1}{3}\sqrt{3}$	
$\{\mathfrak{A}_4, \mathfrak{S}\}$	$(\mathfrak{a}_4\, \mathfrak{s}) = \frac{\pi}{4}$	$\{\mathfrak{A}_2, \mathfrak{A}'_2, \mathfrak{S}\}$
$\{\mathfrak{A}_5, \mathfrak{S}\}$	$(\mathfrak{a}_5\, \mathfrak{s}) = \operatorname{arc\,tg} \frac{-1+\sqrt{5}}{2}$	$\{\mathfrak{A}_2, \mathfrak{A}'_2, \mathfrak{S}\}$

On the other hand it will be practical to choose as generative operations when possible aequivalent operations of the group, as is evident from the following. The octahedral group of SCHOENFLIES f.i.

[1]) Analogous to SCHOENFLIES' distinction between "point groups of the first kind" (the first 5) and "point groups of the second kind" (the rest), we might discern between "finite nuclear groups of the first and of the second kind".

may be represented as well by $\{\mathfrak{A}_4, \mathfrak{A}'_4\}$[1]) as by $\{\mathfrak{A}_2, \mathfrak{A}_4\}$. Adding to the operations $\mathfrak{M}_1$ and $\mathfrak{M}_2\mathfrak{M}_3$ in every combination, we find for the second notation 9, for the first one 6 groups, as evidently $\{\mathfrak{M}_1\mathfrak{A}_4, \mathfrak{A}'_4\}$ and $\{\mathfrak{A}_4, \mathfrak{M}_1\mathfrak{A}'_4\}$, and also $\{\mathfrak{M}_2\mathfrak{M}_3\mathfrak{A}_4, \mathfrak{A}'_4\}$ and $\{\mathfrak{A}_4, \mathfrak{M}_2\mathfrak{M}_3\mathfrak{A}'_4\}$ and also $\{\mathfrak{M}_1\mathfrak{A}_4, \mathfrak{M}_1\mathfrak{M}_2\mathfrak{A}'_4\}$ and $\{\mathfrak{M}_1\mathfrak{M}_2\mathfrak{A}_4, \mathfrak{M}_1\mathfrak{A}'_4\}$ are aequivalent groups. Therefore the first notation is preferable.

Generally both desiderata on the choice of the generative operations cannot be fulfilled at the same time. Then it is desirable to satisfy both separately and to consider the two notations comparatively.

All point groups (between which therefore the 32 classes in question of Schoenflies) are then included in the preceding scheme of 14 kinds of groups for some of which two notations have been given (see table p. 1437).

§ 8. *Investigation of the aequivalence of groups that are found.* Here we can only give some indications, which together with the drawing of a figure or the writing out of the non-aequivalent operations of the group (the two last ways of proceeding are most times superfluous) are at all events sufficient.

By way of illustration we shall directly apply these indications to one of the groups G for which we shall choose $\{\mathfrak{A}_2, \mathfrak{A}'_2\}$.

We must try to reduce to one all notations, by which a group can appear. When therefore a $\mathfrak{M}_1\mathfrak{M}_2$ that is added to one of the generative operations, can be reduced to $\mathfrak{M}$ or even to the identity, we shall do it. Example: $\{\mathfrak{M}_1\mathfrak{A}_2, \mathfrak{M}_2\mathfrak{M}_3\mathfrak{A}'_2\}$.. In the first place we may replace $\mathfrak{M}_2\mathfrak{M}_3$ by $\mathfrak{M}_1\mathfrak{M}_2$ without loss of generality (let this proceeding never be omitted). Further, $(\mathfrak{M}_1\mathfrak{A}_2)^2 = \mathfrak{M}_1$ is an operation of the group. When therefore we consider $\mathfrak{M}_1$ and $\mathfrak{A}_2$ separately as generative operations of the group we have already followed the precept partly. We can however follow it still more completely. As namely $\mathfrak{M}_1$ has now become a generative operation, we may substitute $\mathfrak{M}_2\mathfrak{A}'_2$ for $\mathfrak{M}_1\mathfrak{M}_2\mathfrak{A}'_2$. As then however $(\mathfrak{M}_2\mathfrak{A}'_2)^2 = \mathfrak{M}_2$ is also an operation of the group, we must take as generative operations again $\mathfrak{M}'_2$ and $\mathfrak{A}'_2$ instead of $\mathfrak{M}_2\mathfrak{A}'_2$. The group $\{\mathfrak{M}_1\mathfrak{A}_2, \mathfrak{M}_2\mathfrak{M}_3\mathfrak{A}'_2\}$ is therefore aequivalent with the group $\{\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{A}_2, \mathfrak{A}'_2\}$. When the newly found generative (pure time-) operations had not happened to appear in one of the above given forms $\mathfrak{M}$, $\mathfrak{M}_1\mathfrak{M}_2$ or $\mathfrak{M}_1, \mathfrak{M}_2$ we should have reduced them to one of those forms or,

---

[1]) In the following $\mathfrak{A}_n$ means a rotation through $\frac{1}{n}\,360°$, $\mathfrak{S}$ a reflection in a plane, $\overline{\mathfrak{A}}_n$ a rotatory reflection.

if impossible, we should be already in the above mentioned case that all electrons are at rest, so that we had to omit all added time operations.

From this example it is evident, that we must try to find out all pure time operations that are included in the adding of time operations to generative operations of a group G. Evidenty we find them all by considering all products, of the generative operations of G that are aequivalent to the identity. In most cases these are seen on first sight in a sufficient number. In the above example f. i. $(\mathfrak{A}_3)^3 = 1$ suffices. When however we meet with difficulties we make use of the second notation that has been given for each group. In this way are obtain the following scheme:

$$\{\mathfrak{A}_3, \mathfrak{A}'_3\} = \{\mathfrak{A}_3 (= \mathfrak{A}_3 (\mathfrak{A}'_3)^{-1}), \mathfrak{A}_3\}$$

$$\{\mathfrak{M}\mathfrak{A}_3, \mathfrak{A}'_3\} = \{\mathfrak{M}, \mathfrak{A}_3, \mathfrak{A}'_3\}$$

$$\{\mathfrak{M}_1\mathfrak{A}_3, \mathfrak{M}_2\mathfrak{A}'_3\} = \{\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{A}_3, \mathfrak{A}'_3\}$$

$$\{\mathfrak{M}_1\mathfrak{M}_2\mathfrak{A}_3, \mathfrak{A}'_3\} = \{\mathfrak{M}_1\mathfrak{M}_2, \mathfrak{A}_3, \mathfrak{A}'_3\}$$

$$\{\mathfrak{M}_1\mathfrak{A}_3, \mathfrak{M}_1\mathfrak{M}_2\mathfrak{A}'_3\} = \{\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{A}_3, \mathfrak{A}'_3\}$$

$$\{\mathfrak{M}_1\mathfrak{M}_2\mathfrak{A}_3, \mathfrak{M}_1\mathfrak{M}_2\mathfrak{A}'_3\} = \{(\mathfrak{M}_1\mathfrak{M}_2)^3, (\mathfrak{M}_1\mathfrak{M}_2)^{\pm 1}\mathfrak{A}_3, (\mathfrak{M}_1\mathfrak{M}_2)^{\pm 1}\mathfrak{A}'_3\}_{++}$$

$$\{\mathfrak{M}_1\mathfrak{M}_2, \mathfrak{A}_3, \mathfrak{A}'_3\}$$

$$\{\mathfrak{M}_1\mathfrak{M}_2, \mathfrak{M}_1\mathfrak{A}_3, \mathfrak{A}'_3\} = \{\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{A}_3, \mathfrak{A}'_3\}$$

$$\{\mathfrak{M}_1\mathfrak{M}_2, \mathfrak{M}_1\mathfrak{A}_3, \mathfrak{M}_3\mathfrak{A}'_3\} = \{\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{A}_3, \mathfrak{A}'_3\}$$

$$\{\mathfrak{M}_1\mathfrak{M}_2, \mathfrak{M}_1\mathfrak{M}_3\mathfrak{A}_3, \mathfrak{A}'_3\} = \{\mathfrak{M}_1\mathfrak{M}_2, \mathfrak{A}_3, \mathfrak{A}'_3\}$$

$$\{\mathfrak{M}_1\mathfrak{M}_2, \mathfrak{M}_1\mathfrak{A}_3, \mathfrak{M}_1\mathfrak{M}_3\mathfrak{A}'_3\} = \{\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{A}_3, \mathfrak{A}'_3\}$$

$$\{\mathfrak{M}_1\mathfrak{M}_2, \mathfrak{M}_1\mathfrak{M}_3\mathfrak{A}_3, \mathfrak{M}_1\mathfrak{M}_4\mathfrak{A}'_3\} = \{(\mathfrak{M}_1\mathfrak{M}_2)^3, (\mathfrak{M}_1\mathfrak{M}_2)^{\pm 1}\mathfrak{A}_3, (\mathfrak{M}_1\mathfrak{M}_2)^{\pm 1}\mathfrak{A}'_3\}_{++}$$

$$\{\mathfrak{M}, \mathfrak{A}_3, \mathfrak{A}'_3\}$$

$$\{\mathfrak{M}_1, \mathfrak{M}_2\mathfrak{A}_3, \mathfrak{A}'_3\} = \{\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{A}_3, \mathfrak{A}'_3\}$$

$$\{\mathfrak{M}_1, \mathfrak{M}_2\mathfrak{A}_3, \mathfrak{M}_2\mathfrak{A}'_3\} = \{\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{A}_3, \mathfrak{A}'_3\}$$

$$\{\mathfrak{M}_1, \mathfrak{M}_1\mathfrak{M}_2\mathfrak{A}_3, \mathfrak{A}'_3\} = \{\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{A}_3, \mathfrak{A}'_3\}$$

$$\{\mathfrak{M}_1, \mathfrak{M}_2\mathfrak{A}_3, \mathfrak{M}_1\mathfrak{M}_2\mathfrak{A}'_3\} = \{\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{A}_3, \mathfrak{A}'_3\}$$

$$\{\mathfrak{M}_1, \mathfrak{M}_1\mathfrak{M}_2\mathfrak{A}_3, \mathfrak{M}_1\mathfrak{M}_2\mathfrak{A}'_3\} = \{\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{A}_3, \mathfrak{A}'_3\}$$

$$\{\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{A}_3, \mathfrak{A}'_3\}$$

$$\{\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_3\mathfrak{A}_3, \mathfrak{A}'_3\} = \{\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{A}_3, \mathfrak{A}'_3\}$$

$$\{\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_3\mathfrak{A}_3, \mathfrak{M}_4\mathfrak{A}'_3\} = \{\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{A}_3, \mathfrak{A}'_3\}$$

$$\{\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_1\mathfrak{M}_3\mathfrak{A}_3, \mathfrak{A}'_3\} = \{\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{A}_3, \mathfrak{A}'_3\}$$

$$\{\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_3\mathfrak{A}_3, \mathfrak{M}_1\mathfrak{M}_4\mathfrak{A}'_3\} = \{\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{A}_3, \mathfrak{A}'_3\}$$

$$\{\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_1\mathfrak{M}_3\mathfrak{A}_3, \mathfrak{M}_1\mathfrak{M}_4\mathfrak{A}'_3\} = \{\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{A}_3, \mathfrak{A}'_3\}$$

so that the following 5 groups only remain:

$$\{\mathfrak{A}_3, \mathfrak{A}'_3\} \; \{\mathfrak{M}, \mathfrak{A}_3, \mathfrak{A}'_3\} \; \{\mathfrak{M}_1\mathfrak{M}_2, \mathfrak{A}_3, \mathfrak{A}'_3\} \; \{\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{A}_3, \mathfrak{A}'_3\}$$

and

$$\{(\mathfrak{M}_1\mathfrak{M}_2)^3, (\mathfrak{M}_1\mathfrak{M}_2)^{\pm 1}\mathfrak{A}_3, (\mathfrak{M}_1\mathfrak{M}_2)^{\pm 1}\mathfrak{A}'_3\}_{++}$$

First we derive all groups, in which no pure time operations are added yet to G as generative operations; this simplifies very much the derivation of the other groups. Above we saw that everywhere $\mathfrak{M}\mathfrak{A}_3$ must be replaced directly by $\mathfrak{M}$ and $\mathfrak{A}_3$. As to $\{\mathfrak{M}_1\mathfrak{M}_2\mathfrak{A}_3, \mathfrak{A}'_3\}$ we must attend to two things: firstly that $(\mathfrak{M}_1\mathfrak{M}_2\mathfrak{A}_3)^3 = (\mathfrak{M}_1\mathfrak{M}_2)^3$; secondly that $(\mathfrak{M}_1\mathfrak{M}_2\mathfrak{A}_3\mathfrak{A}'^{-1}_3)^3 = (\mathfrak{M}_1\mathfrak{M}_2\mathfrak{A}_3)^3 = (\mathfrak{M}_1\mathfrak{M}_2)^3$, and therefore also $\mathfrak{M}_1\mathfrak{M}_2$, are operations of the group, so that the generative operation $\mathfrak{M}_1\mathfrak{M}_2\mathfrak{A}_3$ has to be replaced by the generative operations $\mathfrak{M}_1\mathfrak{M}_2$ and $\mathfrak{A}_3$.

In the case of $\{\mathfrak{M}_1\mathfrak{M}_2\mathfrak{A}_3, \mathfrak{M}_1\mathfrak{M}_3\mathfrak{A}'_3\}$ the operations $(\mathfrak{M}_1\mathfrak{M}_2\mathfrak{A}_3)^3 = (\mathfrak{M}_1\mathfrak{M}_2)^3$, and in the same way $(\mathfrak{M}_1\mathfrak{M}_3)^3$, but also $\{\mathfrak{M}_1\mathfrak{M}_2\mathfrak{A}_3(\mathfrak{M}_1\mathfrak{M}_3\mathfrak{A}'_3)^{-1}\}^2 = (\mathfrak{M}_1\mathfrak{M}_2\mathfrak{M}_3\mathfrak{M}_1\mathfrak{A}_3)^2 = (\mathfrak{M}_3\mathfrak{M}_1\mathfrak{M}_1\mathfrak{M}_2)^2 = (\mathfrak{M}_3\mathfrak{M}_2)^2$ belong to the group. Now these three dilations must give together a *G.C.M.* of their periods, by which the new dilation $\mathfrak{M}_4\mathfrak{M}_5$ is determined. Then we must have $(\mathfrak{M}_1\mathfrak{M}_2)^3 = (\mathfrak{M}_4\mathfrak{M}_5)^k$, $(\mathfrak{M}_1\mathfrak{M}_3)^3 = (\mathfrak{M}_4\mathfrak{M}_5)^l$ and $(\mathfrak{M}_3\mathfrak{M}_2)^2 = (\mathfrak{M}_4\mathfrak{M}_5)^m$, ($k$, l and m integers). This involves $\mathfrak{M}_1\mathfrak{M}_2 = (\mathfrak{M}_4\mathfrak{M}_5)^{k/3}$, $\mathfrak{M}_1\mathfrak{M}_3 = (\mathfrak{M}_4\mathfrak{M}_5)^{l/3}$ and $\mathfrak{M}_3\mathfrak{M}_2 = (\mathfrak{M}_4\mathfrak{M}_5)^{m/2}$, so that $\mathfrak{M}_1\mathfrak{M}_3\mathfrak{M}_3\mathfrak{M}_2 = (\mathfrak{M}_4\mathfrak{M}_5)^{l/3+m/2} = \mathfrak{M}_1\mathfrak{M}_2 = (\mathfrak{M}_4\mathfrak{M}_5)^{k/3}$ or $l/3 + m/2 = k/3$ and $l = k + 3n$ (n an integer). Taking the newly found dilation as a generative operation the group becomes therefore $\{\mathfrak{M}_4\mathfrak{M}_5, (\mathfrak{M}_4\mathfrak{M}_5)^{k/3}\mathfrak{A}_3, (\mathfrak{M}_4\mathfrak{M}_5)^{n+k/3}\mathfrak{A}'_3\}$. In the factors of $\mathfrak{A}_3$ and $\mathfrak{A}'_3$ we may of course omit powers of $\mathfrak{M}_4\mathfrak{M}_5$ with integer exponents, so that n vanishes and we can choose for k 1 and 2 only. Here $(\mathfrak{M}_4\mathfrak{M}_5)^{2/3}\mathfrak{A}_3$ means nothing but $(\mathfrak{M}_4\mathfrak{M}_5)^{-1/3}\mathfrak{A}_3$, the time rotation with period opposite to that of $(\mathfrak{M}_4\mathfrak{M}_5)^{1/3}\mathfrak{A}_3$. When besides in order to avoid fractions we substitute $(\mathfrak{M}_1\mathfrak{M}_2)^3$ for $\mathfrak{M}_4\mathfrak{M}_5$, we obtain $\{\mathfrak{M}_1\mathfrak{M}_2)^3, (\mathfrak{M}_1\mathfrak{M}_2)^{\pm 1}\mathfrak{A}_3, (\mathfrak{M}_1\mathfrak{M}_2)^{\pm 1}\mathfrak{A}'_3\}_{++}$, where $++$ means that in the exponents only the $+$ signs or only the $-$ signs may be combined.

Other time operations that must be attended to will not easily be found beside those above mentioned. Here we have a case, that we consider the figure using at the same time the new form for the symbol of the group. Then it is evident, that a simple dilation is allowed.

In the investigation of the following groups the new symbols for the groups obtained until now can be used. (Example: $\{\mathfrak{M}_1\mathfrak{M}_2, \mathfrak{M}_1\mathfrak{M}_2\mathfrak{A}_3, \mathfrak{A}'_3\}$). Not only we attend then to the groups that have been discussed already and to which a generative time operation is joined, but also to the groups treated before that are obtained by joining that time operation to each of the generative operations of the first group.

Finally we may draw the attention to a way of proceeding, which gives us easily a new generative time operation when we are joining a generative time operation. When f.i. $\mathfrak{M}_1$ is joined to $\{\mathfrak{M}_2\mathfrak{A}_4, \mathfrak{M}_3\mathfrak{A}'_4\}$, so that we obtain $\{\mathfrak{M}_1, \mathfrak{M}_2\mathfrak{A}_4, \mathfrak{M}_3\mathfrak{A}'_4\}$, then according to the above we consider also $\{\mathfrak{M}_1, \mathfrak{M}_1\mathfrak{M}_2\mathfrak{A}_4, \mathfrak{M}_3\mathfrak{A}'_4\}$. However not only $\mathfrak{M}_1\mathfrak{M}_2\mathfrak{A}_4$, but also $\mathfrak{M}_2\mathfrak{M}_1\mathfrak{A}_4$, and therefore $(\mathfrak{M}_2\mathfrak{M}_1\mathfrak{A}_4)^{-1} = \mathfrak{A}_4^{-1}\mathfrak{M}_1\mathfrak{M}_2$ is an operation of the group, and so $\mathfrak{M}_1\mathfrak{M}_2\mathfrak{A}_4\mathfrak{A}_4^{-1}\mathfrak{M}_1\mathfrak{M}_2 = (\mathfrak{M}_1\mathfrak{M}_2)^2$ too. In the same way of course $(\mathfrak{M}_1\mathfrak{M}_3)^2$. Further we then proceed in a similar way as has been indicated above for $\{\mathfrak{M}_1\mathfrak{M}_2\mathfrak{A}_3, \mathfrak{M}_1\mathfrak{M}_3\mathfrak{A}'_3\}$.

§ 9. *Final remarks.* With the aid of the preceding we can find in a rather simple way the 167 kinds of groups of space-time

symmetry operations. Each kind of groups as f.i. $\{\mathfrak{A}_n\}$ contains one group for each value given to n. In another paper a list of these kinds of groups will be given. [1])

Finally still the remark that not in all cases that may be found, the nucleus must remain at rest. [2])

---

[1]) Physik. ZS. **22**, p. 457, 1921.

[2]) In contrast with the theory of space symmetry, which has a mathematical character the theory of space time symmetry must be regarded as having a mechanical nature. Born, Landé and Madelung (see for the literature Comm. N°. 7*a*) have already started with the subdivision of this theory called by them dynamics of atoms; then the preceding might be regarded as an introduction to the kinematics of atoms.

Physiology. — "*On Smelling during complete Exhaustion (resp. adaptation) for a given odour.*" By K. KOMURO (Nagasaki). (Communicated by Prof. H. ZWAARDEMAKER.)

(Communicated at the meeting of March 26, 1921).

A rather considerable number of qualities (so-called specific energies) is to be distinguished in the sense of smell. This is borne out by exhaustion-experiments performed long ago by FRÖHLICH, ARONSOHN, ZWAARDEMAKER, HERMANIDES.

FRÖHLICH [1]) first fatigued his olfactory organ by a given odorous substance and then tried to find out whether this organ was still sensitive to other smells. In his discourse he wrote: "Wenn z. B. Valeriana Celtica gerochen wurde, so wurde darauf der so nahe stehende Geruch von Patchouli nicht wahrgenommen; wohl aber erregte Valeriana nach Patchouli noch einen sehr lebhaften Eindruck."

ARONSOHN [2]) has made a more careful study of the subject. He used some phials filled with different odorous substances. He kept smelling at them until his sense-organ had absolutely been blunted for the scent. Then he took another phial and tried to smell its odour, if possible, and he determined its intensity.

By their intensity he groups the odorous substances into three classes, viz. 1°. those of insistent intensity; 2°. those of decreased intensity, and 3°. those that become inodorous. When e.g. his sense-organ was exhausted for the tincture of iodine, it was still fully sensitive to the stimulus of ethereal oils and also ether; only faintly so to that of oil of citron, of sage, of nutmeg-blossom, of turpentine, of bergamot and of cloves. Nothing was smelled, however, of balsam of copaiba and of spiritus. Some more exhaustion-experiments were made for sulphammonium, camhpor and oleum juniperi. He describes his result as follows: "Verschiedene Geruchsqualitäten afficiren verschiedene Bezirke der Geruchsnerven derart, dass eine Classe von Riechstoffen einen Bezirk maximal erregt, einen zweiten Bezirk in niederen Grade, einen dritten gar nicht erregt."

Similar experiments have been performed by ZWAARDEMAKER [3]) in

---

1) FRÖHLICH, E. Sitzungsber. d. Kaiserl. Akad. d. Wissensch. Wien. Bd. VI. S. 322.

2) ARONSOHN, E. Archiv. f. Anat. u. Physiol. 1886. S. 361.

3) ZWAARDEMAKER, H. Die Physiologie des Geruchs. Leipzig. Engelmann. S. 255. 1895.

cases of toxic anosmia evolved by cocain and in a case of post-diphterial anosmia. His results were distributed over two tables. In toxic anosmia evolved by cocain the smell of oil of camphor, of nutmeg, of Roman chamomile, of lavender and of laurel was unimpaired, whereas oil of thyme, tinctura nucis toncae, oil of eucalyptus, of peppermint, oleum spicae, and oil of valerian was more or less obtused. Complete obtusion was attained with oil of cloves, of anise, of hyssop, of rosemary and with asafoetida.

HERMANIDES[1]) examined exhaustion by taking for an index the lengthening of the reaction-time it causes and tabulated his results as follows:

After Exhaustion with	Lengthening of Reaction-time	Constant Reaction-time
Isoamylacetate	Isoamylacetate Valerianic acid	Nitrobenzol
Nitrobenzol	Nitrobenzol Isoamylacetate	Valerianic acid
Valerianic acid	Valerianic acid Isoamylacetate	Nitrobenzol
Scatol	Scatol	Isoamylacetate Nitrobenzol Valerianic acid

E. L. BACKMAN[2]) objects to the idea of "exhaustion" and prefers the word "adaptation", which, however, he also wishes to avoid, because he bases the whole phenomenon upon his differential-hypothesis. He assumes namely that a smell-sensation can be aroused only when a scent can penetrate as far as the olfactory cells in an increasing or decreasing quantity, but not when the accession of the odorous molecules (HEYNINX's odorivectors)[3]) occurs equably through invariable supplies per diffusion. The olfactory stimulus arises only when a certain procentic increase of the number of intruding molecules has taken place. From this hypothesis it follows that, with complete exhaustion (adaptation), only with a certain procentic increase of the concentration of the odorous matter in the air that reaches the olfactory region, a just noticeable sensation must be obtained. BACKMAN's hypothesis on this point has entirely been confirmed by my experiments conducted in ZWAARDEMAKER's camera

[1]) HERMANIDES, J. Onderz. Physiol. Lab. Utrecht.. (5) Deel 10, p. 1.

[2]) BACKMAN, E. L. Exp. Undersökningar öfter Luktsinnets Fysiologi. Upsala. Läkare förendings Föfhandlingar. N. T. Bd. 22, p. 319, 1917.

[3]) HEYNINX, A. Essai d'olfactique physiologique. These Bruxelles, 1919, p. 2.

odorata, in which first exhaustion (adaptation) for a given odour was obtained, and directly after a measurement of smelling was taken. These experiments will be published in another paper.

For terpineol, guajacol, caproic acid, I made tests in which, after complete exhaustion (adaptation), the smell-measurement was made in perfumed air, which of itself did not set up a smell-sensation any longer.

In order to make a comparative estimate of the quantity of smell in the camera of 400 liters, I have determined, at the termination of my experiments, the minimum perceptibile in ZWAARDEMAKER's smelling-box for the odour for which I had exhausted my sense-organ.

I then found that for terpineol the just noticeable concentration was $3.9.10^{-10}$ grms. per c.c. of air. It is called olfactie, so this definite small quantity of odourous matter per cc. represents the value of one olfactie of my individual unexhausted olfactory sense. At the commencement of the experiment the absolute quantity of odorivector per cc. corresponding with the minimum perceptibile, was presumably smaller, seeing that my smelling capacity for the odours used had certainly decreased a little during my experiments. On this account the term "exhaustion" is perhaps more correct than "adaptation". I will, therefore, use it for the present series of experiments [1]), although it should on the other hand be acknowledged that for the lower values the term is quite appropriate in connection with the constancy of the liminal values of differentiation and with the analogy to the sensation of light. (BACKMAN).

After evaporation of one drop of terpineol the camera of 400 liters in which the head of the observer is enclosed, contains per cc. of air 5.10^{-8} grms of odorous substance. This concentration represents 125 of my olfacties as could be established at the termination of my experiments. I remained in the perfumed air for 6 minutes. At the end of this period my olfactory sense had quite adapted itself to the condition, so that I was not aware of any sensation of smell. I did not perceive anything of the kind either when breathing deeply or when sniffing.

[1]) Olfactometrical determinations showed that the obtusion for terpineol and guajacol was rather considerable at the end of my experiments, when compared with the beginning; for caproic acid, however, there was none. This may be inferred from the length to which the cylinder had to be moved out for the minimum perceptibile. It was:

	at the outset	at the end
for terpineol	0.200 c.m.	0.300 c.m.
for guajacol	0.145 c.m.	0.300 c.m.
for caproic acid	0.110 c.m.	0.110 c.m.

In subsequent experiments this was repeated for guajacol.

The minimum perceptibile in ZWAARDEMAKER's camera amounted for my olfactory organ with guajacol to $6.4.10^{-10}$ grms per cc. of air. This concentration corresponds with the value of 1 olfactie.

After one drop of guajácol was evaporated the camera of 400 liters contained $8.2.10^{-8}$ grms per cc. of air, in other words 128 of my olfacties. I remained for 7 minutes in the perfumed air.

The same was repeated with caproic acid in a third series of experiments.

The minimum perceptibile for caproic acid in ZWAARDEMAKER's camera per cc. of air was $3.3.10^{-10}$ grms, corresponding with one olfactie.

After evaporation of 0.5 drop of caproic acid the camera which encloses the observer's head, contains per c.c. of air: $3.35.10^{-8}$ grms of this odorous substance. This concentration represents 118 of my olfacties. I remained for 6 minutes in the perfumed air before making measurements.

In these 6 minutes my organ got completely exhausted, so that no observations could be made either when breathing deeply or when sniffing.

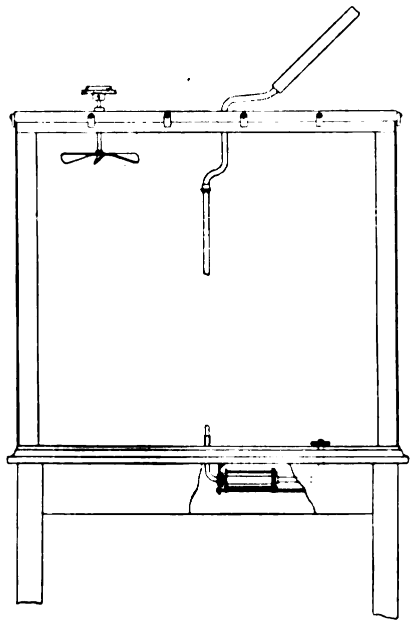
When making a measurement in an inodorous surrounding with the olfactometrical cylinders that I used, the min. perc. for my organ corresponded with

amylacetate	0.11	c.M.
nitrobenzol	0.15	„
terpineol	0.30	„
artif. moschus	0.11	„
allylalcohol	0.20	„
guajacol	0.30	„
caproic acid	0.11	„
pyridin	0.10	„
scatol	0.22	„

However, performing the same determination in the perfumed air of the camera, the results are quite different.

With a view to making these determinations in the camera an olfactometer is attached to the bottom, through which an inspiration-tube of the olfactometer projects. This inside tube of the olfactometer can be freed from adhering odour by means of a current of air passing through the long movable glass-tube, which can be inserted or disconnected at will. During the measurement the turned up part

of the inspiration tube is passed into the forward half of the nostril, while the back part and the other nostril are left open. So the



Smelling Chamber of a capacity of 400 liters arranged for exhaustion-experiments.

Into this chamber, mounted on high legs, the observer's head is thrust through an opening below, which can further be closed by a sliding lid. Under this lid an olfactometer is applied. Through the camera runs a movable glass tube for a current of air to pass through. This current serves to clean the inside tube of the olfactometer in the intervals of the experiments. To the left is a rotating fan. (When the space is used as a camera inodorata a uviol-mercury lamp is burning to destroy the adsorbed odours through ultra-violet light.)

observer inspires the air from the olfactometer with the one nostril, and with the contralateral nostril he inhales the perfumed air of the camera. In order to perfume the air that passes through the olfactometer in the same degree as the air in the large camera a flexible tube is led from the camera to the olfactometer. This tube is taken up by an obturator applied to the outer cylinder of the olfactometer.

As the flexible tube consists of metallic rings held together by caoutchouc, it is necessary first to wash the tube out with tapwater without drying it in order to deodorize the caoutchouc.

These precautions enable us to make the measurements at the olfactometer while being assured that they take place in the perfumed air for which the sense-organ has been fatigued and which does not *per se* impart a smell sensation to the observer.

It is interesting now to determine the sensitiveness of the olfactory organ for guajacol with complete terpineol-exhaustion, for terpineol and guajacol with complete caproic acid-exhaustion.

This is shown in the subjoined tables:

Complete terpineol-exhaustion [1]).

	Min. perc. in olfacties.
Amylacetate	1.7
Nitrobenzol	1.4
Terpineol	∞
Art. Moschus	5.0
Allylalcohol	4.5
Guajacol	4.3
Caproic acid	4.2
Pyridin	1.5
Scatol	5.0

[1]) Time in which complete exhaustion was attained:

with terpineol	1 drop	6 min.,
with guajacol	1 drop	7 min.,
with caproic acid	0.5 drop	8 min.

It appears then that exhaustion is attained sooner for terpineol, then follows caproic acid, while guajacol comes last.

Complete guajacol-exhaustion.

	Min. perc. in olfacties.
Amylacetate	1.7
Nitrobenzol	1.3
Terpineol	4.9
Art. Moschus	5.2
Allylalcohol	8.5
Guajacol	∞
Caproic acid	11.5
Pyridin	1.5
Scatol	5.5

Complete caproic acid exhaustion.	Min. perc. in olfacties.
Amylacetate	1.6
Nitrobenzol	1.3
Terpineol	3.3
Art. Moschus	5.1
Allylalcohol	5.5
Guajacol	5.3
Caproic acid	∞
Pyridin	1.5
Scatol	4.5

GENERAL CONCLUSION.

Amylacetate, nitrobenzol and pyridin are odours for which an olfactory organ that has been exhausted by the smell of terpineol, guajacol or caproic acid is blunted only very slightly. The liminal value is about $1\frac{1}{2}$ times higher than is normally the case. So the anosmia evolved is about $\frac{2}{3}$.

For other qualities of the series of 9 standard-odours (except that for which complete exhaustion exists) the organ is blunted to $\frac{1}{3}$ or $\frac{1}{11}$.

Chemistry. — "*On the Acceleration of Solubility of Metals in Acids by Reducible Compounds.*" By H. J. PRINS. (Communicated by Prof. J. BÖESEKEN.)

(Communicated at the meeting of March 26, 1921).

It is a well-known fact that though there are a great many means available for the reduction of organic compounds, the choice of the reducer greatly contributes to the success of the reduction, so that it is not sufficient to bring hydrogen in status nascens in the presence of the substance that is to be reduced. It ensues from this that there must be a relation between the reducer and the compound that is to be reduced; if this relation were known it would be possible to make a choice with certainty from the available reducers for a definite purpose, or to find new reducers.

Some years ago it was pointed out[1]) that in case of reductions the velocity of solubility of the metal in the acid is enhanced by the reducible substance, and that evidently a cooperation, a coaction must take place between metal, acid, and reducible compound in order to bring about the reduction; which is then accompanied by a more rapid dissolving of the metal.

Definite examples of this have not been recorded, except in the literature[2]) of patents.

Such coactions, which are reckoned among the mutual inductions, are however, known in all kinds of other reactions, especially oxidation reactions. Besides it is known that metals dissolve more rapidly in the presence of oxidizers,[3]) in such cases it is, however, difficult to decide, whether one has to do with a mutual induction or a subsequent reaction, while the formation of primary oxides assumed by some scientists to take place in such reactions as inter-

1) PRINS, Chem. Weekbl. **14**, 72 (also note) and id. 1004 (1917).
Ibid **11**, 476, 477 (1914).
Ibid **12**, 38 et seq. (1915).
Journ. f. prakt. Chem. N. F. **89**. 448 et seq. (1914).

2) LASSAR COHN, Arbeitsmeth. d. org. Chem.

3) VAN NAME, Chem. Centr.bl. (1914). I 20; (1918). I 257, 907.
SALKOWSKI, Chem. Ztg. **40**. 448 (1916).

mediate product, cannot serve as an explanation in the reduction of nitro compounds, aldehydes, and similar substances.[1])

Besides for the knowledge of the phenomena of reduction and undoubtedly also for the knowledge of the phenomena of oxidation, more in particular in organic chemistry, this phenomenon of coaction is of importance in that it is in close relation with catalysis, and can easily pass into it.

The possibility that three substances which in couples do not react on each other or in a very small degree, when brought together, react all three on each other follows from the theory of the mutual activation; the catalysis can then be considered as a special case of the coaction, viz. in which two of the three possible reactions between the three components do not take place or only in a very small degree. At the same time this theory explains how such coactions can be realized, and can be changed into a catalytic action in some cases. If two compounds A and B are placed over against a third one, C, and C is built up of two parts in such a way that one part can be attacked by A, the other by B, a coaction may be expected between A, B and C.

It was now the intention to examine a number of systems, consisting of metal, acid or alkali and reducible compound, first of all qualitatively with regard to the influence that the reducible compound has on the velocity of solution of the metal, either with generation of hydrogen or without.

It now appeared that not only nitrobenzene, but also an aldehyde as benzaldehyde exert a surprisingly accelerating influence on the velocity of solution. It is often, especially in the case of nitrobenzene, so great that the metal dissolves from five hundred to a thousand times more rapidly when nitrobenzene is present. Besides it appears, what was also to be expected, that though nitrobenzene has almost always an accelerating action, benzaldehyde is more selective. (See table).

It is further remarkable that the greatest acceleration was observed in those cases in which in the presence of the reducible compound solution of the metal took place, but no generation of hydrogen, which is probably related to the fact that the contact between metal and reducible compound is prevented by the forming hydrogen, which would also point to the probability that the reduction does

[1]) SKRABAL, Die induzierten Reaktionen. Samm. Chem. und chem. techn. Vorträge. XIII. Bnd. 10. Heft. 1908.

No.	Metal	Acid	Decrease blank × 1000	Decrease nitrob. × 1000	Decrease Benzald. × 1000	Original weight in gr.	Time in min.
1	Iron wire (*h*)	6 % alcoh. HCl	280	790	90	1.98	70
2	"	4 " "	140	720	90	1.98	70
3	"	1 " "	140	200	100	1.98	75
4	"	80 " acetic acid	10	780	10	1.97	60
5	"	86 " formic acid	75	135	0	1.68	70
6	Zinc wire (*e*)	100 " acetic acid	40	120	40	1.86	240
7	"	80 " "	260	470	640	1.12	40
8	"	15 " alcoh. acetic acid	20	920	50	1.12	40
9	Zinc leaf (*h*)	15 " "	30	490	—	1.39	40
10	Lead leaf (*t*)	6 " alcoh. HCl	30	50	50	1.52	270
11	"	80 " acetic acid	0 in 40 min.	1800 in 3 min.	50 in 40 min.	1.80	—
12	"	70 " acetic acid in acetic acid anhydr.	0 in 40 min.	1150 in 10 min.	40 in 40 min.	1.15	—
13	"	25 " alcoh. acetic acid	13 in 180 min.	3080 in 4 min.	13 in 180 min.	3.08	—
14	"	86 " formic acid	0	460	0	1.39	40
15	"	43 " "	10	850	10	2.13	120
16	"	30 " oleic acid in turpentine	330	1020	320	1.15	70
17	"	10 " lauric acid in parafin oil	10	225	0	2.07	120
18	Tin leaf (*f*)	6 " alcoh. HCl	50 in 50 min.	1070 in 10 min.	170 in 50 min.	1.07	—
19	"	80 " acetic acid	10 in 20 min.	690 in 3 min.	6 in 20 min.	0.69	—
20	"	86 " formic acid	0 in 20 min.	470 in 10 min.	0 in 20 min.	0.47	—
21	"	4 " sol. potassium in 90 % alcohol	230	560	—	0.67	2 days.
22	Copper gauze (*h*)	6 " alcoh. HCl	50	1910	—	3.04	200
23	"	6 " alcoh. KCN	250	1330	480	2.57	270
24	Nickel wool (*h*)	6 " alcoh. HCl	11	144	—	0.441	300
25	Aluminium (*h*) leaf	1 " alcoh. HCl	50	125	110	0.985	15
26	id. shiny amalg.	4 " alcoh. acetic acid	50	1450	—	1.45	20
27	Silver wire (*z*)	6 " alcoh. NaCN	4	132	—	0.152	120

(*e*) = electric conduction, is nearly pure.
(*h*) = commercial quality.
(*t*) = so-called tea lead.
(*f*) = tin foil contains a trace of lead.
(*z*) = pure.

To 50 cc. of acid was added 10 cc. of nitrobenzene, resp. benzaldehyde.

not take place through hydrogen in status nascens, but that a coaction between the three components is required.

These preliminary experiments were so executed that only characteristic differences could be expressed; this was desirable in view of the fact that through impurities, presence of different modifications, structure in connection with the treatment, change of the surface in consequence of the solution etc. the metals in themselves can already give differences in the velocity of solution, which might vitiate the conclusions to be drawn.

Use was made of metal in the form of gauze, leaf, or wire, of which a roll or spiral was made, so that the surface that was in contact with the liquid was as much as possible the same. Three rolls of the same weight and surface were now placed in three Erlenmeyer flasks, to each of which the acid was added and after they had assumed the temperature of the waterbath benzaldehyde was added to the one, nitrobenzene to the second, the third containing only an acid by the side of the metal. As soon as a perceptible action had been exerted on one of the three, or as soon as in one of them the metal was quite dissolved, the metal was taken out of them, washed with water, alcohol, and ether, dried and weighed.

The original weight of the metal, the time, and the decrease are recorded in the following table for a number of the principal experiments.

Unless expressly stated 10 c.c. of nitrobenzene or benzaldehyde was added to 50 c.c. of acid (Cf. the table on the next page).

It appears very clearly from the table that the influence of reducible compounds on the velocity of solution of a metal can be very great, in some cases even so great that most probably there is no longer question of an acceleration of an existing reaction, but of a new one.

Though in almost all cases nitrobenzene shows a very great acceleration, also a compound as benzaldehyde appears to be able to exert a strong positive influence (see N°. 7, 11, 12, 18, 23, 25).

It is clear that guided by the theory of these reactions shortly mentioned in the beginning, the number of combinations can be extended, which renders it possible greatly to increase the number of reducers, and to define the conditions which a reducer has to satisfy in a definite case.

Thus metal, a salt of hydrochloric acid, a feeble acid (acetic acid), and nitrobenzene may be taken instead of metal, hydrochloric acid, and nitrobenzene, in this way a coaction is realized in a system of four components.

When nitrobenzene is present in great concentration no hydrogen generation is observed; the solution of the metal and the generation of hydrogen is, however, accelerated by a smaller concentration of the nitrobenzene.

Two strips of zinc-leaf, weighing 10.18 gr., surface 38.6 cm^2, etched with 2 norm. HCl, were placed in 190 cc. of 80°/$_0$ acetic acid, while to one besides 2 cc. of nitrobenzene was added. After having been kept at 73° for 40 min., the zinc was weighed and the generated hydrogen measured. The zinc in the nitrobenzene experiment weighed 7,45 gr., the quantity of hydrogen was 17 cc. (11° and 772 mm.). The zinc in the blank experiment weighed 10.15 gr. and the quantity of hydrogen was 6 cc.

In this experiment there is, however, an unfairness towards the zinc in the blank experiment, because though the two experiments start with the same surface, the nitrobenzene very soon considerably enlarges the surfaces through its strongly corrosive action, which gives it a permanent advantage. Therefore an experiment was made in which the zinc was etched first with diluted hydrochloric acid, then one of the strips of zinc besides with nitrobenzene in 80°/$_0$ acetic acid, and after purification this latter was used for the blank experiment. The result was now as follows:

Weight of the zinc 16.32 gr. Surface 59.2 cm^2. The conc. of the nitrobenzene was taken still smaller, viz. 2,4 gr. to 170 cc. of 80°/$_0$ acetic acid. A third experiment was made with benzophenone, viz. 3.6 gr. to 170 cc. of 80°/$_0$ acetic acid. In connection with the benzophenone experiment the temperature was chosen 20° higher, i.e. 93°.

After 30 min. the condition was as follows: Weight zinc in blank experiment 16,04 gr., generated hydrogen 50 cc. (temp. 11°, barometer 767 mm.). Weight zinc nitrobenzene experiment: 12.50 gr., generated hydrogen 115 cc. Weight zinc benzophenone experiment: 16,09 gr., generated hydrogen 10 cc.

It appears from this, just as from the table that a reducible compound can exert both an accelerating and a retarding influence as well on the dissolving of the metal and on the hydrogen generation.

Finally a similar experiment was made with zinc that had been etched in boiling 30°/$_0$ acetic acid, to which nitrobenzene was added, so that the nature of the zinc surface was the same as in the presence of nitrobenzene for all experiments. The quantity of benzophenone was doubled, so that the quantity of reducible oxygen was equal to that of nitrobenzene.

Weight zinc: 14,32 gr., surface 60 cm^2. Blank experiment contained 170 cc. of 80°/$_0$ acetic acid; the two others resp. 2,4 gr.

of nitrobenzene and 7,2 gr. of benzophenone. The temperature was kept at 65—66°.

After an hour the condition was as follows:

Weight zinc blank experiment: 14,08 gr. Generated hydrogen: 45 cc. (temp. 14°. Barometer 770 mm.).

Weight zinc nitrobenzene experiment: 10,47 gr., generated hydrogen: 142 cc.

Weight zinc benzophenone experiment: 14,19 gr., generated hydrogen 8 cc.

Here too benzophenone gives a retardation, it appearing at the same time from the comparison with the hydrogen generated in the blank experiment that part of the hydrogen has been used for the reduction of the benzophenone. It seems, therefore, that the nature of the surface has no influence on the qualitative result

According to the way indicated above it is further possible to take in these experiments more intricate systems, in which through coaction reactions are brought about or accelerated. Thus in the mixture of metal, hydrochloric acid, and nitrobenzene, the hydrochloric acid can be replaced by a feeble acid, as acetic acid, and a salt of hydrochloric acid.

Then a coaction takes place between four compounds:

1. Copper with 50 cc. 80 % acetic acid and 5 gr. $CaCl_2$.	Decrease 0.180 gr.
2. Copper with as above, and 5 cc. nitrobenzene.	Decrease 0.065 gr.
3. Copper with 50 cc. 70 % alcohol and 5 gr. $CaCl_2$.	Decrease 0.015 gr.
4. Copper with alcoh. as above and $CaCl_2$ as above, and 5 cc. nitrobenzene.	Decrease 0.005 gr.
5. Copper with 50 cc. 80 % acetic acid and 5 gr. $CaCl_2$ and 5 cc. nitrobenzene	Decrease 0.245 gr.

Similar coactions, the explanation of which rests likewise on the considerations given above are met with in the oxidation of organic compounds, and can be most clearly shown by the study of the velocity of dissolving of metal-peroxides in acids whether in presence of oxidyzible compounds or not. Experiments on this subject are still in progress.

It appears from the above preliminary experiments that there is a cooperation, a coaction between metal, acid and reducible compound, which is to be explained by this that a certain mutual activation of the components must take place if the reduction is to appear in a considerable degree. This appears to be the case both in alcoholic and aqueous acetic acid solution, and in anhydrous acetic or paraffin.

CONTENTS.

*